UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2007 1.30 to 4.30

PAPER 29

MODULAR FORMS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

For any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *we write* $f|[\sigma]_k(\tau) = (c\tau + d)^{-k} f(\sigma(\tau))$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that mapping $\tau \in \mathbb{H}$ to the lattice $\Lambda_\tau := \mathbb{Z}\tau + \mathbb{Z}$ induces a bijection between $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and the set of lattices in $\mathbb{C}$ up to homothety.

(b) Prove that the modular invariant $j : \mathbb{H} \to \mathbb{C}$ induces a bijection $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$.

[You may assume the formula $\mathrm{ord}_\infty(f) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \sum_{\tau \neq i,\rho} \mathrm{ord}_\tau(f) = \frac{k}{6}$ for non-zero $f \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.]

(c) Let $\Lambda, \Lambda' \subset \mathbb{C}$ be lattices satisfying $G_4(\Lambda) = G_4(\Lambda')$ and $G_6(\Lambda) = G_6(\Lambda')$. Prove that $\Lambda = \Lambda'$.

2 (a) Define the topology and complex structure of $X(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*$ and prove that $X(1)$ is compact.

(b) Prove using facts about compact Riemann surfaces that the space $M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is finite dimensional.

3 Let $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$ for $q = e^{2\pi i \tau}$.

(a) Using the relation $E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}$ prove that

$$F(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

lies in $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

(b) Let $\Theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Show that, for every $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$,

$$g = (\Theta - \frac{k}{12}E_2)f \in M_{k+2}(\mathrm{SL}_2(\mathbb{Z}))$$

and that $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ if and only if $g \in S_{k+2}(\mathrm{SL}_2(\mathbb{Z}))$.

(c) Show that the coefficients $\tau(n)$ of $F(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$ satisfy

$$(1-n)\tau(n) = 24 \sum_{\ell=1}^{n-1} \sigma_1(\ell)\tau(n-\ell)$$

and

$$\tau(n) \equiv n\sigma_5(n) \pmod 5.$$

[You may use without proof that $\dim M_8(\mathrm{SL}_2(\mathbb{Z})) = \dim S_{12}(\mathrm{SL}_2(\mathbb{Z})) = 1$ and $\dim S_{14}(\mathrm{SL}_2(\mathbb{Z}))) = 0$.]

4 (a) Define the Hecke operators $T_{2k}(n)$ acting on $M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and determine their action on Fourier expansions.

(b) Let $f(\tau) = \sum_{m=0}^{\infty} c(m)q^m \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. If $T_{2k}(n)f = \lambda(n)f$ for all $n \geqslant 1$, show that for all $m, n \geqslant 1$

$$c(n) = c(1)\lambda(n)$$

$$c(0) \neq 0 \Rightarrow \lambda(n) = \sigma_{2k-1}(n)$$

$$\lambda(m)\lambda(n) = \sum_{a|\gcd(m,n)} a^{2k-1}\lambda(\frac{mn}{a^2}).$$

(c) If $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ satisfies $T_{2k}(n)f = \lambda(n)f$ for all $n \geqslant 1$, deduce from (b) that $L(f, s)$ admits an appropriate Euler product.

5 (a) Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Let

$$\mu = \begin{cases} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/2 & \text{if } -I \notin \Gamma \\ [\mathrm{SL}_2(\mathbb{Z}) : \Gamma] & \text{if } -I \in \Gamma. \end{cases}$$

Let ν_2 and ν_3 denote the number of elliptic points of period 2 and 3 in $X(\Gamma) = \Gamma\backslash\mathbb{H}^*$ and ν_∞ the number of cusps of $X(\Gamma)$. Prove that the genus of $X(\Gamma)$ is

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$

(b) Compute all terms in the formulae of (a) for

$$\Gamma = \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

For what values of N is the genus of $X(\Gamma(N))$ equal to zero?

6 (a) Define the space $S_k(\Gamma)$ of cusp forms of weight k with respect to a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

(b) For $f : \mathbb{H} \to \mathbb{C}$ let $\phi(\tau) = |f(\tau)|(\mathrm{Im}(\tau))^{\frac{k}{2}}$. Show that for $\sigma \in \mathrm{SL}_2(\mathbb{Z})$

$$\phi(\sigma(\tau)) = |f|[\sigma]_k(\tau)|(\mathrm{Im}(\tau))^{\frac{k}{2}}.$$

(c) Prove that a function $f : \mathbb{H} \to \mathbb{C}$ is an element of $S_k(\Gamma)$ if and only if the following three conditions hold:

i. f is meromorphic on $\mathbb{H}$,

ii. $f|[\gamma]_k = f$ for all $\gamma \in \Gamma$,

iii. $f(\tau)(\mathrm{Im}(\tau))^{k/2}$ is bounded on $\mathbb{H}$.

END OF PAPER