UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 4 June 2001 9 to 11

PAPER 23

MATHEMATICAL MODELS IN FINANCIAL MANAGEMENT

*Attempt any **THREE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 An investor can invest an initial wealth of T pounds in one of two assets, bank deposits or equity, and is not allowed to borrow for this purpose. Assume that bank deposits do not involve any risk and offer a net return r for each pound invested. Equity holdings are risky. In the good state, which occurs with probability p, equity offers a return g per pound. In the bad state, which occurs with probability $1-p$, the return is b. Assume that $b < r < g$ and $pg + (1-p)b > r$. The investor is an expected utility maximizer, with the von Neumann–Morgenstern utility function in C^2 given by $u(w)$, where w denotes final wealth.

(a) Suppose the investor holds an amount x in bank deposits and the rest of initial wealth in equity. Given this portfolio choice, find expressions for final wealth in the good state (call it w_g) and in the bad state (call it w_b). What is the expected utility of this portfolio?

(b) What is meant by the term *risk neutral*? If this investor were risk neutral, what would the optimal portfolio choice be?

(c) What is meant by the term *risk aversion*? Assuming the investor is risk averse, derive the first-order condition for optimal portfolio choice.

(d) Assume now that the utility function is given by $u(w) = \ln w$ where w is final wealth and p satisfies

$$p > \left[1 + \frac{(g-r)(1+r)}{(r-b)(1+g)}\right]^{-1}.$$

What is the optimal portfolio?

2 (a) Assume that all asset rates of return in a market satisfy the one-factor model

$$\boldsymbol{r}_i = a_i + b_i \boldsymbol{f} \qquad i = 1, \ldots, n.$$

Show that in the absence of arbitrage opportunities

$$\bar{r}_i := \mathbb{E}\boldsymbol{r}_i = \lambda_0 + b_i \lambda_i \qquad i = 1, \ldots, n,$$

for suitable constants $\lambda_0, \lambda_1, \ldots, \lambda_n$.

(b) What is the relationship of this result to the CAPM?

(c) Extend the result of (a) to m factors and interpret the λ_js in financial terms.

(d) Now consider the more realistic model

$$\boldsymbol{r}_i = a_i + \sum_{j=1}^{m} b_{ij} \boldsymbol{f}_j + \boldsymbol{\epsilon}_i \qquad i = 1, \ldots, n,$$

where $\mathbb{E}\boldsymbol{\epsilon}_i = 0$ and $\mathbb{E}\boldsymbol{\epsilon}_i^2 = \sigma_i^2 < s^2$ for all i. Suppose also that a portfolio of these n assets with weights w_i is *well-diversified* in the sense that $w_i < \frac{W}{n}$ for some constant $W \approx 1$. Show that the portfolio return satisfies

$$\boldsymbol{r} \approx a + \sum_{j=1}^{m} b_j \boldsymbol{f}_j$$

asymptotically for suitable constants $a, b_1, \ldots, b_m$ as the number of assets in the market $n \to \infty$ and discuss the implications of this result.

3 (a) Explain precisely what is meant by saying that a security price process S follows a geometric Brownian motion.

(b) Describe briefly another model for such a price process S and indicate whether or not you think it is superior and why.

(c) State the differential form of Ito's lemma and use a discrete time approximation to the underlying diffusion to explain the result heuristically.

(d) Give a heuristic justification for the Black-Scholes partial differential equation for the value f of derivative security:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf + \frac{\partial f}{\partial t} = 0.$$

State precisely the assumptions underlying your derivation and define all parameters involved.

(e) Explain the use of boundary conditions in solving the Black-Scholes PDE for European put and call option values. What is meant by put-call parity?

(f) How is this equation used to value American put and call options?

4 (a) If F has inverse F^{-1}, then for a uniformly distributed sequence, $\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$, show that elements of the sequence $\{\boldsymbol{\eta}_i\} := \{F^{-1}(\boldsymbol{\xi}_i)\}$ satisfy the distribution F.

(b) Given a set of uniformly distributed pseudo-random numbers, how would one derive a set of normally distributed random numbers?

(c) Briefly explain three variance reduction techniques, why they help to improve the accuracy of Monte Carlo estimates, the circumstances under which each is appropriate and their relative effectiveness.

5 (a) In the single factor *Heath-Jarrow-Morton (HJM)* model, given an initial forward rate curve $\mathrm{f}(0,\cdot)$, the forward rate for each maturity T evolves as

$$d\mathbf{f}(t,T) = \boldsymbol{\alpha}(t,T)dt + \boldsymbol{\sigma}(t,T)d\boldsymbol{W}_t \qquad 0 \leqslant t \leqslant T,$$

where $\boldsymbol{W}$ is a Wiener process and $\boldsymbol{\alpha}_t$ and $\boldsymbol{\sigma}_t$ can depend on $\boldsymbol{W}$ and $\mathbf{f}(\cdot,T), T \geqslant t$, up to time t. Give expressions for $\mathbf{P}(t,T)$ and $\boldsymbol{r}_t := \mathbf{f}(t,t)$ and, defining the *cash bond* $\mathbf{B}_t = \exp(\int_0^t \boldsymbol{r}_s dS)$, for the *discounted bond* $\mathbf{Z}_t(t,T) := \mathbf{B}_t^{-1}\mathbf{P}(t,T)$.

(b) Defining $\boldsymbol{\Sigma}(t,T) := -\int_t^T \boldsymbol{\sigma}(t,u)du$, under the risk neutral measure Q, $\mathbf{P}$ has dynamics

$$d\mathbf{P}(t,T) = \mathbf{P}(t,T)\left[\boldsymbol{r}_t dt + \boldsymbol{\Sigma}(t,T)d\boldsymbol{W}_t\right].$$

If $\mathbf{X}$ is the single payoff of a derivative maturing at $T > t$ show that its current price at t is given by

$$\mathbf{V}(t,\boldsymbol{r}) = \mathbb{E}_Q\left[e^{-\int_t^T \boldsymbol{r}_s ds}\mathbf{X}|\mathcal{F}_t\right].$$

(c) Give an expression for the dynamics of the short rate $\boldsymbol{r}$ under Q. Is it necessarily Markov? Explain your answer.