UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 7 June 2002 1.30 to 3.30

PAPER 9

EXTREMAL GRAPH THEORY

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $t(r,\epsilon,n)$ be the maximum value of t such that every graph of order n and size at least

$$\left(1-\frac{1}{r}+\epsilon\right)\binom{n}{2}$$

contains $K_{r+1}(t)$.

Prove that if $r \geq 2$ and $\epsilon > 0$ are fixed then $t(r,\epsilon,n) \to \infty$ as $n \to \infty$.

Describe how $t(r,\epsilon,n)$ grows as a function of n, justifying your answer.

2 State what is meant by a *stable matching* in a bipartite graph.

Prove that every bipartite graph with preference assignment has a stable matching.

Let bg be an edge of a bipartite graph with preference assignment, such that g is b's worst choice and b is g's worst choice. Show that, if bg occurs in some stable matching, then it occurs in every stable matching.

Prove that, if G is a bipartite graph, then $\chi'_\ell(G) = \chi'(G)$, where $\chi'_\ell(G)$ is the list-chromatic-index and $\chi'(G)$ is the chromatic index of G.

3 What is meant by a graph being k-linked? Prove that a $22k$-connected graph is k-linked. [You may assume without proof that a $22k$-connected graph has a subcontraction H with $2\delta(H) \geq |H| + 4k - 1$.]

Show that a $(3k-2)$-connected graph need not be k-linked.

4 State and prove Szemerédi's Regularity Lemma.
[You may assume without proof a form of the Cauchy-Schwarz inequality with deviation, and also that $|d(U',W') - d(U,W)| \leq 2\delta$ for any pair (U,W) with $U' \subset U$, $W' \subset W$, $|U'| \geq (1-\delta)|U|$, $|W'| \geq (1-\delta)|W|$.]

Let F be a graph with $\chi(F) = r+1$ and let $\epsilon > 0$. Show that, if n is large, then every graph G of order n not containing F must have a subgraph H, with $e(G) - e(H) < \epsilon n^2$, such that H contains no K_{r+1}.