UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2007 9.00 to 12.00

PAPER 51

ADVANCED QUANTUM FIELD THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain how for anti-commuting variables θ_i, $\theta_i\theta_j = -\theta_j\theta_i$, integration may be defined so that

$$\int \mathrm{d}\theta_n \ldots \mathrm{d}\theta_1 \ \theta_{i_1} \ldots \theta_{i_n} = \epsilon_{i_1 \ldots i_n} \, , \qquad \epsilon_{12\ldots n} = 1 \, .$$

Show that for an $n \times n$ matrix $\underline{B}$ and with $\underline{\theta} = (\theta_1, \ldots \theta_n)$, $\underline{\bar\theta} = (\bar\theta_1, \ldots, \bar\theta_n)$ anti-commuting n-vectors

$$\int \prod_{i=1}^{n} \mathrm{d}\bar\theta_i \mathrm{d}\theta_i \ \exp\left(-\underline{\bar\theta} \cdot \underline{B}\, \underline{\theta}\right) = \det \underline{B} \, .$$

Let

$$\left\langle O(\underline{\theta}, \underline{\bar\theta}) \right\rangle = \frac{1}{\det \underline{B}} \int \prod_{i=1}^{n} \mathrm{d}\bar\theta_i \mathrm{d}\theta_i \ O(\underline{\theta}, \underline{\bar\theta}) \ \exp\left(-\underline{\bar\theta} \cdot \underline{B}\, \underline{\theta}\right) ,$$

for any polynomial $O(\underline{\theta}, \underline{\bar\theta})$. Show that

$$\left\langle \theta_i \, \bar\theta_j \right\rangle = (\underline{B}^{-1})_{ij} \, .$$

What is $\left\langle \theta_i \, \bar\theta_j \, \theta_k \, \bar\theta_l \right\rangle$? Show that for $n \times n$ matrices $\underline{M}$, $\underline{N}$

$$\left\langle (\underline{\bar\theta} \cdot \underline{M}\, \underline{\theta}) \ (\underline{\bar\theta} \cdot \underline{N}\, \underline{\theta}) \right\rangle = -\mathrm{tr}\big(\underline{M}\underline{B}^{-1}\underline{N}\underline{B}^{-1}\big) + \mathrm{tr}\big(\underline{M}\underline{B}^{-1}\big)\, \mathrm{tr}\big(\underline{N}\underline{B}^{-1}\big) \, .$$

Sketch appropriate Feynman diagrams for this.

For Dirac fermion fields $\psi, \bar\psi$ and a free action $S = -\int \mathrm{d}^d x \, \bar\psi(\gamma \cdot \partial + m)\psi$, with γ_μ gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, show that

$$\left\langle \psi(x) \, \bar\psi(0) \right\rangle = i S_F(x) \, ,$$

where the Feynman propagator has the Fourier transform

$$\tilde S_F(p) = \frac{-i\gamma \cdot p + m}{-p^2 - m^2 + i\epsilon} \, .$$

2 Show that, for k_μ a d-dimensional vector and $k^2 = k_\mu k_\mu$,

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d k\,\frac{1}{(k^2+m^2)^n} = \frac{1}{(n-1)!}\,\frac{1}{(4\pi)^{\frac{1}{2}d}}\,\Gamma(n-\tfrac{1}{2}d)\,(m^2)^{\frac{1}{2}d-n}\,.$$

Using this verify that

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d k\,\frac{1}{(k^2+m^2)((p-k)^2+m^2)} \sim \frac{1}{8\pi^2\varepsilon} \quad \text{as} \quad \varepsilon = 4-d \to 0\,.$$

For a scalar field theory in d-dimensions with Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\,\mu^\varepsilon\lambda\phi^4\,,$$

what are the Feynman rules? Draw the one loop connected one particle irreducible Feynman graphs with two and four external lines. Show that we may cancel the divergences which arise at one loop for $\varepsilon \to 0$ by adding to $\mathcal{L}$

$$\mathcal{L}_{\text{c.t.}} = -\tfrac{1}{2}\,\frac{\lambda m^2}{16\pi^2\varepsilon}\,\phi^2 - \tfrac{1}{24}\,\mu^\varepsilon\,\frac{3\lambda^2}{16\pi^2\varepsilon}\,\phi^4\,.$$

Define the bare coupling λ_0 and determine

$$\hat\beta(\lambda) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda\Big|_{\lambda_0} = -\varepsilon\lambda + \beta(\lambda)\,,$$

to one loop. The running coupling $\lambda(\mu)$ is defined by

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\lambda(\mu) = \hat\beta\big(\lambda(\mu)\big)\,.$$

Assuming that $\hat\beta(\lambda)$ is $\mathrm{O}(\lambda^2)$ sketch a graph for $\varepsilon > 0$ and determine what happens for this case to $\lambda(\mu)$ as $\mu \to 0$ and $\mu \to \infty$.

[You may assume $\int \mathrm{d}^d k\, e^{-\alpha k^2} = (\pi/\alpha)^{\frac{1}{2}d}$ for $\alpha > 0$, and also $\int_0^\infty \mathrm{d}s\, s^{a-1}e^{-s} = \Gamma(a)$ defines the standard Gamma function, $\Gamma(1) = 1$, $\Gamma(a+1) = a\Gamma(a)$.]

3 Consider a quantum field theory with a single scalar field ϕ and a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - V(\phi)\,,$$

and define

$$e^{iW[J]} = \int \mathrm{d}[\phi]\, e^{iS[\phi] + i\int \mathrm{d}^d x\, \phi J}\,, \quad S[\phi] = \int \mathrm{d}^d x\, \mathcal{L}\,.$$

Let

$$\varphi = \frac{\delta}{\delta J} W[J]\,, \qquad \Gamma[\varphi] = -W[J] + \int \mathrm{d}^d x\, \varphi J\,.$$

Describe briefly how $\Gamma[\varphi]$ is related to connected one particle irreducible Feynman diagrams.

Writing $\phi = \varphi + f$ and keeping terms only up to second order in the expansion of $S[\phi]$ show that we then have

$$\Gamma[\varphi] = -S[\varphi] + \frac{1}{2i} \ln\det\Delta\,, \qquad \Delta = -\partial^2 + V''(\varphi)\,.$$

Let φ be a constant and show that after subtracting a divergence in four dimensions we may define an effective potential from the result for Γ which is of the form

$$V_{\text{eff}}(\varphi) = V(\phi) + \frac{1}{64\pi^2} V''(\varphi)^2 \big(\ln V''(\varphi) + \text{const.}\big)\,.$$

Why is $V(\phi)$ a quartic polynomial a special case?

[You may assume, using Wick rotation,

$$\frac{1}{(2\pi)^d i} \int \mathrm{d}^d k\, \ln(k^2 + A) = -\frac{\Gamma(-\frac{1}{2}d)}{(4\pi)^{\frac{1}{2}d}}\, A^{\frac{1}{2}d}\,, \quad A > 0\,.$$
]

4 G is a continuous group with elements $g(\theta)$ for $\theta = (\theta_1, \ldots, \theta_p)$ and $g(0) = 1$, the identity. Near the identity $g(\theta) \approx 1 + \theta_i t_i$. Let $\mathrm{d}\mu(\theta)$ define integration over G so that if $g(\theta) g_0 = g(\theta')$ then $\mathrm{d}\mu(\theta) = \mathrm{d}\mu(\theta')$ and also for $\theta \approx 0$, $\mathrm{d}\mu(\theta) \approx \mathrm{d}^p\theta$.

Suppose $\underline{x}$ is a n-dimensional vector which transforms under the group G so that $\underline{x} \to \underline{x}^g$ with $\mathrm{d}^n\underline{x} = \mathrm{d}^n\underline{x}^g$. Assume $F_j(\underline{x})$, $j = 1, \ldots, p$, are functions such that a solution to $F_j(\underline{x}_0) = 0$ for $\underline{x}_0 = \underline{x}^{g_0}$ is possible for a unique g_0 and any $\underline{x}$. Why must

$$F_j\big(\underline{x}_0{}^{g(\theta)}\big) = M_{ji}(\underline{x}_0)\theta_i \quad \text{as} \quad \theta_i \to 0\,.$$

If $f(\underline{x}) = f(\underline{x}^g)$, for any $g \in G$, show that

$$\int \mathrm{d}^n\underline{x}\, f(\underline{x}) = C \int \mathrm{d}^n\underline{x}\ \delta^p\big(F(\underline{x})\big)\ \det M(\underline{x})\, f(\underline{x})\,, \quad C = \int \mathrm{d}\mu(\theta)\,.$$

Explain why these considerations are relevant to the quantisation of gauge theories. With A_μ a gauge field belonging to the Lie algebra of a gauge group and $F_{\mu\nu}$ the field strength describe how starting from a classical gauge invariant action $\mathcal{L} = -\frac{1}{4}F^{\mu\nu} \cdot F_{\mu\nu}$ we are led to a quantum Lagrangian

$$\mathcal{L}_q = -\tfrac{1}{4}F^{\mu\nu} \cdot F_{\mu\nu} + b \cdot F^\mu A_\mu + \tfrac{1}{2}\xi\, b \cdot b + \bar{c} \cdot F^\mu (D_\mu c)\,,$$

where $F^\mu A_\mu$ is a linear gauge fixing function, D_μ a covariant derivative, ξ a parameter and $b, \bar{c}, c$ are additional fields whose role should be explained.

END OF PAPER