UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 10 June, 2011 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

(i) Let V be a vector space over a field F, and W_1, W_2 subspaces of V. Define the subset $W_1 + W_2$ of V, and show that $W_1 + W_2$ and $W_1 \cap W_2$ are subspaces of V.

(ii) When W_1, W_2 are finite-dimensional, state a formula for $\dim(W_1 + W_2)$ in terms of $\dim W_1$, $\dim W_2$ and $\dim(W_1 \cap W_2)$.

(iii) Let V be the $\mathbb{R}$-vector space of all $n \times n$ matrices over $\mathbb{R}$. Let S be the subspace of all symmetric matrices and T the subspace of all upper triangular matrices (the matrices (a_{ij}) such that $a_{ij} = 0$ whenever $i > j$). Find $\dim S$, $\dim T$, $\dim(S \cap T)$ and $\dim(S + T)$. Briefly justify your answer.

2F Groups, Rings and Modules

A ring R satisfies the descending chain condition (DCC) on ideals if, for every sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ of ideals in R, there exists n with $I_n = I_{n+1} = I_{n+2} = \ldots$. Show that $\mathbb{Z}$ does not satisfy the DCC on ideals.

3E Analysis II

Let $B[0,1]$ denote the set of bounded real-valued functions on $[0,1]$. A distance d on $B[0,1]$ is defined by

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Given that $(B[0,1], d)$ is a metric space, show that it is complete. Show that the subset $C[0,1] \subset B[0,1]$ of continuous functions is a closed set.

4E Complex Analysis

Let $f(z)$ be an analytic function in an open subset U of the complex plane. Prove that f has derivatives of all orders at any point z in U. [You may assume Cauchy's integral formula provided it is clearly stated.]

5A Methods

Use the method of characteristics to find a continuous solution $u(x,y)$ of the equation

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0,$$

subject to the condition $u(0,y) = y^4$.

In which region of the plane is the solution uniquely determined?

6C Quantum Mechanics

Consider the 3-dimensional oscillator with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}(x^2 + y^2 + 4z^2)\,.$$

Find the ground state energy and the spacing between energy levels. Find the degeneracies of the lowest three energy levels.

[You may assume that the energy levels of the 1-dimensional harmonic oscillator with Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2$$

are $(n+\frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$.]

7C Electromagnetism

A plane electromagnetic wave in a vacuum has electric field

$$\mathbf{E} = (E_0 \sin k(z - ct), 0, 0)\,.$$

What are the wavevector, polarization vector and speed of the wave? Using Maxwell's equations, find the magnetic field $\mathbf{B}$. Assuming the scalar potential vanishes, find a possible vector potential $\mathbf{A}$ for this wave, and verify that it gives the correct $\mathbf{E}$ and $\mathbf{B}$.

8B Numerical Analysis

Consider the multistep method for numerical solution of the differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$:

$$\sum_{l=0}^{s} \rho_l \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \quad n = 0, 1, \ldots .$$

What does it mean to say that the method is of order p, and that the method is convergent?

Show that the method is of order p if

$$\sum_{l=0}^{s} \rho_l = 0, \quad \sum_{l=0}^{s} l^k \rho_l = k \sum_{l=0}^{s} l^{k-1} \sigma_l, \quad k = 1, 2, \ldots, p,$$

and give the conditions on $\rho(w) = \sum_{l=0}^{s} \rho_l w^l$ that ensure convergence.

Hence determine for what values of θ and the σ_i the two-step method

$$\mathbf{y}_{n+2} - (1-\theta)\mathbf{y}_{n+1} - \theta \mathbf{y}_n = h[\sigma_0 \mathbf{f}(t_n, \mathbf{y}_n) + \sigma_1 \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \sigma_2 \mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})]$$

is (a) convergent, and (b) of order 3.

9H Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a Markov chain on a state space S, and let $p_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i)$.

(i) What does the term *communicating class* mean in terms of this chain?

(ii) Show that $p_{ii}(m+n) \geqslant p_{ij}(m) p_{ji}(n)$.

(iii) The period d_i of a state i is defined to be

$$d_i = \gcd\{n \geqslant 1 : p_{ii}(n) > 0\}.$$

Show that if i and j are in the same communicating class and $p_{jj}(r) > 0$, then d_i divides r.

SECTION II

10G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space and $f, g : V \to V$ linear transformations. Suppose f is invertible and diagonalisable, and $f \circ g = t \cdot (g \circ f)$ for some real number $t > 1$.

(i) Show that g is nilpotent, i.e. some positive power of g is 0.

(ii) Suppose that there is a non-zero vector $v \in V$ with $f(v) = v$ and $g^{n-1}(v) \neq 0$. Determine the diagonal form of f.

11F Groups, Rings and Modules

State and prove the Hilbert Basis Theorem.

Is every ring Noetherian? Justify your answer.

12E Analysis II

Define a contraction mapping and state the contraction mapping theorem.

Let (X, d) be a non-empty complete metric space and let $\phi : X \to X$ be a map. Set $\phi^1 = \phi$ and $\phi^{n+1} = \phi \circ \phi^n$. Assume that for some integer $r \geqslant 1$, ϕ^r is a contraction mapping. Show that ϕ has a unique fixed point y and that any $x \in X$ has the property that $\phi^n(x) \to y$ as $n \to \infty$.

Let $C[0,1]$ be the set of continuous real-valued functions on $[0,1]$ with the uniform norm. Suppose $T : C[0,1] \to C[0,1]$ is defined by

$$T(f)(x) = \int_0^x f(t)\,dt$$

for all $x \in [0,1]$ and $f \in C[0,1]$. Show that T is not a contraction mapping but that T^2 is.

13G Metric and Topological Spaces

Let X, Y be topological spaces and $X \times Y$ their product set. Let $p_Y : X \times Y \to Y$ be the projection map.

(i) Define the product topology on $X \times Y$. Prove that if a subset $Z \subset X \times Y$ is open then $p_Y(Z)$ is open in Y.

(ii) Give an example of X, Y and a closed set $Z \subset X \times Y$ such that $p_Y(Z)$ is not closed.

(iii) When X is compact, show that if a subset $Z \subset X \times Y$ is closed then $p_Y(Z)$ is closed.

14D Complex Methods

State and prove the convolution theorem for Laplace transforms.

Use Laplace transforms to solve

$$2f'(t) - \int_0^t (t-\tau)^2 f(\tau)\, d\tau = 4tH(t)$$

with $f(0) = 0$, where $H(t)$ is the Heaviside function. You may assume that the Laplace transform, $\widehat{f}(s)$, of $f(t)$ exists for $\operatorname{Re} s$ sufficiently large.

15F Geometry

Suppose that P is a point on a Riemannian surface S. Explain the notion of geodesic polar co-ordinates on S in a neighbourhood of P, and prove that if C is a geodesic circle centred at P of small positive radius, then the geodesics through P meet C at right angles.

16D Variational Principles

Derive the Euler-Lagrange equation for the integral

$$\int_{x_0}^{x_1} f(y, y', y'', x)\, dx$$

where the endpoints are fixed, and $y(x)$ and $y'(x)$ take given values at the endpoints.

Show that the only function $y(x)$ with $y(0) = 1$, $y'(0) = 2$ and $y(x) \to 0$ as $x \to \infty$ for which the integral

$$\int_0^\infty \left(y^2 + (y')^2 + (y' + y'')^2\right) dx$$

is stationary is $(3x+1)e^{-x}$.

17A Methods

Let D be a two dimensional domain with boundary ∂D. Establish Green's second identity

$$\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dA = \int_{\partial D} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds$$

where $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on ∂D.

State the differential equation and boundary conditions which are satisfied by a Dirichlet Green's function $G(\mathbf{r}, \mathbf{r}_0)$ for the Laplace operator on the domain D, where $\mathbf{r}_0$ is a fixed point in the interior of D.

Suppose that $\nabla^2 \psi = 0$ on D. Show that

$$\psi(\mathbf{r}_0) = \int_{\partial D} \psi(\mathbf{r}) \frac{\partial}{\partial n} G(\mathbf{r}, \mathbf{r_0}) ds.$$

Consider Laplace's equation in the upper half plane,

$$\nabla^2 \psi(x, y) = 0, \quad -\infty < x < \infty \quad \text{and} \quad y > 0,$$

with boundary conditions $\psi(x, 0) = f(x)$ where $f(x) \to 0$ as $|x| \to \infty$, and $\psi(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$. Show that the solution is given by the integral formula

$$\psi(x_0, y_0) = \frac{y_0}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{(x - x_0)^2 + y_0^2} \, dx.$$

[*Hint: It might be useful to consider*

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} (\log |\mathbf{r} - \mathbf{r}_0| - \log |\mathbf{r} - \tilde{\mathbf{r}}_0|)$$

for suitable $\tilde{\mathbf{r}}_\mathbf{0}$. *You may assume* $\nabla^2 \log |\mathbf{r} - \mathbf{r}_0| = 2\pi \delta(\mathbf{r} - \mathbf{r}_0)$.]

18D Fluid Dynamics

Show that an irrotational incompressible flow can be determined from a velocity potential ϕ that satisfies $\nabla^2\phi = 0$.

Given that the general solution of $\nabla^2\phi = 0$ in plane polar coordinates is

$$\phi = \sum_{n=-\infty}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n + c \log r + b\theta \,,$$

obtain the corresponding fluid velocity.

A two-dimensional irrotational incompressible fluid flows past the circular disc with boundary $r = a$. For large r, the flow is uniform and parallel to the x-axis ($x = r\cos\theta$). Write down the boundary conditions for large r and on $r = a$, and hence derive the velocity potential in the form

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\kappa\theta}{2\pi} \,,$$

where κ is the circulation.

Show that the acceleration of the fluid at $r = a$ and $\theta = 0$ is

$$\frac{\kappa}{2\pi a^2}\left(-\frac{\kappa}{2\pi a}\mathbf{e}_r - 2U\mathbf{e}_\theta\right).$$

19H Statistics

Consider independent random variables $X_1, \ldots, X_n$ with the $N(\mu_X, \sigma_X^2)$ distribution and $Y_1, \ldots, Y_n$ with the $N(\mu_Y, \sigma_Y^2)$ distribution, where the means μ_X, μ_Y and variances σ_X^2, σ_Y^2 are unknown. Derive the generalised likelihood ratio test of size α of the null hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$ against the alternative $H_1 : \sigma_X^2 \neq \sigma_Y^2$. Express the critical region in terms of the statistic $T = \dfrac{S_{XX}}{S_{XX} + S_{YY}}$ and the quantiles of a beta distribution, where

$$S_{XX} = \sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2 \quad\text{and}\quad S_{YY} = \sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2 .$$

[You may use the following fact: if $U \sim \Gamma(a,\lambda)$ and $V \sim \Gamma(b,\lambda)$ are independent, then $\dfrac{U}{U+V} \sim \mathrm{Beta}(a,b)$.]

20H Optimization

A company must ship coal from four mines, labelled A, B, C, D, to supply three factories, labelled a, b, c. The per unit transport cost, the outputs of the mines, and the requirements of the factories are given below.

	A	B	C	D	
a	12	3	5	2	34
b	4	11	2	6	21
c	3	9	7	4	23
	20	32	15	11	

For instance, mine B can produce 32 units of coal, factory a requires 34 units of coal, and it costs 3 units of money to ship one unit of coal from B to a. What is the minimal cost of transporting coal from the mines to the factories?

Now suppose increased efficiency allows factory b to reduce its requirement to 20.8 units of coal, and as a consequence, mine B reduces its output to 31.8 units. By how much does the transport cost decrease?

END OF PAPER