MA 1972 Discrete Mathematics, Probability and Statistics
May 2006 - Answers

Section A

A1 There are 10 choices for each of the first three characters and 26 for each of the last two so the total number of choices is $10^3 \times 26^2 = 676000$.

A2 (a) Number the balls one to six with the red balls being numbered one to four. Then the sample space Ω is $\{1, 2, 3, 4, 5, 6\}$. Let A be the event that the first ball is red. Then $A = \{1, 2, 3, 4\}$. Since all outcomes are equally likely, $\mathrm{P}(A) = |A|/|\Omega| = 2/3$.

(b) Let B be the event that the second ball is red. Then $B = A$ and the problem is exactly the same as part a. So $\mathrm{P}(B) = |B|/|\Omega| = 2/3$.

(c) We want $\mathrm{P}(A \cap B)$. Now $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B|A)$. We have $\mathrm{P}(B|A) = 3/5$ and so $\mathrm{P}(A \cap B) = 2/3 \times 3/5 = 2/5$.

A3 (a) Since $A \subseteq B$, we have $A \cap B = A$.

(b) We need to show that if $x \in A \cap C$ then $x \in B \cap C$. Suppose $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $A \subseteq B$, we must also have $x \in B$ and so $x \in B \cap C$.

A4 (a) $\mathcal{P}(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ and $\mathcal{P}(\{\emptyset, 1\}) = \{\emptyset, \{\emptyset\}, \{1\}, \{\emptyset, 1\}\}$.

(b) We use the two results that $|X \times Y| = |X||Y|$ and $|\mathcal{P}(X)| = 2^{|X|}$. Hence

$$|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{|A||B|} = 2^{mn}.$$

Section B

B1 (a) i. The number of choices of meal is $2 \times 4 \times 3 = 24$

ii. A meal can consist of a starter and a main course, a main course and a dessert, or a starter and a dessert. There are $2 \times 4 = 8$ of the first type, $4 \times 3 = 12$ of the second and $2 \times 3 = 6$ of the third. Hence the total number is $8 + 12 + 6 = 26$.

(b) i. To select a Korfball team, we must choose two women and two men. There are $\binom{4}{2} = 6$ ways to choose the women and $\binom{5}{2} = 10$ ways to choose the men. Hence the number of possible teams is $6 \times 10 = 60$.

ii. To select an Openball team, we must choose either two women and three men, or three women and two men. The number of teams of the first type is $\binom{4}{2}\binom{5}{3} = 6 \times 10 = 60$. The number of teams of the second type is $\binom{4}{3}\binom{5}{2} = 4 \times 10 = 40$. Hence the total number of possible teams is $60 + 40 = 100$.

iii. To select a Cantorball team, we first choose the three members of the squad and then choose the player to leave out as substitute. There are nine players in total so there are $\binom{9}{3} = 84$ ways to choose the squad of three. Once the squad is chosen there are three ways to choose the substitute. Hence the total number of teams is $3 \times 84 = 252$.

(c) We first show that the statement is true when $n = 0$. If $n = 0$ then

$$\sum_{i=0}^{n}(2i^2 + i) = \sum_{i=0}^{0}(2i^2 + i) = 0$$

and

$$\frac{1}{6}n(n + 1)(4n + 5) = 0,$$

so the statement holds when $n = 0$.

We now show that if the statement holds when $n = m$ then it remains true when $n = m + 1$. We will prove the statement for $n = m + 1$ by using the statement with $n = m$.

When $n = m + 1$, we have

$$\sum_{i=0}^{m+1}(2i^2 + i) = \sum_{i=0}^{m}(2i^2 + i) + 2(m + 1)^2 + (m + 1).$$

Using the inductive hypothesis we have

$$\begin{aligned}
\sum_{i=0}^{m}(2i^2 + i) + 2(m + 1)^2 + (m + 1) &= \frac{1}{6}m(m + 1)(4m + 5) + 2(m + 1)^2 + (m + 1) \\
&= \frac{1}{6}(m + 1)(m(4m + 5) + 12(m + 1) + 6) \\
&= \frac{1}{6}(m + 1)(4m^2 + 17m + 18) \\
&= \frac{1}{6}(m + 1)(m + 2)(4(m + 1) + 5).
\end{aligned}$$

This is the statement with $n = m + 1$ and so by induction the result is true for all $n \in \mathbb{N}$.

B2 (a) i. Let X denote the number of matches missed by Michael.

$$\mathrm{P}(X = 2) = \binom{6}{2}0.4^2(1 - 0.4)^{6-2} = 0.31104.$$

ii.

$$\begin{aligned}
\mathrm{P}(X \leq 2) &= \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2) \\
&= \binom{6}{0}0.4^0(1 - 0.4)^{6-0} + \binom{6}{1}0.4^1(1 - 0.4)^{6-1} + \binom{6}{2}0.4^2(1 - 0.4)^{6-2} \\
&= 0.046656 + 0.186624 + 0.31104 \\
&= 0.54432.
\end{aligned}$$

iii. We require $\mathrm{P}(X = 2|X \leq 2)$. However

$$\mathrm{P}(X = 2|X \leq 2) = \frac{\mathrm{P}(X = 2 \cap X \leq 2)}{\mathrm{P}(X \leq 2)} = \frac{\mathrm{P}(X = 2)}{\mathrm{P}(X \leq 2)} = \frac{0.31104}{0.54432} = \frac{4}{7}.$$

(b) i. Let Y denote the number of suits bought by a random customer.

$$\mathrm{P}(Y = 2) = \frac{e^{-0.4}0.4^2}{2!} \simeq 0.0536.$$

ii.

$$\begin{aligned}
\mathrm{P}(Y \geq 2) &= 1 - \mathrm{P}(Y < 2) \\
&= 1 - (\mathrm{P}(Y = 0) + \mathrm{P}(Y = 1)) \\
&= 1 - e^{-0.4} - 0.4e^{-0.4} \\
&\simeq 0.0616
\end{aligned}$$

iii. The sum of 10 independent random variables having the Poisson distribution with parameter 0.4 is a random variable having the Poisson distribution with parameter 4.

iv. Let E_1 and E_2 be the events that the first two customers to the shop do not buy a suit. We require $\mathrm{P}(E_1 \cap E_2)$. Note that E_1 and E_2 are independent. Hence

$$\mathrm{P}(E_1 \cap E_2) = \mathrm{P}(E_1)\,\mathrm{P}(E_2) = (e^{-0.4})^2 \simeq 0.4493.$$

v. Let E be the event that no-one buys a suit between 9.00 and 10.00. Let X_0, X_1, X_2 be the events that there are respectively 0, 1, 2 customers. Then

$$\begin{aligned}
\mathrm{P}(E) &= \mathrm{P}(E|X_0)\,\mathrm{P}(X_0) + \mathrm{P}(E|X_1)\,\mathrm{P}(X_1) + \mathrm{P}(E|X_2)\,\mathrm{P}(X_2) \\
&= 1 \times 0.5^2 + e^{-0.4} \times 2 \times 0.5^2 + 0.4493 \times 0.5^2 \\
&= 0.6975.
\end{aligned}$$