**UNIVERSITY OF ABERDEEN** **SESSION 2004–05**

Degree Examination

MX4505 Chaos and Fractals

Monday 23 May 2005 (9 am to 11 am)

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*Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.*

*Marks may be deducted for answers that do not show clearly how the solution is reached.*

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*Answer THREE questions. All questions carry equal weight.*

**1.** **(a)** Consider the fractal curve $\gamma : [0,1] \longrightarrow \mathbb{R}$ with generator

$$x_0 = (0,0), \quad x_1 = (\tfrac{1}{6}, 0), \quad x_2 = (\tfrac{1}{3}, \tfrac{\sqrt{3}}{6}), \quad x_3 = (\tfrac{2}{3}, \tfrac{\sqrt{3}}{6}), \quad x_4 = (\tfrac{5}{6}, 0), \quad x_5 = (1,0).$$

(i) Sketch $\gamma_0, \gamma_1$ and $\gamma_2$.

(ii) Find orientation-preserving similarities $f_1, \ldots, f_5 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ mapping $(0,0)$ to $x_{i-1}$ and $(1,0)$ to $x_i$, where $1 \leq i \leq 5$. Describe $f_1, \ldots, f_5$ geometrically in terms of scaling, rotation, and translation.

**(b)** Let $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f_1(x) = \tfrac{1}{4}x, \quad f_2(x) = \tfrac{1}{4}x + \tfrac{3}{4}.$$

Let $C_0 = [0,1]$. Let $C_n = f_1(C_{n-1}) \cup f_2(C_{n-1})$. Let $C = \bigcap_{n=0}^{\infty} C_n$.

(i) Sketch $C_0$, $C_1$ and $C_2$.

(ii) Calculate the similarity dimension of $C$ (with respect to the iterated function system given by $f_1, f_2$).

(iii) Calculate the box dimension of $C$.

(iv) Calculate the Hausdorff distance between $C_0$ and $C_1$.

(v) Calculate the Hausdorff distance between $C_0$ and $C$.

**2.** Three orientation-preserving similarities $f_1, f_2, f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, are defined by $f_i(x) = \frac{1}{3}x + b_i$, where

$$b_1 = (0,0), \quad b_2 = (\tfrac{1}{3}, \tfrac{1}{3}), \quad b_3 = (\tfrac{2}{3}, \tfrac{2}{3}).$$

Let $K(\mathbb{R}^2)$ be the collection of non-empty, compact subsets of $\mathbb{R}^2$. Define $F : K(\mathbb{R}^2) \longrightarrow K(\mathbb{R}^2)$ by $F(A) = f_1(A) \cup f_2(A) \cup f_3(A)$. You may assume that

$$d(F(A, B)) \leq \frac{1}{3} d(A, B)$$

for any $A, B \in K(\mathbb{R}^2)$, where $d$ is the Hausdorff distance. Let $M_0$ be the closed square

$$M_0 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Let $M_n = F^n(M_0)$.

(i) Sketch $M_0, M_1, M_2$.

(ii) For each $n \in \mathbb{N}$, show that $d(M_{n-1}, M_n) \leq \dfrac{1}{3^{n-1}}\, d(M_0, M_1)$.

(iii) Let $m, n \in \mathbb{N}$ and assume that $m > n$. Show that $d(M_n, M_m) \leq \dfrac{1}{2 \cdot 3^{n-1}}\, d(M_0, M_1)$. Deduce that the sequence $(M_n)$ is a Cauchy sequence.

(iv) Since $K(\mathbb{R}^2)$ is a complete metric space, the Cauchy sequence $(M_n)$ converges to a set $M \in K(\mathbb{R}^2)$. Show that $M$ is a fixed point of $F$. That is, show that $F(M) = M$.

(v) Let $D_0$ be the closed disc of radius 1, centred at the origin. Let $D_n = F^n(D_0)$. You may assume that $(D_n)$ is a Cauchy sequence and converges to a set $D \in K(\mathbb{R}^2)$. Briefly explain why $D = M$.

(vi) Prove that $M$ equals the straight line segment joining the points $(0,0)$ and $(1,1)$.

**3.** **(a)** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Suppose $a \in \mathbb{R}$ is a fixed point of $f$. Suppose $b \in \mathbb{R}$ is a periodic point of $f$ of period $k > 1$.

(i) What is meant by saying that $a$ is a *hyperbolic* fixed point? What is meant by saying that the orbit of $b$ is *hyperbolic*?

(ii) State a criterion which determines when a hyperbolic fixed point of $f$ is attracting or repelling.

(iii) What is meant by saying that the orbit of $b$ is attracting? Let $b_0 = b$. For $1 \leq i < k$, let $b_i = f^i(b)$. Suppose the orbit of $b$ is hyperbolic. Prove that the orbit of $b$ is attracting if

$$\mid f'(b_{k-1}) \cdot f'(b_{k-2}) \cdots f'(b_0) \mid < 1.$$

(You may assume the criterion for when a hyperbolic fixed point is attracting.)

**(b)** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the map defined by $f(x) = x^2 + 2x - 6$.

(i) Find the hyperbolic fixed points of $f$. Determine whether they are attracting or repelling.

(ii) Find the hyperbolic period 2 orbit of $f$. Determine whether it is attracting or repelling.

**4.** **(a)** Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous maps. What is meant by saying that $f$ and $g$ are *topologically conjugate*?

**(b)** Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be the logistic maps $f(x) = \frac{5}{4}x(1-x)$ and $g(x) = \frac{3}{4}x(1-x)$. Find constants $\alpha$ and $\beta$ such that $f$ and $g$ are topologically conjugate by a homeomorphism of the form $\phi(x) = \alpha x + \beta$.

**(c)** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous map. Suppose $a \in \mathbb{R}$ is a fixed point of $f$. Define what is meant by the *basin of attraction* of $a$.

**(d)** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the logistic map $f(x) = \frac{5}{4}x(1-x)$. You may assume that $f(x)$ has two fixed points, an attracting fixed point at $x = \frac{1}{5}$ and a repelling fixed point at $x = 0$. Find the basins of attraction for each of the fixed points.