UNIVERSITY OF ABERDEEN **SESSION 2004–05**

DEGREE EXAMINATION
MX3002 Rings and Fields
Monday 17 January 2005 (3pm to 5pm)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define the term *subring* of a ring R. Show that the subset

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

of $\mathbb{C}$ is a subring of $\mathbb{C}$.

(b) Consider the standard norm function

$$N : \mathbb{Z}[i] \longrightarrow \mathbb{Z}^+$$

defined by $N(a+bi) = a^2 + b^2$. You may assume that $\mathbb{Z}[i]$ together with this norm function forms a Euclidean domain.

In $\mathbb{Z}[i]$, let $b = 4+12i$ and $a = 3+4i$. Use the division algorithm to find $q, r \in \mathbb{Z}[i]$ such that $b = qa + r$ with $N(r) < N(a)$. Check that $N(r) < N(a)$ does in fact hold. Find $\gcd(b, a)$.

(c) Let R be a commutative ring. Define what is meant by a *unit* in R. Define what is meant by saying that two elements $a, b \in R$ are *associates*.

(d) Let R and S be commutative rings and suppose that $\varphi : R \longrightarrow S$ is a ring homomorphism.

(i) Show that if u is a unit in R then $\varphi(u)$ is a unit in S.

(ii) Suppose that two elements $a, b \in R$ are associates. Show that $\varphi(a)$ and $\varphi(b)$ are associates in S.

(iii) Suppose R is a field. Show that its image $\varphi(R)$ is also a field. (You may assume that $\varphi(R)$ is a subring of S.)

2. **(a)** Let A be an integral domain. Define what is meant by an *irreducible* element in A, and what is meant by a *prime* element in A.

(b) Consider the integral domain $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$ with the norm function $N(a + b\sqrt{-3}) = a^2 + 3b^2$. You may assume that $a + b\sqrt{-3}$ is a unit if and only if $N(a + b\sqrt{-3}) = 1$. Observe that

$$(1 + \sqrt{-3})(1 - \sqrt{-3}) = 4 = 2 \cdot 2.$$

(i) Show that $1 + \sqrt{-3}$ is irreducible.

(ii) Show that $1 + \sqrt{-3}$ is not prime.

(iii) Assuming that $1 - \sqrt{-3}$ and 2 are also irreducible, briefly explain why $\mathbb{Z}[\sqrt{-3}]$ is not a Euclidean domain.

(c) Define the term *ring homomorphism.*

(d) A map

$$\varphi : \mathbb{Z}[\sqrt{-3}] \longrightarrow \mathbb{F}_7$$

is defined by $\varphi(a + b\sqrt{-3}) = a + 2b \pmod 7$. Show that φ is a ring homomorphism.

(e) Suppose A is a commutative ring and B is an integral domain. Suppose there is a ring homomorphism $\varphi : A \longrightarrow B$ which is also a monomorphism. Prove that A is also an integral domain.

3. **(a)** Define the term *ideal* in a commutative ring.

(b) Suppose I and J are ideals in a commutative ring R. Prove that the intersection $I \cap J$ is also an ideal.

(c) Let $\varphi : \mathbb{Q}[x] \longrightarrow \mathbb{R}$ be the ring homomorphism defined by $\varphi(f(x)) = f(\sqrt{2})$. (You may assume that φ is in fact a ring homomorphism.) Find the kernel of φ, expressing it as a principal ideal. Justify your answer.

(d) Define what is meant by a *prime ideal* in a commutative ring R, and what is meant by a *maximal ideal* in a commutative ring R.

(e) Consider the ring $R = \mathbb{Z}/16\mathbb{Z}$.

(i) List all of the distinct principal ideals in $\mathbb{Z}/16\mathbb{Z}$.

(ii) Show that the principal ideal (4) is neither prime nor maximal.

(iii) Show that the principal ideal (2) is both prime and maximal.

4. **(a)** State *Eisenstein's Test.*

(b) Use Eisenstein's Test to show that the following polynomials are irreducible in $\mathbb{Q}[x]$.

(i) $f(x) = x^5 + 6x^3 + 15x - 21$,

(ii) $f(x) = x^4 + x^3 + x^2 + x + 1$.

(c) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial. Briefly explain why the quotient ring $\mathbb{Q}[x]/(f(x))$ is a field.

(d) Let $f(x) \in \mathbb{F}_3[x]$ be defined by $f(x) = 2x^2 + x + 1$.

(i) Show that $f(x)$ is irreducible in $\mathbb{F}_3[x]$.

(ii) Use the polynomial $f(x)$ to construct, as a quotient ring, a field $\mathbb{K}$ with 9 elements. Write a list of the elements in $\mathbb{K}$. Find the inverse of each nonzero element in $\mathbb{K}$.