UNIVERSITY OF ABERDEEN **SESSION 2003–04**

Degree Examination

MX3002 Rings and Fields

Tuesday 20 January 2004 (9am to 11am)

Only calculators approved by the Department of Mathematical Sciences may be used in this examination. Calculator memories must be clear at the start of the examination.

Marks may be deducted for answers that do not show clearly how the solution is reached.

Answer THREE questions. All questions carry equal weight.

1. **(a)** Define the term *subring* of a ring R. Show that the subset

$$\mathbb{Z}[\sqrt{-2}] = \{s + t\sqrt{-2} \mid s, t \in \mathbb{Z}\}$$

of $\mathbb{C}$ is a subring of $\mathbb{C}$.

(b) Consider the standard norm function

$$N : \mathbb{Z}[\sqrt{-2}] \longrightarrow \mathbb{Z}^+$$

defined by $N(s + t\sqrt{-2}) = s^2 + 2t^2$. You may assume that $\mathbb{Z}[\sqrt{-2}]$ together with this norm function forms a Euclidean domain.

In $\mathbb{Z}[\sqrt{-2}]$, let $b = 7 + 2\sqrt{-2}$ and $a = 4 - \sqrt{-2}$. Use the division algorithm to find $q, r \in \mathbb{Z}[\sqrt{-2}]$ such that $b = qa + r$ with $N(r) < N(a)$. Check that $N(r) < N(a)$ does in fact hold. Find $\gcd(b, a)$.

(c) Let R be a commutative ring.

(i) Define what is meant by a *unit* in R.

(ii) Prove that an element $u \in R$ is a unit if and only if $(u) = R$, where (u) is the principal ideal generated by u.

(iii) Define what is meant by a *prime ideal* in R.

(iv) Consider the ring $R = \mathbb{Z}/8\mathbb{Z}$. List all of the distinct principal ideals in $\mathbb{Z}/8\mathbb{Z}$. Determine which of these ideals is prime, justifying your answer.

2. **(a)** Let A be an integral domain. Define what is meant by an *irreducible* element in A, and what is meant by a *prime* element in A.

(b) Consider the integral domain $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$, with norm function

$$N(a + b\sqrt{-5}) = a^2 + 5b^2.$$

You may assume that $a + b\sqrt{-5}$ is a unit if and only if $N(a + b\sqrt{-5}) = 1$ if and only if $a + b\sqrt{-5} = \pm 1$. Observe that

$$(1 + \sqrt{-5})(1 - \sqrt{-5}) = 6 = 2 \cdot 3.$$

(i) Show that $1 + \sqrt{-5}$ is irreducible.

(ii) Show that $1 + \sqrt{-5}$ is not prime.

(iii) Assuming that $1 - \sqrt{-5}$, 2, and 3 are also irreducible, briefly explain why $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean domain.

(c) Let A be an integral domain.

(i) If $x \in A$ is prime, prove that x is irreducible.

(ii) Suppose $x \in A$ is prime. Show that the principal ideal (x) generated by x is a prime ideal.

3. **(a)** Define the terms *ring homomorphism* and the *kernel* of a ring homomorphism.

(b) A map $\varphi : \mathbb{Z}[\sqrt{5}] \longrightarrow \mathbb{F}_{11}$ is defined by $\varphi(a + b\sqrt{5}) = a + 4b \pmod{11}$. Show that φ is a ring homomorphism.

(c) Define the term *ideal* in a commutative ring.

(d) Let R and S be commutative rings, and suppose $\varphi : R \longrightarrow S$ is a ring homomorphism. Prove that the kernel of φ is an ideal of R.

(e) Let $\varphi : \mathbb{Q}[x] \longrightarrow \mathbb{R}$ be the ring homomorphism defined by $\varphi(f(x)) = f(\sqrt{5})$. (You may assume that φ is in fact a ring homomorphism.) Find the kernel of φ, expressing it as a principal ideal. Justify your answer.

(f) Suppose R and S are rings and $\varphi : R \longrightarrow S$ is a ring homomorphism which is also an epimorphism. Prove that if R is commutative then so is S.

4. **(a)** Define the term *field*.

(b) Prove that a commutative ring R is a field if and only if the only ideals in R are the zero ideal $\{0\}$ and the whole ring R.

(c) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial. Briefly explain why the quotient ring $\mathbb{Q}[x]/(f(x))$ is a field.

(d) Let $p \in \mathbb{N}$ be a prime. Let $n \in \mathbb{N}$. Show that the polynomial $x^n - p \in \mathbb{Q}[x]$ is irreducible.

(e) Let $f(x) \in \mathbb{F}_3[x]$ be defined by $f(x) = x^2 + 2x + 2$.

(i) Show that $f(x)$ is irreducible in $\mathbb{F}_3[x]$.

(ii) Use the polynomial $f(x)$ to construct, as a quotient ring, a field $\mathbb{K}$ with 9 elements. Write a list of the elements in $\mathbb{K}$. Find, in simplest terms, the coset representative in $\mathbb{K}$ corresponding to the polynomial $x^3 \in \mathbb{F}_3[x]$. Find the inverse of this coset representative.