Answers to Practice Questions

Sheet 1

1.1 $a = 1, b = 2, c = 2$. Therefore the centre is $(1, 2)$ and the radius is 2.

1.2 $y = -\frac{x}{3} + 10$

1.8 $\frac{2}{x-2} - \frac{1}{x-1}$

1.9 $2\frac{17}{60}$

Sheet 2

2.5 $y = 2x - 1$

2.6 5

2.7 $\frac{\pi}{4}$

2.8 $t = 2m\pi$ secs, where $m = 0, 1, 2, ...$

2.9 $72ms^{-2}$

Sheet 3

3.1 $\frac{2x}{1+x^2}$

3.2 $\frac{1-x^2}{(1+x^2)^2}$ **3.3** $4x^3\text{sinh}(x^4)$

3.4 $2\text{ln}x + 3$

3.5 $-\frac{x}{(1+x^2)^{\frac{3}{2}}}$

3.8 -1

3.9 -5

Sheet 4

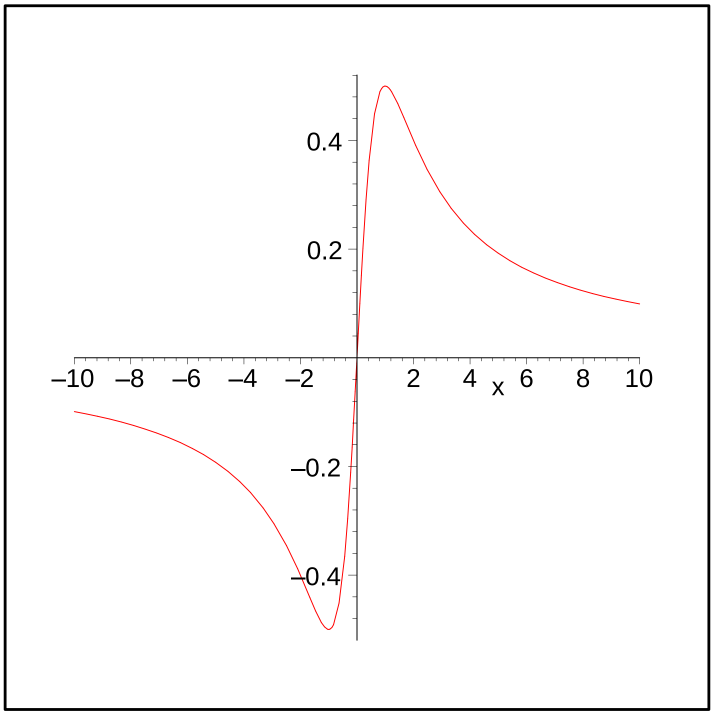
4.1 $-c^2\text{sin}(x - ct)$

4.2 At $x = \text{e}^{\frac{1}{2}}$ there is a maximum.

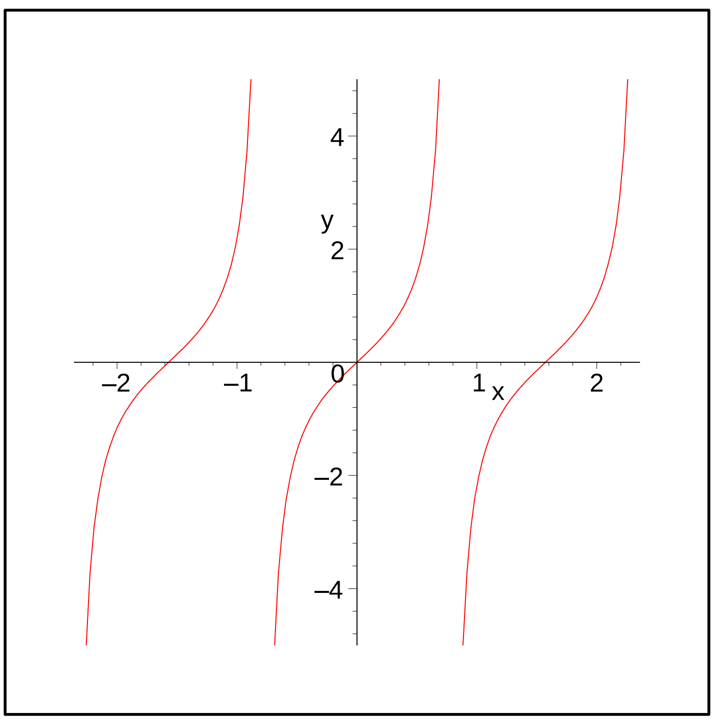
4.3 $y(\frac{3}{2}) = -\frac{1}{4}$ is a minimum.

4.4 $x = \frac{2n-1}{4}\pi, y = \frac{3-2n}{4}\pi, n = ..., -2, -1, 0, 1, 2, ...$

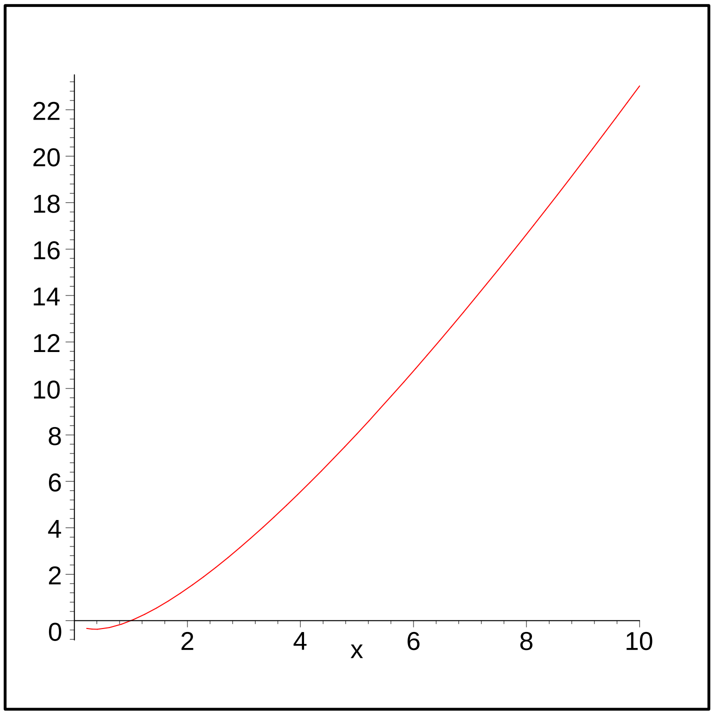
4.6



4.7



4.8


22
20
18
16
14
12
10
8
6
4
2
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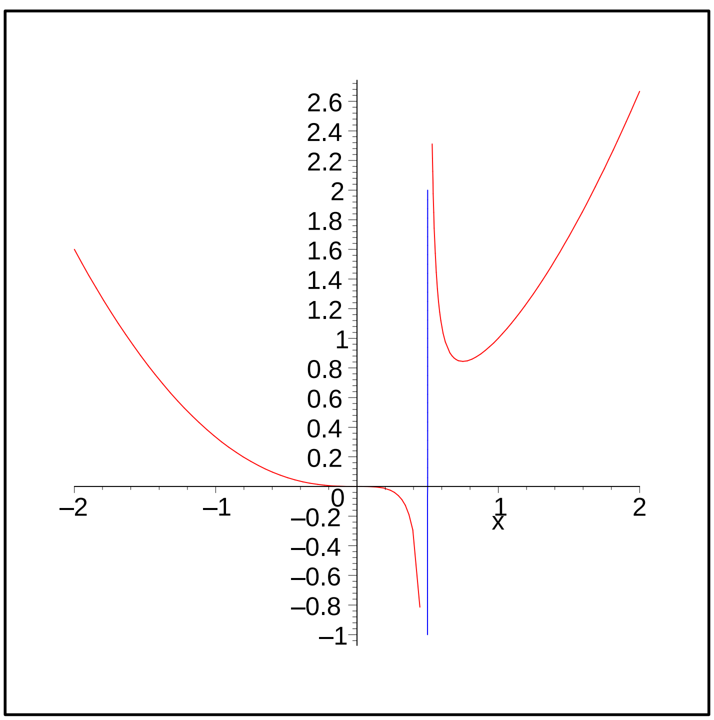
2
4
x
6
8
10

4.9


2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
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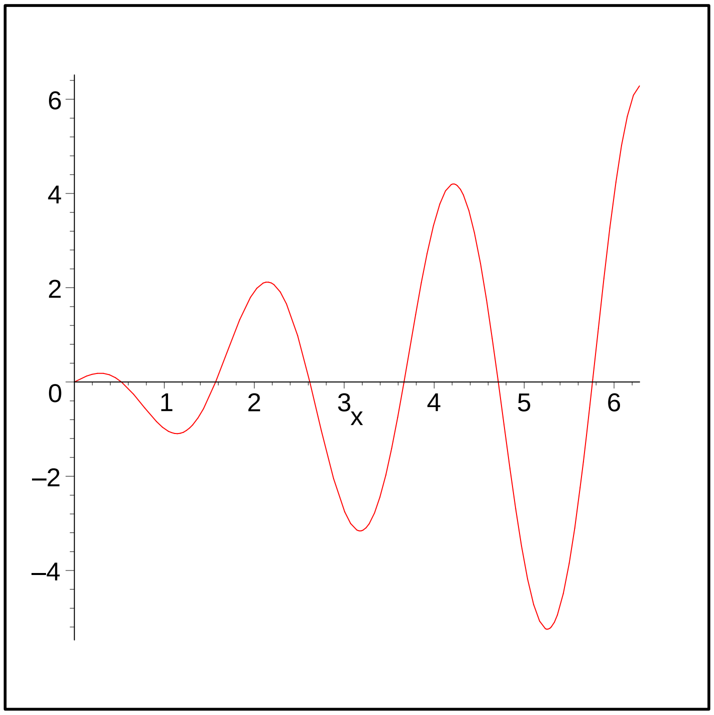
–2
–1
0
1
2
x
–0.2
–0.4
–0.6
–0.8
–1

4.10



Sheet 5

5.1 Recall, $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$.

(a) $f(x) = e^x, f(0) = 1, f^{(n)}(0) = 1$ (prove by induction), therefore $\frac{f^n(0)x^n}{n!} = \frac{x^n}{n!}$

(b) $f(0) = 0, f^{(2n)}(0) = 0, f^{(2n+1)}(0) = (-1)^n$ (prove by induction), therefore $\frac{f^{(2n)}(0)x_n}{(2n)!} = 0, \frac{f^{(2n+1)}(0)x^{2n+1}}{(2n+1)!} = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(c) $f(0) = 1, f^{(2n)}(0) = (-1)^n, f^{(2n+1)}(0) = 0$ (prove by induction), therefore, $\frac{f^{(2n)}(0)x^{2n}}{(2n)!} = \frac{(-1)^n x^{2n}}{(2n)!}, \frac{f^{(2n+1)}(0)x^{2n+1}}{(2n+1)!} = 0$

(d) $f(0) = 1, f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))$ (prove by induction)

(e) $f(0) = 1, f^{(n)}(0) = (-1)^{n-1}(n-1)!$ (prove by induction), therefore $\frac{f^{(n)}(0)x^n}{n!} = \frac{(-1)^{n-1}x^n}{n}$

5.2 (a) $(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$

(b) $\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots$

(c) $ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} - \dots$

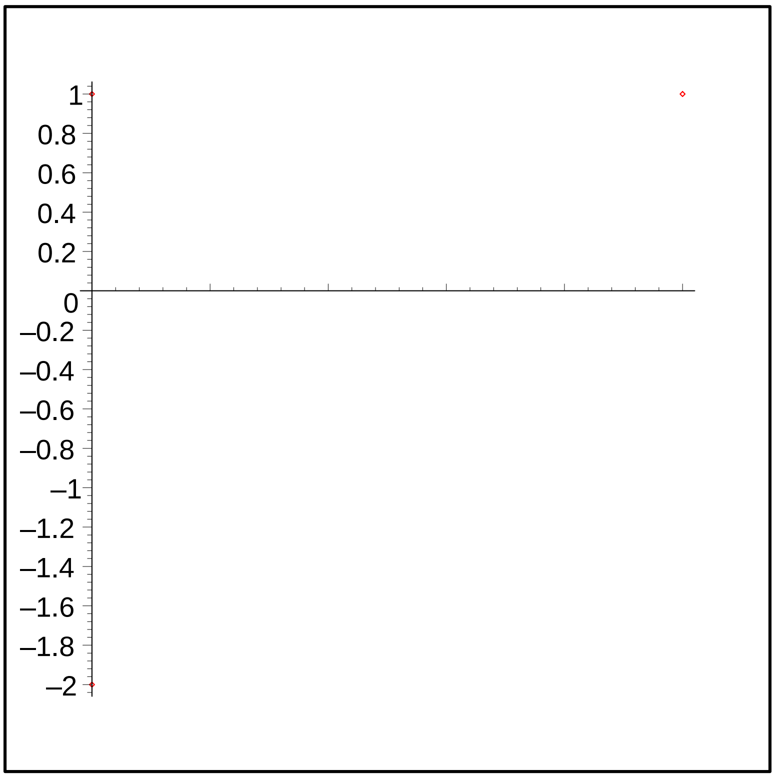
Sheet 6

6.1 (a) $\frac{11+7i}{17}$ (b) $-4-4i$

6.2 (a) $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}, |z_1| = r_1, |z_2| = r_2$, so $|z_1 z_2| = |r_1 r_2 e^{i(\theta_1+\theta_2)}| = r_1 r_2 = |z_1||z_2|$ (b)$|\frac{z_1}{z_2}| = |\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}| = \frac{r_1}{r_2} = |\frac{z_1}{z_2}|$

6.3 $e^{i(A+B)}$ =cos$(A+B)+i$sin$(A+B) = e^{iA}e^{iB} = \cos A + i\sin A)(\cos B + i\sin B)$. Equating real and imaginary parts: $\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\sin(A+B) = \sin A\cos B + \sin B\cos A$

6.4 (a) $\bar{z}^2 = -2i$, so modulus $= 2$, and principal argument $= -\frac{\pi}{2}$ (b)$\frac{z}{\bar{z}} = i$, so modulus $= 1$, and principal argument $= \frac{\pi}{2}$



6.5 (a) Modulus is $\sqrt{8}$, principal argument is $\frac{3\pi}{4}$ (b) Modulus is 5, principal argument is 0.93

6.6 (a) cosh $z = 1$, so $e^z = 1$. Therefore $z = 2n\pi i, n = 0, \pm1, \pm2, \ldots$ (b) sinh $z = 1$, so $e^z = 1 \pm \sqrt{2}$. Therefore $z = ln(1+\sqrt{2}) + 2n\pi i, n = 0, \pm1, \pm2, \ldots$, or $z = ln(1\sqrt{2} - 1) + \pi i(2n+1), n = 0, \pm1, \pm2, \ldots$ (c) $e^z = -1$, so $z = (2n+1)\pi i, n = 0, \pm1, \pm2, \ldots$ (d) cos $z = \sqrt{2}$, so $e^{iz} = \sqrt{2} \pm 1$. Therefore, $z = iln(\frac{1}{\sqrt{2}+1}) + 2n\pi, n = 0 \pm 1, \pm 2 \ldots$, or $z = iln(\frac{1}{\sqrt{2}-1}) + 2n\pi, n = 0 \pm 1, \pm 2, \ldots$

6.7 $w = x + iy + \frac{c(x-iy)}{x^2+y^2}$. So, $u = x + \frac{cx}{x^2+y^2}, v = y - \frac{cy}{x^2+y^2}$. Now $|z| = 1$ implies $x^2+y^2 = 1$, therefore $u = (1+c)x, v = (1-c)y$. Hence $(\frac{u}{1+c})^2 + (\frac{v}{1-c})^2 = x^2+y^2 = 1$, so $\frac{u^2}{(1+c)^2} + \frac{v^2}{(1-c)^2} = 1$

which is the equation of an ellipse.

6.8 $\cos n\theta = \frac{1}{2}(z^n + \frac{1}{z^n})$, where $z = e^{i\theta}$. Therefore $\cos^6\theta = \frac{1}{2^6}(z + \frac{1}{z})^6 = \frac{1}{64}(z^6 + \frac{1}{z^6} + 6(z^4 + \frac{1}{z^4}) + 15(z^2 + \frac{1}{Z^2}) + 20) = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

Sheet 7

7.1 $a_{13} = 3, a_{31} = 2$

7.4 Let $A + A^T = [\alpha_{ij}]$. Then $\alpha_{ij} = a_{ij} + a_{ji} = a_{ji} + a_{ij} = \alpha_{ji}$. Therefore $A + A^T$ is symmetric. Let $A - A^T = [\beta_{ij}]$. Then $\beta_{ij} = a_{ij} - a_{ji} = -(a_{ji} - a_{ij}) = -\beta_{ji}$. Therefore $A - A^T$ is skew-symmetric.

$$\begin{aligned}
A &= \frac{1}{2}(A - A^T) + \frac{1}{2}(A - A^T) \\
&= \frac{1}{2}\left\{\begin{bmatrix} 2 & 1 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right\} \\
&+ \frac{1}{2}\left\{\begin{bmatrix} 2 & 1 & 3 \\ -2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} 2 & -\frac{1}{2} & 3 \\ -\frac{1}{2} & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & 0 \\ -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}
\end{aligned}$$

7.5

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2b + ac & 0 & 1 \end{bmatrix}$$

$A^2 = I$ if $2b + ac = 1$. $A^{-1} = A$. Inverse of A^{2n-1} is A

7.6 $\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

7.7 $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA) = A^{-1}BBA$ (since $AA^{-1} = I$ and using distributive law) $= A^{-1}B^2A$.

Sheet 8

8.1

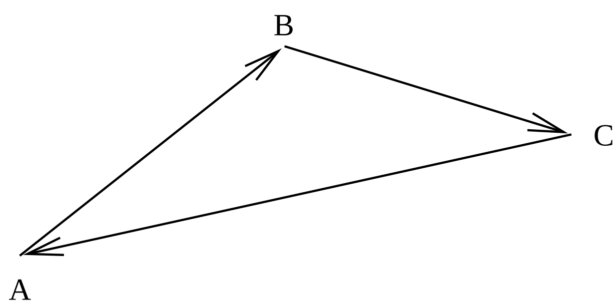
(a) $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$

(b) $(\frac{-3\sqrt{3}}{2}, \frac{-3}{2})$

8.2

$$\begin{aligned} \overline{QB} &= \overline{QA} + \overline{AB} = (6,5) \\ \overline{QE} &= \overline{QC} + \overline{CD} + \overline{DE} = (6,1) \\ \overline{BE} &= \overline{QE} - \overline{QB} \\ &= (0,-4) \end{aligned}$$

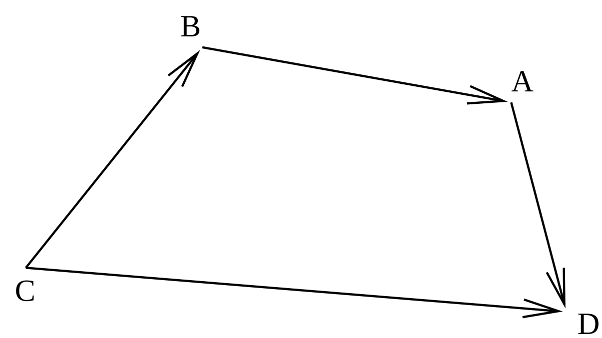
8.3



$\overline{AB} + \overline{BC} + \overline{CA} = \overline{0}$

$\overline{AB_0} + \overline{B_0B_1} + \overline{B_1B_2} + \cdots + \overline{B_{n-1}B_n} + \overline{B_nA} = \overline{0}$

8.4



$\overline{CD} = \overline{CB_0} + \overline{B_0B_1} + \overline{B_1B_2} + \cdots + \overline{B_{n-1}B_n} + \overline{B_nD}$

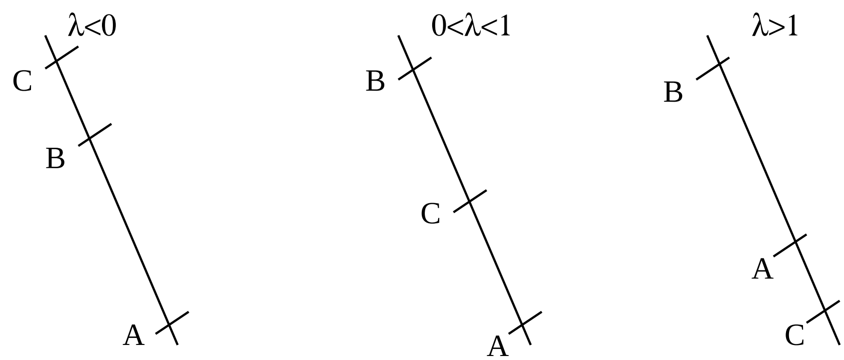
8.5

(a) $\frac{(\boldsymbol{a}+\boldsymbol{b})}{2}$

(b) $\frac{3\boldsymbol{a}+\boldsymbol{b}}{4}$

8.6 Any point on AB can be written

$$\begin{aligned} \text{as } \boldsymbol{r} &= \boldsymbol{b} + \lambda(\boldsymbol{a} - \boldsymbol{b}) \\ &= \lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b} \end{aligned}$$



8.7

$$\begin{aligned} D = \text{ dist}^2 &= (1+t)^2 + (2+t)^2 + (3+t)^2 \\ \frac{dD}{dt} &= 2(1+t) + 2(2+t) + 2(3+t) \\ &= 0 \text{ at } \ t = -2 \\ \frac{d^2D}{dt^2} &= 6 > 0 \\ \text{therefore min at } \ t &= -2 \\ D_{t=-2} &= (-1)^2 + 0^2 + 1^2 \\ &= 2 \\ \text{therefore min} &= \sqrt{2} \end{aligned}$$

8.8

$$\begin{aligned} \boldsymbol{p} &= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}), \quad \boldsymbol{q} = \frac{1}{2}(\boldsymbol{b}+\boldsymbol{c}) \\ \boldsymbol{r} &= \frac{1}{2}(\boldsymbol{c}+\boldsymbol{d}), \quad \boldsymbol{s} = \frac{1}{2}(\boldsymbol{d}+\boldsymbol{a}) \\ \text{therefore} \boldsymbol{q} - \boldsymbol{p} &= \frac{1}{2}(\boldsymbol{c}-\boldsymbol{a}) \\ \text{and } \boldsymbol{r} - \boldsymbol{s} &= \frac{1}{2}(\boldsymbol{c}+\boldsymbol{a}) \\ \text{therefore} \overline{PQ} &= \overline{RS} \end{aligned}$$

therefore$PQRS$ is a parallelogram.

8.9

$$\begin{aligned} \boldsymbol{p} &= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}) \text{ etc.} \\ \text{equ of medians } \boldsymbol{r} &= \boldsymbol{a}+\lambda_1[\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})-\boldsymbol{a}] \\ \boldsymbol{r} &= \boldsymbol{b}+\lambda_2[\frac{1}{2}(\boldsymbol{c}+\boldsymbol{a})-\boldsymbol{b}] \\ \boldsymbol{r} &= \boldsymbol{c}+\lambda_3[\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})-\boldsymbol{c}] \end{aligned}$$

Observe that the point $\boldsymbol{r} = \frac{1}{3}(\boldsymbol{c}+\boldsymbol{b}+\boldsymbol{c})$ lies on all 3 lines (by taking $\lambda_1 = \lambda_2 = \lambda_3 = \frac{2}{3}$).

8.10

$$\overline{OC} = 2\overline{OA} + 3\overline{OB}$$

Sheet 9

9.1

$$\begin{aligned} \overline{AC} &= (3,1) \\ \overline{BD} &= (-1,1) \\ \overline{AC}.\overline{BD} &= -2 \\ |\overline{AC}||\overline{BD}| &= \sqrt{20} \\ |\overline{AC}||\overline{BD}|\cos\alpha &= \overline{AC}.\overline{BD} \end{aligned}$$

where α is the required angle.

$$\text{therefore}\alpha = -\frac{2}{\sqrt{20}} = -\frac{1}{\sqrt{5}}$$

9.2

$$\begin{aligned} \boldsymbol{a}.\boldsymbol{b} &= -2+18-16=0 \\ \boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 3 & 4 \\ -2 & 6 & -4 \end{vmatrix} \\ &= \hat{\boldsymbol{\imath}}(-14-24) - \hat{\boldsymbol{\jmath}}(-4+8) + \hat{\boldsymbol{k}}(6+6) \\ &= -38\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}} \end{aligned}$$

9.3

$$\begin{aligned} \text{The dot product} &= \lambda+2+3\lambda \\ &= 0 \;\text{ iff }\; \lambda = -\frac{1}{2} \end{aligned}$$

9.4 $\boldsymbol{r} = (1,2,1) + \lambda(-1,3,1)$. Therefore $(-1,3,1)$ is parallel to the line.

9.5 $\hat{\boldsymbol{a}}.\boldsymbol{r} = |\hat{\boldsymbol{a}}||\boldsymbol{r}| \cos\alpha, \quad |\hat{\boldsymbol{a}}| = 1$. Therefore the Cartesian equation is

$$2x - 3y - 6z = \frac{7}{2}\sqrt{x^2 + y^2 + z^2}$$

Sheet 10

10.1

(a)

$$\begin{aligned} |\boldsymbol{a}+\boldsymbol{b}|^2 &= (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b}) \\ &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} \\ \text{similarly}\quad |\boldsymbol{a}-\boldsymbol{b}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} \end{aligned}$$

Hence result.

(b) From (a),

$$|\boldsymbol{a}+\boldsymbol{b}|^2 - |\boldsymbol{a}-\boldsymbol{b}|^2 = 4\boldsymbol{a}\cdot\boldsymbol{b}$$

Hence result.

10.2

(a)

$$\begin{aligned} \boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & -2 & 2 \\ 3 & -1 & -1 \end{vmatrix} \\ &= (4,7,5) \end{aligned}$$

(b)

$$\begin{aligned} \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= \begin{vmatrix} 1 & -2 & 2 \\ 3 & -1 & -1 \\ -1 & 0 & -1 \end{vmatrix} \\ &= 1(1) + 2(-3-1) + 2(0-1) \\ &= -9 \end{aligned}$$

(c)

$$\begin{aligned} \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) &= \begin{vmatrix} -1 & 0 & -1 \\ 1 & -2 & 2 \\ 3 & -1 & -1 \end{vmatrix} \\ &= -1(2+2) - 1(-1+6) \\ &= -9 \end{aligned}$$

10.3 $\boldsymbol{a}$ parallel to $\boldsymbol{b}$

10.4 $\boldsymbol{c} \cdot \boldsymbol{b} = 12 + 6 - 18 = 0$ therefore perpendicular

$$\begin{aligned} \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{vmatrix} \\ &= -21\hat{\boldsymbol{\imath}} + 42\hat{\boldsymbol{\jmath}} - 14\hat{\boldsymbol{k}} \\ &= 7(-3\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}) \end{aligned}$$

10.5 $\boldsymbol{v} = X\boldsymbol{a} + Y\boldsymbol{b} + Z\boldsymbol{c}$
$\boldsymbol{v} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = XD$ etc.

Sheet 11

11.1 $\frac{1}{3}\sin(3x+4) + C$ where C is an arbitrary constant.

11.2 $-\frac{(1-2x)^{11}}{22} + C$ where C is an arbitrary constant.

11.3 $\frac{e^{4x-1}}{4} + C$ where C is an arbitrary constant.

11.4 $y = x^2 + 1$.

11.5 $\frac{1}{4}\ln(4x+3) + C$ where C is an arbitrary constant.

11.6 $21ms^{-1}$.

11.7 1.

11.8 ln2.

11.9 12.

11.10 $y = -\sin x - \frac{2}{x} + Ax + B$ where A and B are arbitrary constants.