United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

Organised by the United Kingdom Mathematics Trust

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method.

The mark allocation on Mathematics Olympiad papers is different from what you are used to at school. To get any marks, you need to make significant progress towards the solution. This is why the rubric encourages candidates to try to finish whole questions rather than attempting lots of disconnected parts.

Each question is marked out of 10.

3 or 4 marks roughly means that you had most of the relevant ideas, but were not able to link them into a coherent proof.

8 or 9 marks means that you have solved the problem, but have made a minor calculation error or have not explained your reasoning clearly enough. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap?

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1. All the digits 1 to 9 are to be placed in the circles in Figure 1, one in each, so that the total of the numbers in any line of four circles is the same. In the example shown in Figure 2, the total is equal to 20.

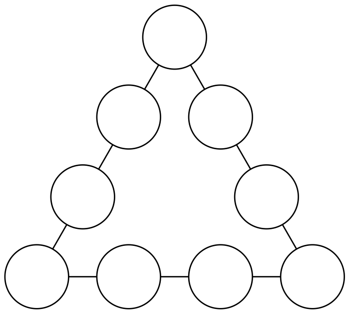


Figure 1

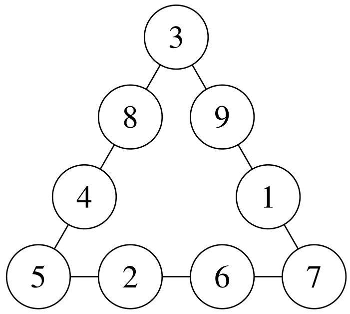


Figure 2

Prove that if the total T is possible then the total $40 - T$ is possible.

SOLUTION

COMMENTARY

A sensible way to start is to try and create different totals. For example, try filling in the numbers so that the total on each line is 23; then try to make 17. This may lead you to spot a connection between the two configurations.

It is natural to ask which totals T are possible. That question can be answered (see the note after the solution), but it turns out that the answer is not needed to solve the actual problem.

The key is to look for a connection between a configuration with total T and one with total $40 - T$. Since there are four numbers in each line, if each number d is replaced by $10 - d$, the total of the four numbers in each line will change from T to $40 - T$. It is important to check that this new configuration satisfies the conditions of the problem: that all the digits are different and each one is still between 1 and 9. (Even if this seems obvious, you should mention in your solution that you have checked it.)

Consider a numbering in which the total is T. In each circle replace the number d with the number $10 - d$. The digits 1 to 9 will still all be used once each, and now each line has total $40 - T$.

NOTE

You can find which totals are possible by considering the sum of all nine numbers: $1 + 2 + \cdots + 9 = 45$. This should equal $3T$ minus the sum of three corner numbers which are counted twice.

Denote the corner numbers a, b and c. Then

$$3T - (a + b + c) = 45 \implies 3T = 45 + (a + b + c).$$

Since a, b and c are all different numbers between 1 and 9, the smallest and largest possible values of $a + b + c$ are 6 and 24, respectively. This means that $17 \leq T \leq 23$.

The question given an example with $T = 20$. Try to construct examples with $T = 17$ and $T = 19$. You can then use the method described in the solution to find examples with $T = 23$ and $T = 21$.

To get $T = 18$ we need $a + b + c = 9 = 1 + 2 + 6 = 1 + 3 + 5 = 2 + 3 + 4$. For each of those options there are only a few possibilities to check, and none of them give a valid configuration. So $T = 18$ is not possible. It follows that $T = 22$ is not possible either.

In conclusion, the only possible totals are 17, 19, 20, 21 and 23.

2. A positive integer is said to be *jiggly* if it has four digits, all non-zero, and no matter how you arrange those digits you always obtain a multiple of 12.

How many jiggly positive integers are there?

SOLUTION

COMMENTARY

To solve this problem you need to know various divisibility criteria. For an integer to be a multiple of 12 it needs to be a multiple of both 3 and 4. An integer is a multiple of 3 if the sum of its digits is a multiple of 3. It is a multiple of 4 if it ends in a two-digit number which is a multiple of 4. (Have you ever tried to prove these divisibility criteria?)

Probably the first thing to notice is that all the digits must be even (because any rearrangement must end in an even digit). There aren't very many two-digits multiples of four which are made up of two even digits; checking them all shows that the only possibilities where both 'ab' and 'ba' are multiples of 4 are 44, 48, 84 and 88. Hence each digit is either 4 or 8.

You then need to check which combinations of four such digits have a sum which is a multiple of 3, and then count how many different four-digit numbers can be made from those digits.

Since rearrangement always gives an even integer, all the digits must be even. Numbers ending '$a2$' or '$a6$' are not divisible by 4 when a is even. Therefore all the digits must be multiples of 4, that is, every digit is either 4 or 8. A number is divisible by 3 if and only if the sum of its digits is divisible by 3. Therefore, if the number is to be divisible by 12 then neither 4 nor 8 can occur 3 times or 4 times, and so it is an arrangement of 4488, of which there are 6. Thus there are precisely 6 jiggly positive integers.

NOTE

In this case, the number of arrangements of the digits 4488 is sufficiently small for you to write out and count all of them. If you know about combinatorics, you can also calculate it as $\frac{4!}{2!2!}$.

3. Four different points A, B, C and D lie on the curve with equation $y = x^2$.

Prove that $ABCD$ is *never* a parallelogram.

Solution

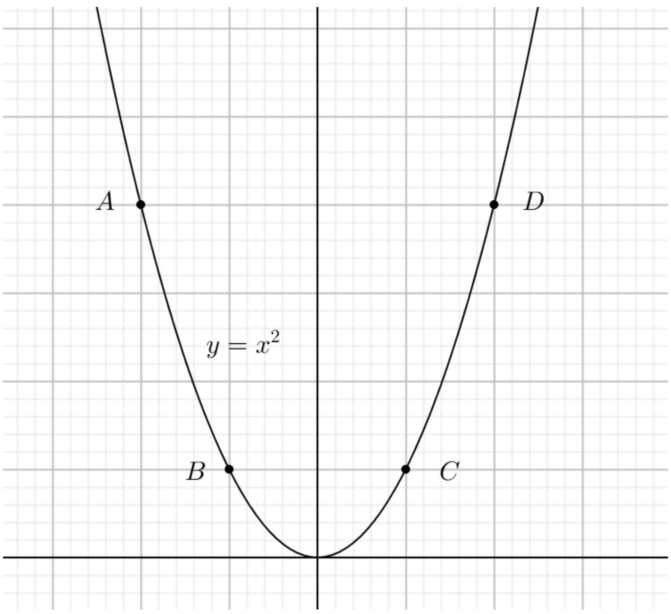
Commentary

All we know about the four points is that they lie on the given curve. It thus makes sense to work with their coordinates. Two sides of the quadrilateral $ABCD$ are parallel if they have the same slope (gradient).

You need to prove that it is not possible for the quadrilateral $ABCD$ to have two pairs of parallel sides. We present two slightly different proofs. The first picks a particular vertex and uses inequalities to show that at most one of the lines passing through it can be parallel to another line. The second is an example of *proof by contradiction*: we assume that there are two pairs of parallel sides and show that this contradicts one of the assumptions in the question (in this case, that all four points have different x-coordinates).

The question says nothing about the order in which the four points are arranged on the curve, so in your solution you need to be clear which pairs of sides you are considering.

Let the points A, B, C, D be (a, a^2), (b, b^2), (c, c^2), (d, d^2) respectively, where, without loss of generality, $a < b < c < d$. We investigate which of the sides through A could be parallel to another side. The slope of the line-segment AB is $(b^2 - a^2)/(b - a)$, which is $a + b$. The slopes of AC, AD, BC, BD, and CD are given by the analogous expressions. Now $a + b < c + d$ and $a + c < b + d$ and so $AB \nparallel CD$ and $AC \nparallel BD$.



Thus at most one side through A (namely AD) can be parallel to another side, and therefore $ABCD$ is not a parallelogram.

Alternative

Let the coordinates of the four points be defined as above. If $ABCD$ is a parallelogram then AB is parallel to DC and BC is parallel to DA.

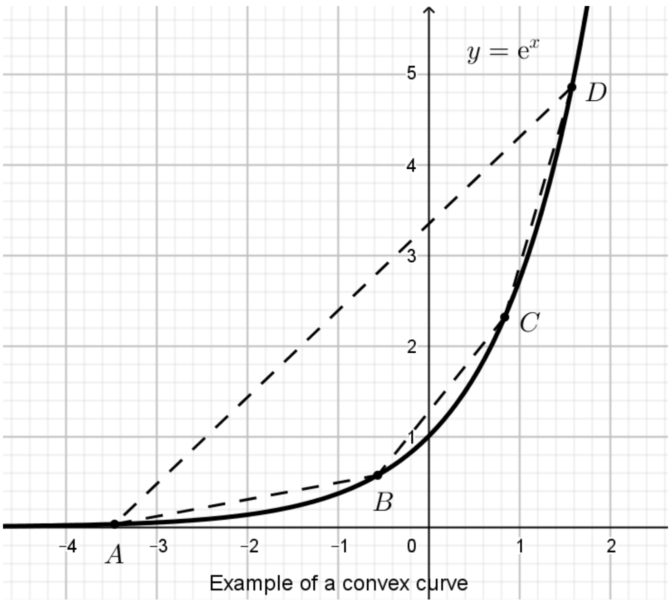
From the calculation of gradients given above, this means that $a + b = c + d$ and $b + c = a + d$. Subtracting the second equation from the first gives $a - c = c - a$ which implies that $a = c$. But this is not possible, as the four points are different.

Thus our initial assumption that two pairs of sides are parallel must be false, so $ABCD$ is not a parallelogram.

Note

The result in question can be much generalised. A curve with equation $y = f(x)$ is said to be *convex* if whenever $u < v < w$, the point $(v, f(v))$ on the curve lies below (in the sense that its y-coordinate is smaller than that of) the corresponding point on the chord joining $(u, f(u))$ to $(w, f(w))$. We may write $v = tu+(1-t)w$ where $0 < t < 1$ since $t = (w-v)/(w-u)$. Algebraically convexity may be expressed as the inequality $f(v) < tf(u) + (1 - t)f(w)$. Equivalently, the curve is convex if its slope is always increasing. The parabola with equation $y = x^2$ is just one example. There are many others such as the exponential curve with equation $y = \mathrm{e}^x$.

In general, if four different points A, B, C, D lie on a convex curve with equation $y = f(x)$, then the quadrilateral $ABCD$ is never a parallelogram. Without loss of generality we may take the points A, B, C, D to be $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $(d, f(d))$ respectively, where $a < b < c < d$. The convexity condition tells us that B and C lie below the chord AD, B lies below the chord AC and C lies below the chord BD. It follows that the chords AC and BD meet (above the curve) and the chords AB and CD (both extended) meet (below the curve). Thus, as in the case of the parabola, at most one of the sides through A (namely AD) can be parallel to another side, and therefore $ABCD$ is not a parallelogram.



Example of a convex curve

4. Let n be an odd integer greater than 3 and let $M = n^2 + 2n - 7$.

Prove that, for all such n, at least four different positive integers (excluding 1 and M) divide M exactly.

SOLUTION

> COMMENTARY
>
> This question is about factors of M so it makes sense to try and factorise the expression. The expression in terms of n does not factorise, but we have not yet used the fact that n is odd. Writing $n = 2k + 1$ gives an expression that can be factorised.
>
> Once you have identified four factors of M you need to show that they are all different, and that neither of them equal 1.

Since n is odd and $n > 3$ we may write $n = 2k + 1$ where k is an integer and $k \geq 2$. Then $M = 4k^2 + 8k - 4 = 4(k^2 + 2k - 1)$. Even if $k^2 + 2k - 1$ is prime, we see that M has the four positive integral divisors 2, 4, $k^2 + 2k - 1$ and $2(k^2 + 2k - 1)$, all of them different from 1 and M. Clearly they are all different since (given that $k \geq 2$) $k^2 + 2k - 1 \geq 7$.

NOTE

When $k^2 + 2k - 1$ is prime (as happens, for example, when k is 2 or 4 or 6 or 8), M has *exactly* four different factors (excluding 1 and itself). But it can have more than four different factors: for example, when k is 3, M is divisible by $2, 4, 7, 8, 14$ and 28. These examples show that every algebraic factor of an expression gives a numerical factor, but there may be numerical factors that don't come from an algebraic factor.

5. Claire and Stuart play a game called *Nifty Nines*:

(i) they take turns to choose one number at a time, with Claire choosing first;
(ii) numbers can only be chosen from the integers 1 to 5 inclusive;
(iii) the game ends when n numbers have been chosen (repetitions are permitted).

Stuart wins the game if the sum of the chosen numbers is a multiple of 9, otherwise Claire wins.

Find all values of n for which Claire can ensure a win, whatever Stuart's choices were. You must prove that you have found them all.

Solution

Commentary

This problem is about finding a winning strategy in a game. This means describing a way that Claire can choose her moves so that, no matter what Stuart does, she always wins. This may involve having some fixed moves (for example, having a fixed number as her first move), but also "responding" to Stuart's moves.

In general, it maybe be that neither player has a winning strategy. For example, in *Noughts and Crosses (Tic-Tac-Toe)* if both players play correctly the game always ends in a draw. In this problem, the question suggests that Claire has a winning strategy for some, but not all, values of n.

It is useful to start thinking about the last few moves of the game. This "working backwards" is unlikely to lead to the full solution, but it gives you a good feel for what it going on.

It should be pretty clear that, if Claire has the final move, she can ensure that the total is not a multiple of 9 (as she has a choice of five numbers). Can you see for which values of n this happens?

If Claire does not go last, she needs to put Stuart in a position where he cannot win on his final move. You can start working backwards from there, but you will probably quickly realise that there are too many options.You need to look for some sort of strategy that Claire can repeat, no matter how large n is.

You may have seen this type of problem before. For example, you may be familiar with the following game. Two players take turns counting from 1, saying either one or two numbers at a time. The winner is the person who says 21. Here the second player can always win by saying a multiple of 3 (if the first person says one number the second says two, and vice versa).

In *Nifty Nines* the person going second can ensure that after their move the total is always a multiple of 6. This means that Stuart can ensure that after every six moves (three lots of two moves) the total is a multiple of 18. Thus, if n is a multiple of 6, Stuart has a winning strategy.

Are there any other values of n where Claire can ensure a win? You have probably found that, if Stuart is going last, Claire needs her final move to give a sum which leaves remainder 0, 1, 2 or 3 when divided by 9. After her first move she can play as if she is the second player, "responding" to Stuart's move to increase the sum by 6, and so ensuring that after every six moves the sum increases by 18. Therefore, if there are $6k + 2$ moves, she can start with 1, 2 or 3 and force a win.

The only possibility we have not yet covered is when n is of the form $6k + 4$. This is similar to the $6k + 2$ case, but there are additional two moves so Claire can add another 6 to the total. Thus Claire can ensure that after her final move the sum leaves the remainder 7, 8 or 0. Out of those, only the last one prevents Stuart from winning. So if she starts with 3, Claire can ensure a win in this case as well.

Your solution needs to include three things:

(i) State for which values of n Claire has a winning strategy.
(ii) Describe the winning strategy for each of those values and explain why it works.
(iii) Explain why for all other values of n she does not have a winning strategy.

Let s_k be the sum of the numbers chosen up to and including the k^{th} step, and let r_k be the remainder when s_k is divided by 9. If Claire makes the last choice, she can choose the number 1 or 2 to ensure that $r_n \neq 0$, so that s_n is not a multiple of 9. She makes the last choice if and only if n is odd; therefore if n is odd then Claire can force a win.

She can also ensure that she wins if n is even and not a multiple of 3, that is, n is of one of the forms $6m + 2$, $6m + 4$ for some m. To do this, on her first move Claire chooses 3. Thereafter, at step $2k + 1$, if Stuart has chosen d at stage $2k$ then Claire chooses the number $6 - d$. If $n = 6m + 2$, after $3m$ pairs of steps s_{6m+1} will be $3 + 3m \times 6$, so $r_{6m+1} = 3$ and Stuart will be unable to make a multiple of 9 at the n^{th} step; and if $n = 6m + 4$, after $3m + 1$ pairs of steps s_{6m+3} will be $3 + (3m + 1) \times 6$, so $r_{6m+3} = 0$ and again Stuart will be unable to make a multiple of 9 at the n^{th} step.

If, however, n is a multiple of 6 then Stuart can ensure that he wins: whatever number d Claire chooses at stage r (where $r < n$ and r is odd), Stuart chooses $6 - d$ at stage $r + 1$. This ensures that $s_{k+6} = s_k + 18$ and Stuart can thereby ensure that $r_k = 0$ whenever k is a multiple of 6.

The conclusion is that Claire can force a win if n is not a multiple of 6; she cannot force a win if n is a multiple of 6.