

UK Junior Mathematical Olympiad 2017

Organised by The United Kingdom Mathematics Trust

Tuesday 13th June 2017

RULES AND GUIDELINES : **READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING**

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. **Write in blue or black pen or pencil.**
For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work.
For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.
Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.
Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 30 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
This means that each answer must be accompanied by clear explanations and proofs.
Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you are not able to do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π , fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

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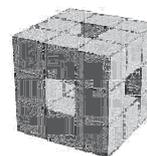
Section A

Try to complete Section A within 30 minutes or so. Only answers are required.

A1. How many centimetres are there in 1 km 2 m 3 cm 4 mm?

A2. The solid shown is formed by taking a $3\text{ cm} \times 3\text{ cm} \times 3\text{ cm}$ cube and drilling a $1\text{ cm} \times 1\text{ cm}$ square hole from the centre of each face to the centre of the opposite face.

What is the volume in cm^3 of the solid?



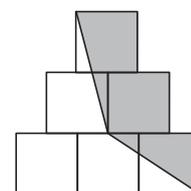
A3. Howard is out running. He is now $\frac{3}{5}$ of the way through the second half of his run. What fraction of the whole run has he completed?

A4. A bookmark-maker sells bookmarks for £1 each or 7 for £6. What is the smallest amount you could pay for 2017 of her bookmarks?

A5. In 1866, the yacht *Henrietta* – with Gordon Bennett aboard – won the Great Ocean Yacht Race, travelling a distance of approximately 3000 nautical miles. The winning time was 13 days and 22 hours, to the nearest hour.

What was the yacht's average speed in nautical miles per hour, to the nearest integer?

A6. The diagram shows six identical squares arranged symmetrically. What fraction of the diagram is shaded?



A7. A fully-grown long-tailed tit – *Aegithalos caudatus* – weighs only 9 g, whereas a £1 coin weighs 9.5 g.

To the nearest 1 %, what percentage of the weight of a £1 coin is the weight of a fully-grown long-tailed tit?

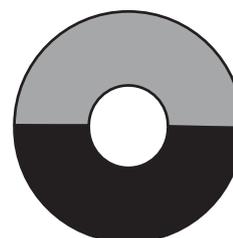
A8. A jar contains red and white marbles in the ratio 1 : 4. When Jenny replaces 2 of the white marbles with 7 red marbles, the ratio becomes 2 : 3.

What is the ratio of the total number of marbles in the jar now to the total number in the jar before?

A9. How many multiples of 3 that are less than 1000 are not divisible either by 9 or by 10?

A10. Two concentric circles are drawn, as shown in the diagram. Concentric circles share the same point as their centre. The radius of the smaller circle is a third of the radius of the larger circle. The top half of the larger circle which is outside the smaller circle, is shaded in grey.

The ratio of the grey shaded area to the area of the smaller circle in its simplest form is $a : b$. What are the values of a and b ?



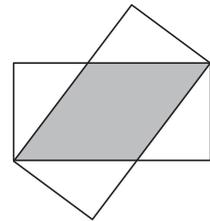
Section B

Your solutions to Section B will have a major effect on your JMO result. Concentrate on one or two questions first and then **write out full solutions** (not just brief ‘answers’).

B1. An amount of money is to be divided equally between a group of children. If there was 20p more than this amount, then there would be enough for each child to receive 70p. However, if each child was to receive 60p, then £2.10 would be left over. How many children are there in the group?

B2. A 3-digit integer is called a ‘V-number’ if the digits go ‘high-low-high’ – that is, if the tens digit is smaller than both the hundreds digit and the units (or ‘ones’) digit. How many 3-digit ‘V-numbers’ are there?

B3. Two identical rectangles overlap in such a way that a rhombus is formed, as indicated in the diagram. The area of the rhombus is five-eighths of the area of each rectangle. What is the ratio of the length of the longer side of the rectangle to the length of the shorter side?

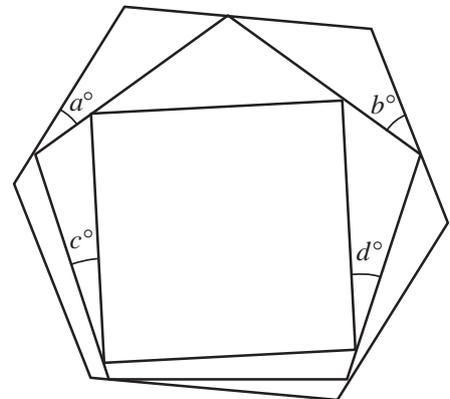


B4. My uncle lives a long way away and his letters always contain puzzles. His three local teams are the Ants (A), the Bees (B), and the Cats (C), who play each other once a year. My uncle claimed that the league table part way through the year looked like this:

	Played	Won	Drawn	Lost	Goals for	Goals against
A	1	0	0	1	4	2
B	2	1	1	0	2	2
C	2	1	0	1	3	1

When we complained that this is impossible, he admitted that every single number was wrong but he excused himself because every number was exactly ‘1 out’. Find the correct table, explaining clearly how you deduced the corrections.

B5. The diagram shows a square whose vertices touch the sides of a regular pentagon. Each vertex of the pentagon touches a side of a regular hexagon. Find the value of $a + b + c + d$.



B6. The 9-digit positive integer N with digit pattern $ABCABCBBB$ is divisible by every integer from 1 to 17 inclusive. The digits A , B and C are distinct. What are the values of A , B and C ?

UK Junior Mathematical Olympiad 2017 Solutions

A1. 100203.4 When you convert all the distances to cm, you obtain 1 km = 100000 cm, 2 m = 200 cm and 4 mm = 0.4 cm. Therefore 1 km 2 m 3 cm 4 mm is equal to 100203.4 cm.

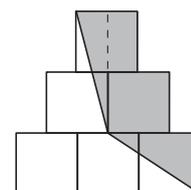
A2. 20 The volume of the large cube is 27 cm^3 . To make the hole, one small cube is removed from the centre of each face and one from the centre of the large cube. Each small cube has a volume of 1 cm^3 . Hence seven small cubes are removed and the remaining volume is 20 cm^3 .

A3. $\frac{4}{5}$ Howard has completed $\frac{1}{2}$ of the run. He is now $\frac{3}{5}$ of the way through the second half. Hence he has completed $\frac{1}{2} + \frac{3}{5} \times \frac{1}{2} = \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$ of the whole run.

A4. £1729 Since $2017 = 288 \times 7 + 1$, you can buy 288 lots of 7 bookmarks at £6 each. Hence the smallest amount you could pay for 2017 of the bookmarks is $£6 \times 288 + £1 \times 1 = £1729$.

A5. 9 Since 13 days and 22 hours are equivalent to 334 hours, the yacht travels 3000 nautical miles in 334 hours. Therefore the yacht travels $\frac{3000}{334}$ nautical miles in 1 hour. Hence the yacht's average speed in nautical miles per hour is 9 (to the nearest integer).

A6. $\frac{11}{24}$ There are several ways to solve this problem. This solution is just one example. Without loss of generality, let each square have a side length of 2 units. Hence the six identical squares have a total area of $6 \times 2^2 = 24$ square units. The grey shaded area can be thought of as consisting of two triangles, one rectangle and one square, as shown on the diagram. Therefore the grey shaded area is



$$\frac{1}{2} \times 1 \times 4 + \frac{1}{2} \times 3 \times 2 + 1 \times 2 + 2^2 = 11 \text{ square units.}$$

Hence $\frac{11}{24}$ of the diagram is shaded.

A7. 95% The required percentage is $\frac{9}{9.5} \times 100 = \frac{18}{19} \times 100 = \frac{1800}{19} \approx 94.7$ by long division. This is 95% to the nearest 1%.

A8. 6 : 5 Let m be the original number of marbles in the jar. Therefore, as Jenny replaces 2 of the white marbles with 7 red marbles, there are now $m + 5$ marbles in the jar. We know that $\frac{4}{5}$ of the original number of marbles were white, that 2 white marbles were removed and that now $\frac{3}{5}$ of the jar's marbles are white. Hence

$$\frac{4}{5}m - 2 = \frac{3}{5}(m + 5)$$

Solving this equation, gives $m = 25$. Therefore the ratio of the total number of marbles in the jar now to the number in the jar before is $30 : 25 = 6 : 5$.

A9. 200 There are 333 multiples of 3 less than 1000, and there are 111 multiples of 9 less than 1000. As numbers that are multiples of both 3 and 10 are multiples of 30, consider the 33 multiples of 30 that are less than 1000. The lowest common multiple of 9 and 30 is 90 and there are 11 multiples of 90 less than 1000. Hence the number of multiples of 3 that are less than 1000 but not divisible by either 9 or 10 is $333 - 111 - 33 + 11 = 200$.

A10. 4, 1 Let the radius of the smaller circle be r , so the radius of the larger circle is $3r$. The area of the smaller circle is πr^2 and the grey shaded area is $\frac{1}{2}\pi(3r)^2 - \frac{1}{2}\pi r^2 = 4\pi r^2$. Therefore the ratio of the grey shaded area to the area of the smaller circle is $4\pi r^2 : \pi r^2 = 4 : 1$, and hence $a = 4$ and $b = 1$.

- B1.** An amount of money is to be divided equally between a group of children. If there was 20p more than this amount, then there would be enough for each child to receive 70p. However, if each child was to receive 60p, then £2.10 would be left over. How many children are there in the group?

Solution

Let C be the number of children in the group and let A be the total amount of money in pence to be divided between the children.

Then $\frac{A + 20}{C} = 70$, so that $A + 20 = 70C$ and therefore $A = 70C - 20$.

Also $A = 60C + 210$.

Hence $70C - 20 = 60C + 210$.

Solving this last equation we obtain $C = 23$. Therefore there are 23 children in the group.

It is good practice to check the solution works. The total amount of money is £15.90, and $(£15.90 + £0.20) \div 23 = £0.70$; $£0.60 \times 23 = £13.80$, and $£13.80 + £2.10 = £15.90$.

- B2.** A 3-digit integer is called a ‘V-number’ if the digits go ‘high-low-high’ – that is, if the tens digit is smaller than both the hundreds digit and the units (or ‘ones’) digit. How many 3-digit ‘V-numbers’ are there?

Solution

The smallest ‘V-number’ is 101 and the largest ‘V-number’ is 989.

Consider the tens digits. The smallest tens digit is 0 and the largest tens digit is 8.

If the tens digit is 0, the hundreds digit can be 1 to 9, and the units digit can be 1 to 9, giving 9×9 possible ‘V-numbers’.

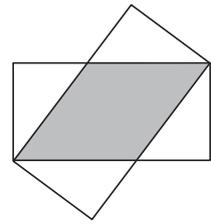
If the tens digit is 1, then the hundreds digit can be 2 to 9 and the units digit can be 2 to 9, giving 8×8 possible ‘V-numbers’.

If the tens digit is d , where d can be any digit from 0 to 8, the hundreds digit can be $(d + 1)$ to 9 and the units digit can be $(d + 1)$ to 9, giving $(9 - d) \times (9 - d)$ possible ‘V-numbers’.

The greatest value of d is 8. In this case, the hundreds digit can only be 9 and the units digit can only be 9, which gives just 1×1 possibilities.

This gives the total number of possible ‘V-numbers’ to be $9 \times 9 + 8 \times 8 + \dots + 1 \times 1 = 285$, which is the sum of the squares from 1 to 9 inclusive.

- B3.** Two identical rectangles overlap in such a way that a rhombus is formed, as indicated in the diagram. The area of the rhombus is five-eighths of the area of each rectangle. What is the ratio of the length of the longer side of the rectangle to the length of the shorter side?



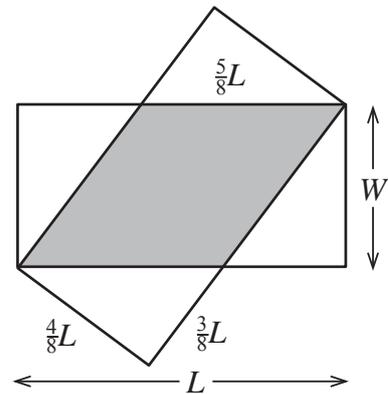
Solution

Let the length of the longer side and of the shorter side of the rectangle be L and W respectively.

Since the area of the rhombus is $\frac{5}{8}$ of the area of each rectangle, the area of the rhombus is $\frac{5}{8}LW$. Also, since the area of a rhombus is equal to base \times perpendicular height and the perpendicular height of the shaded rhombus is W , the length of each side of the rhombus is $\frac{5}{8}L$.

Consider one of the white right-angled triangles.

The length of the hypotenuse is $\frac{5}{8}L$ and the length of one other side is $L - \frac{5}{8}L = \frac{3}{8}L$. Therefore, using Pythagoras' Theorem, we can find the length of the third side in terms of L since the triangle is a 3:4:5 triangle. Hence $W = \frac{4}{8}L$ and the ratio of the length of the longer side of the rectangle to the length of the shorter side of the rectangle is $L : \frac{4}{8}L = 2 : 1$.



- B4.** My uncle lives a long way away and his letters always contain puzzles. His three local teams are the Ants (A), the Bees (B), and the Cats (C), who play each other once a year. My uncle claimed that the league table part way through the year looked like this:

	Played	Won	Drawn	Lost	Goals for	Goals against
A	1	0	0	1	4	2
B	2	1	1	0	2	2
C	2	1	0	1	3	1

When we complained that this is impossible, he admitted that every single number was wrong but he excused himself because every number was exactly '1 out'.

Find the correct table, explaining clearly how you deduced the corrections.

Solution

The maximum number of games any team can play is 2, as each team only plays another team once in the year and there are only 3 teams.

Therefore team B (the Bees) and team C (the Cats) played 1 game each.

Team A (the Ants) played 2 games (because if they had played 0 games they would have 0 goals for). Since each figure is '1 out', team A won 1, drew 1 and lost 0 (so that the total number of matches played is 2).

Since team A won 1 match, team B lost 1 match (since team B had lost 0 games originally and team C cannot have lost 2 games). Therefore, team B won 0 games and drew 0 games.

So team A drew against team C. Therefore the number of games resulting in a draw for team C is 1 and, as they only played 1 match, they won and lost 0 games.

Since team C's only game resulted in a draw, team C's goals for and against are equal. Therefore, team C's goals for and against are 2 each.

Team B's only match resulted in a loss, so team B's goals against are greater than its goals for. Therefore, the number of team B's goals for is 1 and the number of its goals against is 3.

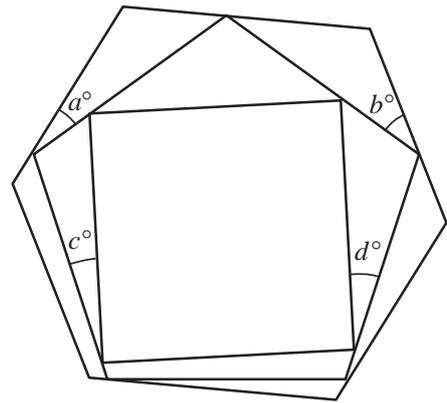
Because team B and team C both only played team A, the number of team A's goals for is equal to the sum of the number of team B's goals against and the number of team C's goals against. Hence the number of team A's goals for is 5. Similarly, the number of team A's goals against is the sum of the number of team B's goals for and the number of team C's goals for. Therefore the number of team A's goals against is 3.

So the correct table is

Team	Played	Won	Drawn	Lost	Goals for	Goals against
A	2	1	1	0	5	3
B	1	0	0	1	1	3
C	1	0	1	0	2	2

- B5.** The diagram shows a square whose vertices touch the sides of a regular pentagon. Each vertex of the pentagon touches a side of a regular hexagon.

Find the value of $a + b + c + d$.



Solution

Each interior angle of a regular pentagon is 108° and each interior angle of a regular hexagon is 120° .

Consider the two shaded triangles in the diagram alongside, which contain the angles a° and b° and the side of the hexagon that these triangles have in common.

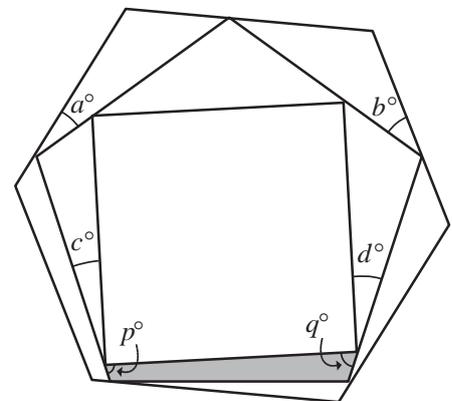
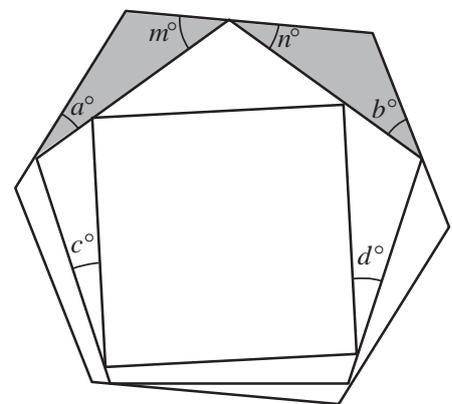
Let the two angles shown be m° and n° . Since the sum of the interior angles in a triangle is 180° , we have $120 + a + m = 180$ and $120 + b + n = 180$. Therefore, $m = 60 - a$ and $n = 60 - b$. Now, since the sum of the angles on a straight line is 180° , we have $m + 108 + n = 180$. Hence $60 - a + 108 + 60 - b = 180$. Therefore $a + b = 48$.

The region outside the square but inside the pentagon consists of a quadrilateral, shown shaded in the diagram alongside, and three triangles. We now consider the quadrilateral.

Let the two angles shown be p° and q° . Since the interior angles of a square are 90° and the sum of the angles on a straight line is 180° , we have $c + 90 + p = 180$ and $d + 90 + q = 180$. Therefore $p = 90 - c$ and $q = 90 - d$. Now, since the sum of the angles in a quadrilateral is 360° and two of these angles are interior angles of a regular pentagon, we have

$108 + p + q + 108 = 360$. Hence $216 + 90 - c + 90 - d = 360$. Therefore $c + d = 36$. Hence

$$\begin{aligned} a + b + c + d &= 48 + 36 \\ &= 84. \end{aligned}$$



- B6.** The 9-digit positive integer N with digit pattern $ABCABCBBB$ is divisible by every integer from 1 to 17 inclusive.

The digits A , B and C are distinct. What are the values of A , B and C ?

Solution

Since N is divisible by both 2 and 5, N is divisible by 10 and hence $B = 0$. Therefore N is of the form $A0CA0C000 = A0C \times 1001 \times 1000$.

Now $1001 = 7 \times 11 \times 13$ and $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$. Hence $A0CA0C000$ is certainly divisible by 1, 2, 4, 5, 7, 8, 11, 13 and 14.

We are told that N is divisible by every integer from 1 to 17. Hence, in particular, N is divisible by 9. Therefore, since the rule for divisibility by 9 is that the sum of the digits of the number is also divisible by 9, we have $2A + 2C$ is a multiple of 9 and hence $A + C$ is a multiple of 9. Also, since A and C are distinct, $A + C = 9$.

Once A and C are chosen so that N is divisible by 9, N will also be divisible by 3, 6, 12 and 15.

Since $N = A0C \times 1001 \times 1000$ is divisible by 16, $A0C$ is divisible by 2 and hence C is even. Therefore, the only options for A and C (in that order) are 1 and 8, 3 and 6, 5 and 4, or 7 and 2.

To ensure N is divisible by 17, we must now ensure that A and C are chosen so that $A0C$ is divisible by 17. When we look at each case in turn, we find that $108 = 6 \times 17 + 6$, $306 = 18 \times 17$, $504 = 29 \times 17 + 11$ and $702 = 41 \times 17 + 5$. Therefore $A = 3$, $B = 0$ and $C = 6$.