Junior Mathematical Challenge

Thursday 1 May 2014

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct.Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the Junior Mathematical Olympiad and similar competitions).

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments to:

JMC Solutions, UKMT, School of Mathematics Satellite,
University of Leeds, Leeds LS2 9JT

☎ *0113 343 2339* enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D	B	C	E	A	A	B	A	E	D	D	B	A	D	E	E	C	E	C	B	D	C	D	C	A

1. What is $(999 - 99 + 9) \div 9$?

A 91 B 99 C 100 D 101 E 109

SOLUTION

D We can calculate the value in more than one way.

One method is to first work out the value of the expression in the brackets and then divide the result by 9. This gives

$(999 - 99 + 9) \div 9 = 909 \div 9 = 101.$

Alternatively, we can first divide each number in the bracket by 9 and then evaluate the resulting expression. This gives

$(999 - 99 + 9) \div 9 = (111 - 11 + 1) = 101.$

Of course, both methods give the same answer.

2. How many minutes are there in $\dfrac{1}{12}$ of a day?

A 240 B 120 C 60 D 30 E 15

SOLUTION

B There are 24 hours in a day. So in $\frac{1}{12}$ of a day the number of hours equals $\frac{1}{12} \times 24 = 2$. In 1 hour there are 60 minutes. Hence the number of minutes in 2 hours is $2 \times 60 = 120$. So there are 120 minutes in $\frac{1}{12}$ of a day.

FOR INVESTIGATION

2.1 How many seconds are there in $\frac{1}{120}$ of a day?

2.2 What fraction of a day is 3 hours?

2.3 What fraction of a day is 250 seconds?

3. In my row in the theatre the seats are numbered consecutively from T1 to T50. I am sitting in seat T17 and you are sitting in seat T39. How many seats are there between us?

A 23 B 22 C 21 D 20 E 19

SOLUTION

C The seats between us are numbered from T18 up to T38. So the seats between us are the 38 seats from T1 up to T38, other than the 17 seats from T1 up to T17. So there are $38 - 17 = 21$ seats between us.

For investigation

3.1 How many integers are there between 100 and 200 (not including either 100 or 200)?

3.2 Suppose that k is an integer greater than 27, and that there are 50 integers between 27 and k, not including 27 and k. What is the value of k?

3.3 Find a formula for the number of integers in the sequence

$$m, m+1, \ldots, n-1, n,$$

where m and n are integers, with $m < n$.

4. The number 987 654 321 is multiplied by 9.

How many times does the digit 8 occur in the result?

A 1 B 2 C 3 D 4 E 9

Solution

E There does not seem to be any better way to answer this question than to do the multiplication:

$$\begin{array}{r} 9\;8\;7\;6\;5\;4\;3\;2\;1 \\ \times\;9 \\ \hline 8\;8\;8\;8\;8\;8\;8\;8\;8\;9 \end{array}$$

From this we see that the digit 8 occurs 9 times in the answer.

For investigation

4.1 Try the following multiplications which have surprising answers:

(a) 127×9721,

(b) $32\,800\,328 \times 271$,

(c) 239×4649,

(d) $21\,649 \times 513\,239$.

4.2 Find some other examples of a similar type.

5. What is the difference between the smallest 4-digit number and the largest 3-digit number?

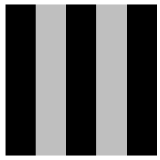
A 1 B 10 C 100 D 1000 E 9899

Solution

A The smallest 4-digit number is 1000. The largest 3-digit number is 999. So their difference is equal to $1000 - 999 = 1$.

6. The diagram shows a square divided into strips of equal width. Three strips are black and two are grey.

What fraction of the perimeter of the square is grey?



A $\frac{1}{5}$ B $\frac{1}{4}$ C $\frac{4}{25}$ D $\frac{1}{3}$ E $\frac{2}{5}$

SOLUTION

A Two sides of the square are wholly black, and $\frac{2}{5}$ of two sides are grey. So the length of the perimeter that is grey is equal to $2 \times \frac{2}{5} = \frac{4}{5}$ of the length of one side. The length of the perimeter is 4 times the length of one side. So the fraction of the perimeter that is grey is

$$\frac{\frac{4}{5}}{4} = \frac{1}{5}.$$

FOR INVESTIGATION

6.1 A square is divided into seven strips of equal width. Four of the strips are black and three are grey. What fraction of the perimeter of the square is grey?

6.2 A square is divided into an odd number of strips of equal width. The strips are alternately black and grey, with one more black strip than grey strip. The fraction of the perimeter that is grey is $\frac{6}{25}$. How many strips are there?

7. What is 2014 − 4102?

A −2012 B −2088 C −2092 D −2098 E −2112

SOLUTION

B It is easier to subtract the smaller number, 2014, from the larger number, 4102. Now

$$4102 - 2014 = 2088$$

and so

$$2014 - 4102 = -2088.$$

8. How many prime numbers are there in the list

$$1,\ 12,\ 123,\ 1234,\ 12\,345,\ 123\,456?$$

A 0 B 1 C 2 D 3 E 4

SOLUTION

A It is important to remember that we do *not* regard 1 as a prime number.

We see that 12, 1234 and 123 456 are not prime numbers becauses they are divisible by 2. Also, 123 is not a prime number because it is divisible by 3, and 12 345 is not a prime number because it is divisible by 5.

So there are no prime numbers in the list.

FOR INVESTIGATION

8.1 Of course, there are other ways to see that some of the numbers in the list are not prime numbers. For example, there is a useful test for divisibility by 3:

An integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

For example, $1 + 2 + 3 + 4 + 5 + 6 = 21$, and 21 is divisible by 3. It follows, using this test, that 123 456 is divisible by 3.

So, if the sum of the digits of a positive integer, other than 3 itself, is a number that is divisible by 3, we can deduce that the integer is not a prime number.

(a) Is the number 123 456 789 divisible by 3?

(b) Why does the test for divisibility by 3 in terms of the sum of the digits of an integer work?

8.2 Is 1 234 567 a prime number? [Hint: Look at Investigation **4.1** (a).]

8.3 Is the number 12 345 678 a prime number?

8.4 Find a test for a number to be divisible by 9 in terms of the sum of its digits.

8.5 Find a test for a number to be divisible by 11 in terms of its digits. [*Note:* this test is a little more complicated than those for divisibility by 3 and by 9.]

NOTE

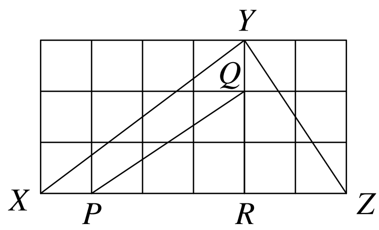
It is only a *convention* that 1 does not count as a prime number. However, it is a very useful convention and one used by all mathematicians.

One reason, among many, why this is a useful convention is that it enables us to state *The Fundamental Theorem of Arithmetic* in a simple way. This theorem says that: *Each positive integer greater than 1 can be expressed as a product of prime numbers in just one way.*

Of course, we can often write the prime factors of a given integer in more than one order. For example, 12 may be expressed as $2 \times 2 \times 3$, $2 \times 3 \times 2$ and $3 \times 2 \times 2$, but in each case the prime factor 2 occurs twice, and the prime factor 3 occurs once. This is what the Fundamental Theorem means by "in just one way". We would need to state this Theorem in a more complicated way if we regarded 1 as a prime number.

9. Triangles XYZ and PQR are drawn on a square grid. What fraction of the area of triangle XYZ is the area of triangle PQR?

A $\frac{1}{4}$ B $\frac{7}{18}$ C $\frac{1}{2}$ D $\frac{5}{18}$ E $\frac{1}{3}$



Solution

E We assume that we have chosen units so that the sides of the squares in the grid have length 1. We calculate the areas of the triangles PQR and XYZ by using the formula

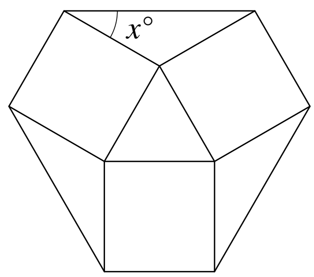
$$\text{area} = \tfrac{1}{2}(\text{base} \times \text{height})$$

for the area of a triangle.

The triangle PQR has base PR of length 3, and height QR of length 2. So the area of triangle PQR is $\frac{1}{2}(3 \times 2) = 3$. The triangle XYZ has base XZ of length 6 and height YR of length 3. So the area of triangle XYZ is $\frac{1}{2}(6 \times 3) = 9$. Therefore the area of triangle PQR is $\frac{3}{9} = \frac{1}{3}$ of the area of triangle XYZ.

10. An equilateral triangle is surrounded by three squares, as shown. What is the value of x?

A 15 B 18 C 24 D 30 E 36



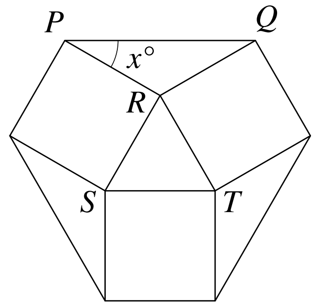
Solution

D We have labelled some of the vertices in the diagram, as shown below, so that we can refer to them.

We have $PR = RS$ because they are sides of the same square, $RS = RT$ because they are sides of an equilateral triangle, and $RT = RQ$ because they are sides of the same square. Therefore $PR = RQ$, and so triangle PRQ is isosceles. Hence $\angle QPR = \angle PQR$.

Now $\angle PRS$ and $\angle QRT$ are both 90° because they are angles of a square, and $\angle SRT = 60°$, because it is the angle of an equilateral triangle. Since the angles at the point R have sum 360°, it follows that

$$\angle PRQ = 360° - 90° - 90° - 60° = 120°.$$



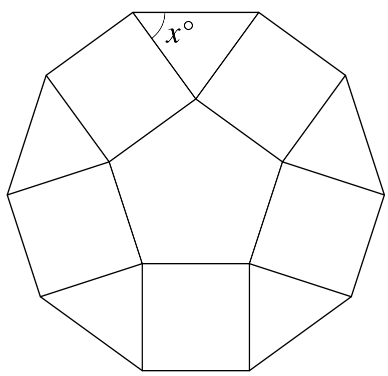
Therefore, since the angles in a triangle have sum 180°,

$$\angle QPR + \angle PQR = 180° - 120° = 60°.$$

Since $\angle QPR = \angle PQR$, it follows that $\angle QPR = 30°$. So $x = 30$.

For investigation

10.1 What is the value of x in the case of a regular pentagon surrounded by squares, as shown in the diagram?



10.2 Find a formula which gives the value of x in the analogous diagram in which a regular pentagon is replaced by a regular polygon with n sides.

10.3 Check that the formula you have obtained in **10.2** gives the correct values for the cases of an equilateral triangle, as in **Question 10**, and a regular pentagon, as in **10.1**.

11. The first two terms of a sequence are 1 and 2. Each of the following terms in the sequence is the sum of all the terms which come before it in the sequence.

Which of these is *not* a term in the sequence?

A 6 B 24 C 48 D 72 E 96

SOLUTION

D The third term of the sequence is $1 + 2 = 3$, the fourth term is $1 + 2 + 3 = 6$, the fifth term is $1 + 2 + 3 + 6 = 12$, and so on. So the sequence begins

$$1, 2, 3, 6, 12, \ldots$$

In fact it is not difficult to see that, from the fourth term onwards, *each term is double the previous term.* We show this as follows. Suppose a is a term of the sequence other than the first two, and that b is the term that immediately follows a. Then a is the sum of all the terms that come before it. The term b is also the sum of all the terms that become before it. Hence b is the sum of all the terms that come before a, and a. So, $b = a + a = 2a$. Therefore each term from the fourth term onwards is double the previous term.

Since from the fourth term onwards each term is double the previous term, the first eight terms are

$$1, 2, 3, 6, 12, 24, 48, 96.$$

Since the terms continue to increase we see that of the given options, 6, 24, 48 and 96 occur in the sequence, but 72 does not occur.

NOTE

The proof that from the fourth term onwards each term is twice the previous one can be given more formally, as follows.

We use the notation x_n for the nth term of the sequence. We can then define the sequence by $x_1 = 1$, $x_2 = 2$, and for $n > 2$,

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

It follows that, for $n > 2$,

$$\begin{aligned} x_{n+2} &= x_1 + x_2 + \cdots + x_n + x_{n+1} \\ &= (x_1 + x_2 + \cdots + x_n) + x_{n+1} \\ &= x_{n+1} + x_{n+1} \\ &= 2x_{n+1}. \end{aligned}$$

This shows that from the fourth term onwards each term is double the previous term.

For investigation

11.1 Find a formula in terms of n that for $n > 2$ gives the nth term of the sequence defined in the question.

11.2 Suppose that the first two terms of a sequence are 2 and 3, and that each of the following terms in the sequence is the sum of all the terms that come before it in the sequence. What is the 10th term of the sequence? Find a formula for the nth term, for $n > 2$.

11.3 Suppose that the first two terms of a sequence are a and b, and that each of the following terms is the sum of all the terms that come before it in the sequence. Find a formula for the nth term of the sequence, for $n > 2$.

12. In this subtraction, P, Q, R, S, T represent single digits.

What is the value of $P + Q + R + S + T$?

$$\begin{array}{r} 7\ Q\ 2\ S\ T \\ -\ P\ 3\ R\ 9\ 6 \\ \hline 2\ 2\ 2\ 2\ 2 \end{array}$$

A 30 B 29 C 28 D 27 E 26

Solution

B Given any three numbers x, y and z,

$$x - y = z \quad \text{if, and only if,} \quad x = y + z.$$

It follows that the subtraction sum in the question is equivalent to the addition sum shown on the right. We now work out the values of P, Q, R, S and T by working from right to left.

$$\begin{array}{r} P\ 3\ R\ 9\ 6 \\ +\ 2\ 2\ 2\ 2\ 2 \\ \hline 7\ Q\ 2\ S\ T \end{array}$$

From the units column we see that $T = 6 + 2 = 8$. Next, we look at the tens column. Since $9 + 2 = 11$, we have $S = 1$ (and 1 is carried to the hundreds column). Therefore, from the hundreds column we see that, because $R + 2 + 1$ cannot be 2, we have $R + 2 + 1 = 12$. Therefore $R = 9$ (and 1 is carried to the thousands column). It follows from the thousands column that $Q = 3+2+1 = 6$. Finally, from the ten-thousands column, we see that $P+2 = 7$ and so $P = 5$.

Therefore

$$P + Q + R + S + T = 5 + 6 + 9 + 1 + 8 = 29.$$

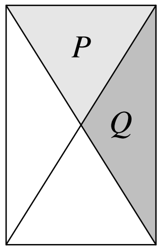
For investigation

12.1 Repeat this question with the first two rows unchanged, but with each 2 in the final row replaced by (i) 3, (ii) 4 and (c) 5, respectively.

12.2 What goes wrong if you try to repeat the question with each 2 in the final row replaced by 6, or by 7?

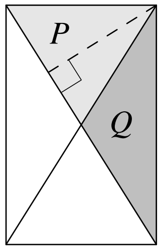
13. A rectangle is split into triangles by drawing in its diagonals. What is the ratio of the area of triangle P to the area of triangle Q?

A 1 : 1 B 1 : 2 C 2 : 1 D 2 : 3
E the ratio depends on the lengths of the sides of the rectangle



SOLUTION

A We see from the diagram on the right that the triangles P and Q have the same height, as is shown by the broken line. Because the diagonals of a rectangle bisect each other the triangles have bases with the same length. The area of a triangle is half the base multiplied by the height. It follows that the triangles have the same area. So the ratio of their areas is 1:1.



FOR INVESTIGATION

13.1 This solution assumes that the diagonals of a rectangle bisect each other. How could you prove this?

14. Which of these is equal to one million millimetres?

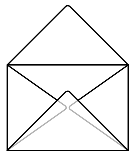
A 1 metre B 10 metres C 100 metres D 1 kilometre
E 10 kilometres

SOLUTION

D Since there are 1000 millimetres in one metre, one million millimetres make

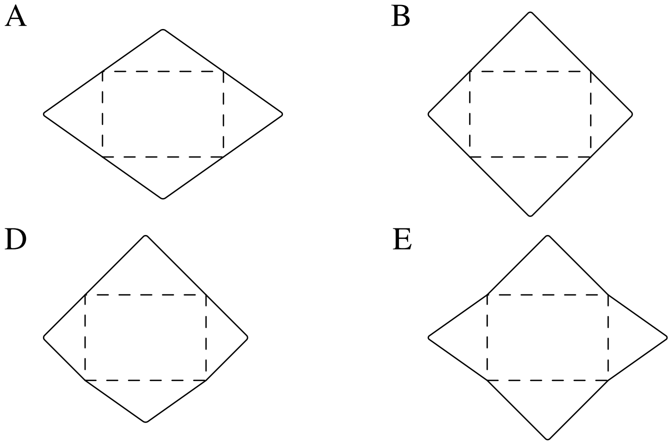
$$\frac{1\,000\,000}{1000} \text{ metres} = 1000 \text{ metres} = 1 \text{ kilometre.}$$

15. The diagram shows a rectangular envelope made by folding (and gluing) a single sheet of paper.



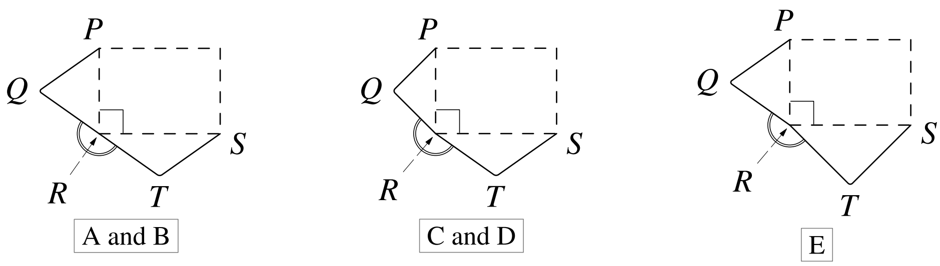
What could the original unfolded sheet of paper look like?

(The dashed lines are the fold lines.)



Solution

E Think about the left-hand corner of the envelope which is labelled R in the diagrams below.



In the following argument by $\angle QRT$ we mean the marked angle which is exterior to the piece of paper.

The triangles PQR and RST overlap after they are folded. It follows that $\angle PRQ + \angle SRT > 90°$. Therefore, $\angle QRT < 180°$, because the angles at R add up to $360°$. Only in option E is this the case (in options A and B we see that $\angle QRT = 180°$ and in options C and D that $\angle QRT > 180°$). So E is the correct option.

16. Only one of the following statements is true. Which one?

A 'B is true'
B 'E is false'
C 'Statements A to E are true'
D 'Statements A to E are false'
E 'A is false'

Solution

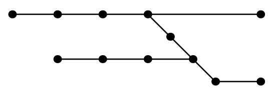
E For statement A to be true, B would also have to be true. But we are told that only one statement is true, so this is not possible. We deduce that statement A is false. Therefore statement E is true.

In the context of the JMC you could stop here, because we have found one statement that must be true. However, we really ought to satisfy ourselves that statement E is the *only* one that is true.

As statement E is true, statement B is false. Statement D can never be true, because if statement D were true, all the statements, including D, would be false, and we would have a contradiction. Because statement D must be false, statement C cannot be true.

So statement E is true and all the other statements are false.

17. The diagram is a "map" of Jo's local rail network, where the dots represent stations and the lines are routes. Jo wants to visit all the stations, travelling only by train, starting at any station and ending at any station, with no restrictions on which routes are taken.

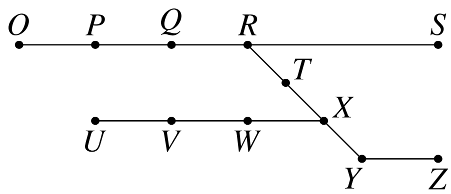


What is the smallest number of stations that Jo must go to more than once?

A 1 B 2 C 3 D 4 E 5

Solution

C We label the stations as shown in the diagram on the right so that we can refer to them. Jo can visit the stations in the order



O, P, Q, R, S, R, T, X, Y, Z, Y, X, W, V, U.

In this way she visits all the stations and goes through just three of them, R, Y and X more than once.

It remains only to show that Jo cannot visit all the stations without going through at least three of them more than once.

Jo must go through the junction stations R and X more than once. If Jo does not start or finish at O, she will have to go through P more than once. If she does not start or finish at U she will have to go through V more than once. If she starts at O and finishes at U, or *vice versa* , she will have to go through Y more than once. It follows that Jo has to go through at least three stations more than once. So this is the smallest number of stations Jo must go through more than once.

18. Which of these statements is true?

A $15614 = 1 + 5^6 - 1 \times 4$

B $15615 = 1 + 5^6 - 1 \times 5$

C $15616 = 1 + 5^6 - 1 \times 6$

D $15617 = 1 + 5^6 - 1 \times 7$

E $15618 = 1 + 5^6 - 1 \times 8$

Solution

E In the context of the JMC we can assume that just one of the given options is correct, so we can find which it is by eliminating the ones that are wrong. We can do this by just considering the last digit (the units digit) of the given options.

The last digit of 5^6 is 5. Since $1 + 5^6 - 1 \times 4 = 1 + 5^6 - 4$ its last digit is the same as that of $1 + 5 - 4$, that is, 2. So option A is not the correct answer. In a similar way, it follows that the last digit of $1 + 5^6 - 1 \times 5$ is 1, the last digit of $1 + 5^6 - 1 \times 6$ is 0, and the last digit of $1 + 5^6 - 1 \times 7$ is 9. So options B, C and D are also not correct This leaves E as the only possible correct option.

In this way there is no need to evaluate 5^6. However, to give a complete answer we would need to check that E is correct. This is straightforward:

$$1 + 5^6 - 1 \times 8 = 1 + 15\,625 - 8 = 15\,626 - 8 = 15\,618.$$

19. Jack and Jill played a game for two people. In each game, the winner was awarded 2 points and the loser 1 point. No games were drawn. Jack won exactly 4 games and Jill had a final score of 10 points. How many games did they play?

A 5 B 6 C 7 D 8

E impossible to determine

Solution

C Jack won exactly 4 games. So Jill lost 4 games and gained 4 points from these losses. Jill gained 10 points altogether and so gained $10 - 4 = 6$ points from her wins. Since there are 2 points for a win, Jill won 3 games. So Jack won 4 games, Jill won 3 games and altogether $4 + 3 = 7$ games were played.

For investigation

19.1 Suppose Jack wins 5 games and Jill gains 15 points. How many games were played altogether?

19.2 Suppose Jack does not win any games and Jill gains 6 points. How many points did they gain between them?

19.3 Suppose Jack wins exactly a games, and Jill gains b points. Find a formula, in terms of a and b, for the total number of games they played. Are there any restrictions that you need to put on the values of a and b?

20. Box P has p chocolates and box Q has q chocolates, where p and q are both odd and $p > q$. What is the smallest number of chocolates which would have to be moved from box P to box Q so that box Q has more chocolates than box P?

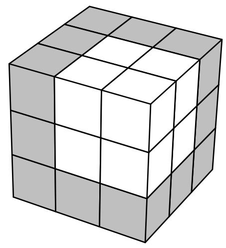
A $\dfrac{q-p+2}{2}$ B $\dfrac{p-q+2}{2}$ C $\dfrac{q+p-2}{2}$ D $\dfrac{p-q-2}{2}$ E $\dfrac{q+p+2}{2}$

Solution

B Because p and q are both odd, $p+q$ is even. If the chocolates were shared equally between the boxes there would be $\frac{1}{2}(p+q)$ chocolates in each box. So the least number of chocolates there must be in box Q if it is to have more chocolates than box P is $\frac{1}{2}(p+q)+1$. Because box Q starts with q chocolates in it, to end up with $\frac{1}{2}(p+q)+1$ in box Q the number of chocolates we need to transfer from box P to box Q is

$$\left(\frac{p+q}{2}+1\right)-q=\frac{p-q}{2}+1=\frac{p-q+2}{2}.$$

21. Pablo's teacher has given him 27 identical white cubes. She asks him to paint some of the faces of these cubes grey and then stack the cubes so that they appear as shown. What is the largest possible number of the individual white cubes which Pablo can leave with no faces painted grey?



A 8 B 12 C 14 D 15 E 16

Solution

D We can see 19 of the 27 cubes. Of these there are 12 which we can see have at least one grey face. The remaining cubes could have all their faces white. So the maximum number of cubes that could be all white is $27-12=15$.

22. In the division calculation $952\,473 \div 18$, which two adjacent digits should be swapped in order to increase the result by 100?

A 9 and 5 B 5 and 2 C 2 and 4 D 4 and 7 E 7 and 3

Solution

C If after division by 18 the number has to be increased by 100, then before the division it needs to be increased by $18 \times 100 = 1800$. So we need to find two adjacent digits in the number 952 473 which when swapped increase it by 1800.

If 952 473 is increased by 1800 it becomes $952\,473 + 1\,800 = 954\,273$. We obtain 954 273 from 952 473 by swapping the adjacent digits 2 and 4.

FOR INVESTIGATION

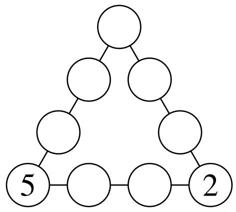
22.1 Which pair of adjacent digits of 952 473 should be swapped to decrease the result of the division calculation by 1500 ?

22.2 Which pair of digits (not necessarily adjacent) should be swapped to increase the result of the division calculation by 275 ?

22.3 A number has the digit a in the thousands place, and the digit b in the hundreds place. By how much is this number changed if these digits are swapped?

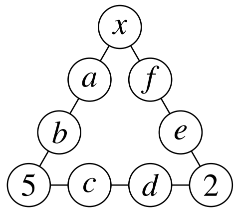
23. Sam wants to complete the diagram so that each of the nine circles contains one of the digits from 1 to 9 inclusive and each contains a different digit. Also, the digits in each of the three lines of four circles must have the same total. What is this total?

A 17 B 18 C 19 D 20 E 21



SOLUTION

D We suppose that x, a, b, c, d, e, and f are the numbers in the circles, as shown in the diagram. These numbers are the digits 1, 3, 4, 6, 7, 8, and 9, in some order.



Let the common total of the digits in the three lines be k.

Then

$$(x + a + b + 5) + (x + e + f + 2) + (5 + c + d + 2) = 3k,$$

that is,

$$(x + a + b + c + d + e + f) + x + 14 = 3k.$$

Now

$$x + a + b + c + d + e + f = 1 + 3 + 4 + 6 + 7 + 8 + 9 = 38$$

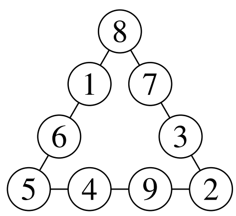
and therefore

$$52 + x = 3k.$$

It follows that $52 + x$ is divisible by 3. Now x is one of 1, 3, 4, 6, 7, 8, 9, and the only one of these possible values for which $52 + x$ is divisible by 3 is $x = 8$. This gives $52 + x = 60$. Hence $60 = 3k$, and so $k = 20$.

In the context of the JMC, we can stop here. But to give a full mathematical solution we should satisfy ourselves that it is possible to place the digits in the circles so that each line adds up to 20.

One way this can be done is shown in the diagram alongside.



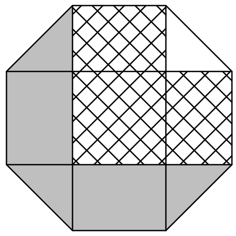
FOR INVESTIGATION

23.1 In how many different ways can the digits 1, 3, 4, 6, 7, 8 and 9 be placed in the circles so that the digits in each line have the total 20?

23.2 There are lots of other pairs of digits which can replace the 5 and the 2 in the bottom corners of the diagram in the question, so that the remaining digits can be placed in the other circles with the totals of the digits in each line being equal. See how many of these you can find.

24. The diagram shows a regular octagon with sides of length 1. The octagon is divided into regions by four diagonals.

What is the difference between the area of the hatched region and the area of the region shaded grey?

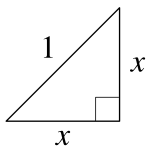


A 0 B $\frac{1}{8}$ C $\frac{1}{4}$ D $\frac{1}{2}$ E 1

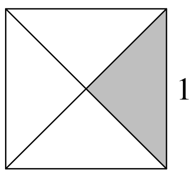
SOLUTION

C The octagon is divided into the central square, which has sides of length 1, four congruent rectangles and four congruent right-angled triangles. The hatched region is made up of two of the rectangles, and the square. The grey area is made up of two of the rectangles and three of the triangles. Therefore the difference between their areas is the difference between the area of the square and the area of three of the triangles.

The square has area 1. Each of the triangles is a right-angled isosceles triangle with hypotenuse of length 1. Let x be the length of each of the other two sides. By Pythagoras' Theorem, $x^2 + x^2 = 1^2$, that is $2x^2 = 1$ and so $x^2 = \frac{1}{2}$. So the area of each triangle is $\frac{1}{2}x^2 = \frac{1}{4}$. Therefore the difference between the area of the square and the area of three of the triangles is $1 - 3(\frac{1}{4}) = \frac{1}{4}$.



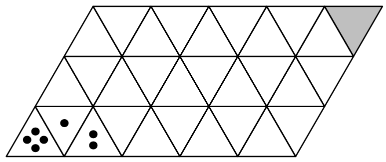
We can also see this geometrically. The diagram on the right shows that four of the triangles fit together to make a square whose side has length 1. So the difference between the area of the square and that of three of the triangles is the area of one-quarter of the square.



FOR INVESTIGATION

24.1 What is the area of the octagon?

25. A die has the shape of a regular tetrahedron, with the four faces having 1, 2, 3 and 4 pips. The die is placed with 4 pips 'face down' in one corner of the triangular grid shown, so that the face with 4 pips precisely covers the triangle marked with 4 pips.



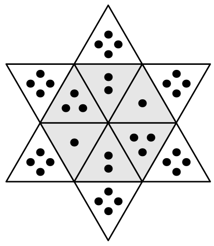
The die is now 'rolled', by rotating about an edge without slipping, so that 1 pip is face down. It is rolled again, so that 2 pips are face down, as indicated. The rolling continues until the die rests on the shaded triangle in the opposite corner of the grid.

How many pips are now face down?

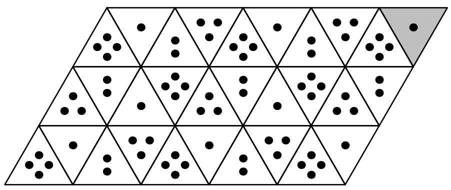
A 1 B 2 C 3 D 4
E it depends on the route taken

Solution

A We first consider what happens if one vertex of the tetrahedral die remains fixed on the grid and the die rolls about this fixed point. In doing this the die covers in succession six faces making up a hexagon. The faces adjacent to the fixed vertex are each face down twice in this hexagon. If the die is rolled about any of the edges on the perimeter of the hexagon, the remaining face is then face down. This is shown in the diagram on the right for the case where the faces with 1, 2 and 3 pips are adjacent to the fixed vertex, and the remaining face has 4 pips on it.



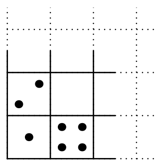
The pattern shown above can be extended to the whole of the given triangular grid, as shown in the diagram on the right. (Indeed, this pattern could be extended to a triangular grid filling the entire plane.)



For any triangle in the grid, the face then face down together with the two preceding faces form part of a hexagon, and determine the orientation of the die. (Here, by the *orientation* of the die, we mean the positions of all four of its faces.) The die can roll to this triangle across any of its edges. However, the pattern implies that the same face is face down in all these cases. From this we see that, however the die reaches a given triangle the same face is face down. In particular, the face with 1 pip is face down when the die reaches the shaded triangle.

For investigation

25.1 Investigate the similar problem for a standard cubic die rolling on a square grid. In particular, consider what happens if the die begins in the bottom left-hand corner of the grid with 1 pip face down, and so oriented that when it rolls to the squares above and to the right, the 2 pip face and the 4 pip face are face down, respectively, as shown.

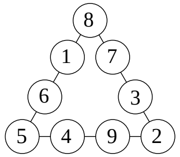


20. **B** Let the smallest number of chocolates required be n. Then $q + n > p - n$, that is $2n > p - q$. Therefore $n > \frac{1}{2}(p - q)$. Since $p > q$ and p and q are both odd, $\frac{1}{2}(p - q)$ is a positive integer. So the smallest possible value of n is $\frac{1}{2}(p - q) + 1 = \frac{1}{2}(p - q + 2)$.

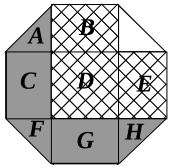
21. **D** Both the top and bottom layers of 9 cubes can be seen to contain 5 cubes with at least one face printed grey. The bottom layer could contain more than 5. In the middle layer, two cubes with grey faces are visible and there could be more. Therefore at least 12 cubes must have at least one face painted grey, which means that the largest number of cubes which Pablo can leave with no faces painted grey is $27 - 12 = 15$.

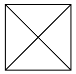
22. **C** In order to increase the result of the calculation (the quotient) by 100, the number to be divided (the dividend) must be increased by 100×18, that is 1800. So the new dividend needs to be $952\,473 + 1800$, that is 954 273. So the two digits which need to be swapped are 2 and 4.

23. **D** Note first that the sum of the first 9 positive integers is 45. Therefore, when the four numbers in each of the three lines are added together the total is 45 plus the sum of the numbers in the three corner circles, each of which contributes to the sum of two lines of circles. So if the number in the top circle is x, the total of all 3 lines is $45 + 2 + 5 + x = 52 + x$. As all three lines of circles must have the same total, $52 + x$ must be a multiple of 3. The possible values of x are 2, 5 and 8 but 2 and 5 have already been assigned. So $x = 8$ and the sum of each line is $60 \div 3 = 20$. The diagram shows one way of completing the task.

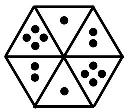


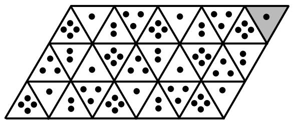
24. **C** Note that rectangles B, C, E and G are all congruent. Two of these are shaded grey and two are hatched, so the difference between the area of the hatched region and the area shaded grey is the difference between the area of square D of side 1 and the sum of the areas of triangles A, F and H. These are all isosceles right-angled triangles with hypotenuse 1 and the lower diagram shows how a square of side 1 may be divided into 4 such triangles. So the required difference in area is $1 - \frac{3}{4} = \frac{1}{4}$.





25. **A** If the die is rolled around a single vertex it covers, in turn, 6 small triangles making up a regular hexagon. It uses three different faces, repeated twice. An example is shown on the right. However, if it is rolled out from that hexagon in any direction, that will use the fourth face. The face that ends up covering each small triangle in the grid is always the same, regardless of the path taken to reach that triangle. Using these facts, it is easy to complete the diagram as shown. So, whichever route through the grid is taken, the '1' is face down when it reaches the shaded triangle.





UK Junior Mathematical Challenge

THURSDAY 1st MAY 2014

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

Institute and Faculty of Actuaries

SOLUTIONS LEAFLET

This solutions leaflet for the JMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

The UKMT is a registered charity

1. **D** $(999 - 99 + 9) \div 9 = (900 + 9) \div 9 = 909 \div 9 = 101$.

2. **B** There are 24 hours in one day, so $\frac{1}{12}$ of a day is 2 hours. Therefore the number of minutes in $\frac{1}{12}$ of a day is $2 \times 60 = 120$.

3. **C** The seats between us are T18 to T38 *inclusive*, that is, all the seats before seat 39 except for seats 1 to 17. So the number of seats is $38 - 17 = 21$.

4. **E** $987\,654\,321 \times 9 = 8\,888\,888\,889$.

5. **A** The smallest 4-digit number is 1000 and the largest 3-digit number is 999. They differ by 1.

6. **A** Let the width of each strip be 1. Then the square has side 5 and perimeter 20. The grey strips contribute 4 to the perimeter, so the fraction of the perimeter which is grey is $\frac{4}{20} = \frac{1}{5}$.

7. **B** $2002 - 4102 = -2100$. So $2014 - 4102 = -2100 + 12 = -2088$.

8. **A** Prime numbers have exactly two distinct factors, so 1 is not a prime number as it has exactly one factor. Of the others, 12, 1234 and 123 456 are all even numbers, so are not prime as the only even prime is 2. Also, $123 = 3 \times 41$ and 12 345 is clearly a multiple of 5, so neither of these is prime. Therefore none of the numbers in the list is prime.

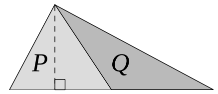
9. **E** The area of a triangle $= \frac{1}{2} \times$ base $\times$ height. If we let the length of the sides of each square in the grid be 1, then the area of triangle PQR is $\frac{1}{2} \times 3 \times 2 = 3$. The area of triangle XYZ is $\frac{1}{2} \times 6 \times 3 = 9$. So the required fraction is $\frac{3}{9} = \frac{1}{3}$.

10. **D** The angles at a point sum to 360°, so the largest angle in the triangle which includes the angle marked $x°$ is equal to $(360 - 90 - 90 - 60)° = 120°$. This triangle is isosceles as the sides of the three squares in the figure are equal to the sides of the equilateral triangle. So the triangle has angles 120°, $x°$ and $x°$. Therefore $x = \frac{1}{2}(180 - 120) = 30$.

11. **D** The third term of the sequence equals $1 + 2 = 3$. Now consider the fourth term: it is the sum of the first three terms. However, as the first two terms sum to the third term, the sum of the first three terms is twice the third term, i.e. $2 \times 3 = 6$. So the fourth term is twice the third term. Similar reasoning applies to each subsequent term, i.e. each term after the third term is equal to twice the term which precedes it. Therefore the sequence is 1, 2, 3, 6, 12, 24, 48, 96, ….

12. **B** As $7Q2ST - P3R96 = 22222$, it follows that $7Q2ST = P3R96 + 22222$. Looking at the units column: $2 + 6 = T$, so $T = 8$. Looking at the tens column, as $2 + 9 = 11$, we deduce that $S = 1$ and that 1 is carried to the hundreds column. Looking at the hundreds column: the carry of $1 + 2 + R$ must equal 12 since the sum has 2 in the hundreds column. So $R = 9$ and there is a carry of 1 to the thousands column. Looking at this column: the carry of $1 + 2 + 3 = Q$, so $Q = 6$. Finally, since there is no carry to the next column, $2 + P = 7$, so $P = 5$. Therefore the calculation is $76218 - 53996 = 22222$ and $P + Q + R + S + T = 5 + 6 + 9 + 1 + 8 = 29$.

13. **A** The diagram shows part of the given diagram after a rotation so that the diagonal shown is horizontal. The perpendicular height of triangle P is shown and it can be seen that this is also the perpendicular height of triangle Q. The diagonals of a rectangle bisect each other, so triangles P and Q have bases of equal length and the same perpendicular height. Therefore their areas are equal.



14. **D** One million millimetres is $(1\,000\,000 \div 1000)$ m $= 1000$ m $= 1$ km.

15. **E** Consider, for example, the bottom left-hand corner of the envelope (see Figure 1). The two flaps overlap, so that the sum of the angles marked x and y is greater than 90°. So when the flaps are unfolded, as in Figure 2, the angle marked z is less than 180°. Therefore the correct answer is E.

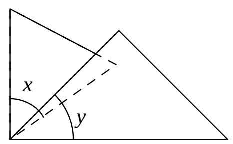


Figure 1

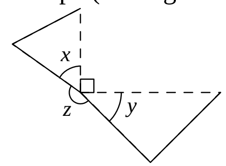
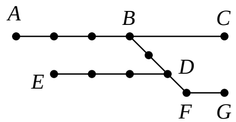


Figure 2

16. **E** If A is true then B is true which cannot be so since we are told only one statement is true. Hence A is false which is what E says. So E is the one true statement. [For completeness, we note that C and D must be false because we are told that exactly one statement is true; and B is false because A is false.]

17. **C** Whichever route is chosen, it must include section BD. We will divide the route into two sections. The first will include stations A, B, C and will finish at D. The second will start at D and include stations E, F, and G. Clearly the first section cannot be traversed without visiting at least one station more than once and the route $A – B – C – B – D$ visits only B more than once so it is an optimal solution. Also, traversing the second section involves visiting D more than once as two branches lead from it. If $D – E – D$ is part of the route then two stations are visited more than once. However, if $D – G – D$ is part of the route then only F is visited more than once. So to traverse the second section, it is necessary to visit at least two stations (one of which is D) more than once. Therefore, the complete route must involve visiting at least 3 stations more than once. An example of an optimum route is $A – B – C – B – D – F – G – F – D – E$. The stations visited twice are B, D, and F.



18. **E** The units digit of any power of 5 is 5 so the units digit of $1 + 5^6$ is 6. Therefore the units digits of the calculations in the 5 options are 2, 1, 0, 9, 8 in that order. So the only calculation which could be correct is E. Checking this gives $1 + 5^6 - 8 = 1 + 15\,625 - 8 = 15\,626 - 8 = 15\,618$.

19. **C** Since Jack won 4 games, Jill lost 4 games for which she was awarded 4 points. So the number of games she won is $(10 - 4) \div 2 = 3$. Therefore, they played 7 games in total.