Next Selection Test: Paper 4

Oundle School, Northamptonshire

6^{th} June 2012

1. Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

2. Let ABC be an acute triangle with circumcircle Γ. Let B' be the midpoint of AC and let C' be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of ABC. Let ω be a circle through B' and C' that is tangent to Γ at a point X distinct from A. Prove that D, G and X are collinear.

3. Let n be a positive integer and let $W = \ldots x_{-1}, x_0, x_1, x_2, \ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

   A finite nonempty word U is said to *appear in* W if there exist indices $k \leq l$ such that $U = x_k x_{k+1} \ldots x_l$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Each question is worth seven marks.
Time permitted: 4 hours, 30 minutes.