Next Selection Test: Paper 3

Oundle School

31st May 2011

1. If X is a set of integers, define $D(X)$ to be the set of differences between elements of X:

$$D(X) = \{n - m \mid m, n \in X, m < n\}.$$

Find the largest natural number N, for which the following is true: if we partition the natural numbers into N disjoint infinite sets $X_1, \ldots, X_N$, then the intersection

$$D(X_1) \cap \cdots \cap D(X_N)$$

must have infinitely many elements.

2. (a) Given a positive integer n, prove that there do not exist two distinct integers strictly between n^2 and $(n+1)^2$, whose product is a square.

   (b) Given an integer $a > 2$, prove that there exist a distinct integers strictly between n^a and $(n+1)^a$, whose product is an ath power, for all but a finite number of positive integers n.

3. Let $a, b, c > 0$ and $a + b + c = 3$. Prove that

$$\frac{1}{a\sqrt{2(a^2+bc)}} + \frac{1}{b\sqrt{2(b^2+ca)}} + \frac{1}{c\sqrt{2(c^2+ab)}} \geq \frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab}.$$

Each question is worth seven marks.
Time: 4 hours, 30 minutes.