Next Selection Test: Paper 2

Oundle School, Northamptonshire

4th June 2012

1. Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

2. Determine the greatest positive integer k that satisfies the following property: the set of positive integers can be partitioned into k subsets $A_1, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

3. Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Show that p divides m.

Each question is worth seven marks.
Time permitted: 4 hours, 30 minutes.