UK IMO Next Selection Test 2

Oundle 2005

1. Let $n \geq 2$ be a natural number. A pyramid $\mathcal{P}$ has base $A_1A_2\cdots A_{2n}$ and apex O. The polygon $A_1A_2\cdots A_{2n}$ is regular and the point C is its centre. The line OC is perpendicular to the plane of the base of $\mathcal{P}$. A sphere passes through O and meets each of the line segments OA_i internally. For each $i = 1, 2, \ldots, 2n$ let X_i be the point (other than O) where the sphere meets OA_i. Prove

$$OX_1 + OX_3 + \cdots + OX_{2n-1} = OX_2 + OX_4 + \cdots + OX_{2n}.$$

2. Find the number of subsets B of $\{1, 2, 3, \ldots, 2005\}$ such that the sum of the elements of B is congruent to 2006 modulo 2048.

3. Let $n \geq 3$ be an integer. Consider positive real numbers $a_1, a_2, \ldots, a_n$ such that $a_1a_2\cdots a_n = 1$. Show that the following inequality holds

$$\frac{a_1+3}{(a_1+1)^2} + \frac{a_2+3}{(a_2+1)^2} + \cdots + \frac{a_n+3}{(a_n+1)^2} \geq 3.$$

Time allowed $4\frac{1}{2}$ *hours.*