Next Selection Test: Paper 1

Oundle School, Northamptonshire

3^{rd} June 2012

1. For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so, for example, we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that, for every integer $k \geq 0$, the number $f(2^k)$ divides $f(2^{k+1})$.

2. Let p be a prime and n a positive integer. We write $\mathbb{Z}/p^n\mathbb{Z}$ for the set of congruence classes modulo p^n. Determine the number of functions $f : (\mathbb{Z}/p^n\mathbb{Z}) \to (\mathbb{Z}/p^n\mathbb{Z})$ satisfying the condition

$$f(a) + f(b) = f(a + b + pab)$$

for all $a, b \in \mathbb{Z}/p^n\mathbb{Z}$.

3. Let ABC be a triangle with incentre I and circumcircle Γ. Let D and E be the second intersection points of Γ with the lines AI and BI respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to Γ at A and at B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

Each question is worth seven marks.
Time permitted: 4 hours, 30 minutes.