# Next Selection Test: Paper 1

Oundle School

29th May 2011

1. Circles $\Gamma_1$ and $\Gamma_2$ meet at $M$ and $N$. Let $A$ be on $\Gamma_1$ and $D$ on $\Gamma_2$. The lines $AM$ and $AN$ meet $\Gamma_2$ again at $B$ and $C$ respectively; the line $DM$ and $DN$ meet $\Gamma_1$ again at $E$ and $F$, respectively. Assume that $M, N, F, A, E$ are in cyclic order around $\Gamma_1$, and that $AB$ and $DE$ are congruent. Prove that $A$, $F$, $C$ and $D$ lie on a circle whose centre does not depend on the position of $A$ and $D$ on the circles.

2. Let $n \geq 2$ be an integer, and let $a_1, \ldots, a_n$ be positive reals. We define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ from the positive reals to the positive reals by the formula

$$f(x) = \frac{a_1 + x}{a_2 + x} + \frac{a_2 + x}{a_3 + x} + \cdots + \frac{a_{n-1} + x}{a_n + x} + \frac{a_n + x}{a_1 + x}.$$

   Show that $f$ is a decreasing function of $x$.

3. Find the smallest number $n$ such that there exist polynomials $f_1, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \cdots + f_n(x)^2.$$

*Each question is worth seven marks.*
*Time: 4 hours, 30 minutes.*