First Selection Test: Paper 2

Trinity College, Cambridge

18$^{\text{th}}$ April 2011

1. For any positive integer n, let a_n be the exponent of the largest power of 2 which occurs as a factor of $5^n - 3^n$. Also, let b_n be the exponent of the largest power of 2 which divides n.

   Show that $a_n \leq b_n + 3$ for all n.

2. Let $ABCD$ be a cyclic quadrilateral, whose circumcircle has centre O. Let E be the midpoint of AB and F be the midpoint of AD. Show that if the area of the quadrilateral $ABCD$ is four times the area of the triangle OEF, then one of BC and DC is a diameter.

3. Each point of the plane is coloured either red or blue. A triangle is said to be *monochromatic* if its three vertices have the same colour. Given an odd integer $n \geq 3$, prove that there exist two monochromatic triangles (not necessarily the same colour), which are similar with size ratio $n : 1$.

Each question is worth seven marks.
Time: 4 hours, 30 minutes.