FST2 2006

Trinity College Cambridge

1. Each integer is coloured either red, blue, green or white. Let x and y be odd integers such that $|x| \neq |y|$. Show that there exist two natural numbers of the same colour whose difference is one of $x, y, x + y$ or $x - y$.

2. A point P is in the interior of the cyclic quadrilateral $ABCD$ and has the property that

$$\angle BPC = \angle BAP + \angle PDC.$$

The feet of the perpendiculars from P to AB, AD and DC are respectively denoted E, F and G. Show that $\triangle FEG$ and $\triangle PBC$ are similar.

3. Find all real numbers x such that there is a positive integer n with

$$15\lfloor x \rfloor^2 + 10\lfloor x \rfloor x + 18\lfloor x \rfloor + 2x + 6 - 4^n = 0.$$

Here $\lfloor y \rfloor$ denotes the integer part of the real number y, the greatest integer which is not more than y.