First Selection Test 1

3-iv-2004

1. Three distinct points A, B and C are fixed on a line in that order. Let Γ be a circle passing through A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose that Γ meets the line segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

2. Let $D_1, D_2, \ldots, D_n$ be distinct closed disks in the plane. (A closed disk consists of the union of a circle and the region in its interior.) Suppose that every point of the plane is contained in at most 2004 disks D_i. Prove that there exists a disk D_k which intersects at most $7 \cdot 2004 - 1$ other disks D_i.

3. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

   (i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

   (ii) $f(x) < f(y)$ for all $1 \le x < y$.