First Selection Test

Trinity IMO Training Camp

7 April 2002

1. We are given a circle Γ and a straight line l which is tangent to the circle at the point B. Through any point A on Γ we drop the perpendicular AP to l where $P \in l$. Let M be the point which is the reflection of P in the line AB. Determine the locus of M as A varies on Γ.

2. Let $a_1, a_2, \ldots, a_{2n}$ be distinct integers such that the equation

$$(x - a_1)(x - a_2) \cdots (x - a_{2n}) + (-1)^{n-1}(n!)^2 = 0$$

has an integer solution r. Show that

$$r = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n}.$$

3. For a positive integer n define a sequence of zeros and ones to be *balanced* if it contains n zeros and n ones. Two balanced sequences a and b are *neighbours* if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbours because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most

$$\frac{1}{n+1}\binom{2n}{n}$$

balanced sequences such that every balanced sequence is equal to or is a neighbour of at least one sequence in S.

4. Let $k \geq 2$ be a fixed integer. Show that there is an irrational number r such that

$$\lfloor r^m \rfloor \equiv -1 \bmod k$$

for every natural number m. Here $\lfloor y \rfloor$ denotes the integer part of the real number y.