Intermediate Mathematical Challenge 2015

1. What is the value of $1 - 0.2 + 0.03 - 0.004$?

A 0.826 B 0.834 C 0.926 D 1.226 E 1.234

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1. A $1 - 0.2 + 0.03 - 0.004 = 0.8 + 0.026 = 0.826$.

2. Last year, Australian Suzy Walsham won the annual women's race up the 1576 steps of the Empire State Building in New York for a record fifth time. Her winning time was 11 minutes 57 seconds. Approximately how many steps did she climb per minute?

A 13 B 20 C 80 D 100 E 130

2. E The number of steps climbed per minute $\sim \dfrac{1600}{12} = \dfrac{400}{3} \sim 130$.

3. What is a half of a third, plus a third of a quarter, plus a quarter of a fifth?

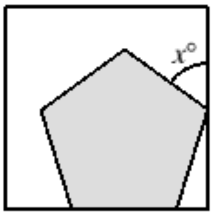
A $\frac{1}{1440}$ B $\frac{3}{38}$ C $\frac{1}{30}$ D $\frac{1}{3}$ E $\frac{3}{10}$

3. E Half of a third, plus a third of a quarter, plus a quarter of a fifth equals $\frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{10 + 5 + 3}{60} = \frac{18}{60} = \frac{3}{10}$.

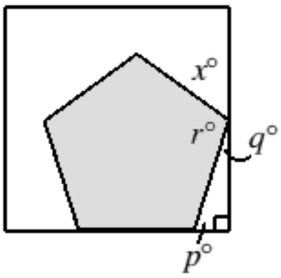
4. The diagram shows a regular pentagon inside a square.

What is the value of x?

A 48 B 51 C 54 D 60 E 72



4. C The sum of the exterior angles of a convex polygon equals 360°. The angle marked $p°$ is the exterior angle of a regular pentagon. So $p = 360 \div 5 = 72$. The angle sum of a triangle equals 180°, so $q = 180 - 90 - 72 = 18$. The angle marked $r°$ is the interior angle of a regular pentagon, so $r = 180 - 72 = 108$. The angles marked $q°$, $r°$ and $x°$ lie along a straight line, so $x = 180 - (q + r) = 180 - (18 + 108) = 54$.



5. Which of the following numbers is not a square?

A 1^6 B 2^5 C 3^4 D 4^3 E 5^2

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5. B $1^6 = (1^3)^2$, $3^4 = (3^2)^2$, $4^3 = 64 = 8^2$ and 5^2 are all squares. However, $2^5 = 32$ and is not a square.

6. The equilateral triangle and regular hexagon shown have perimeters of the same length.
What is the ratio of the area of the triangle to the area of the hexagon?

A 5 : 6 B 4 : 5 C 3 : 4 D 2 : 3 E 1 : 1

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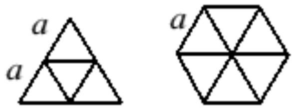
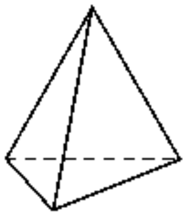
6. D Let the length of the side of the regular hexagon be a. Then its perimeter is $6a$. Therefore the perimeter of the equilateral triangle is also $6a$, so the length of each of its sides is $2a$. The diagrams show that the equilateral triangle may be divided up into 4 equilateral triangles of side a, whereas the regular hexagon may be divided into 6 such triangles. So the required ratio is $4 : 6 = 2 : 3$.



7. A tetrahedron is a solid figure which has four faces, all of which are triangles.
What is the product of the number of edges and the number of vertices of the tetrahedron?

A 8 B 10 C 12 D 18 E 24



7. E The tetrahedron has 6 edges and 4 vertices, so the required product is $6 \times 4 = 24$.

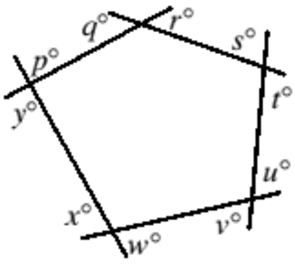
8. How many two-digit squares differ by 1 from a multiple of 10?

A 1 B 2 C 3 D 4 E 5

8. B The two-digit squares are 16, 25, 36, 49, 64 and 81. Of these, only 49 and 81 differ by 1 from a multiple of 10.

9. What is the value of $p + q + r + s + t + u + v + w + x + y$ in the diagram?

A 540 B 720 C 900 D 1080 E 1440



9. B The sum of the exterior angles of a convex polygon equals 360°. Therefore $p + r + t + v + x = 360°$. Similarly, $q + s + u + w + y = 360°$. Therefore $p + q + r + s + t + u + v + w + x + y = 720°$.

10. What is the remainder when $2^2 \times 3^3 \times 5^5 \times 7^7$ is divided by 8?

A 2 B 3 C 4 D 5 E 7

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10. C $2^2 \times 3^3 \times 5^5 \times 7^7$ is of the form $2^2 \times$ an odd number. It therefore has the form $4(2n+1) = 8n+4$ where n is a positive integer and so leaves a remainder of 4 when divided by 8.

11. Three different positive integers have a mean of 7. What is the largest positive integer that could be one of them?

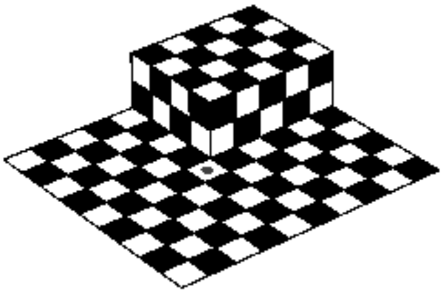
A 15 B 16 C 17 D 18 E 19

11. D As the 3 numbers have mean 7, their sum equals $3 \times 7 = 21$. For one of the numbers to be as large as possible the other two numbers must be as small as possible. They must also be different and so must be 1 and 2. Hence the largest possible of the three numbers equals $21 - (1 + 2) = 18$.

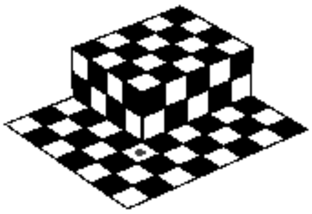
12. An ant is on the square marked with a black dot. The ant moves across an edge from one square to an adjacent square four times and then stops.

How many of the possible finishing squares are black?

A 0 B 2 C 4 D 6 E 8

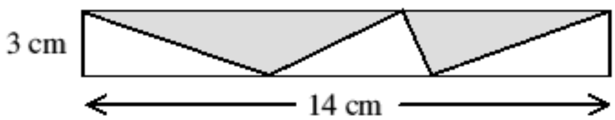


12. D If the ant moves alternately from white square to black square and from black to white, then it will end on a white square after 4 moves. So it must find a way to move from white to white or from black to black. However, there is only one pair of adjacent black squares and only one of white. To reach that pair of black squares, the ant must move to one side then climb up to one of the pair. That uses up 3 moves, and the fourth must take it to the other black square of that pair. Thus the two black squares in that pair are possible end points.



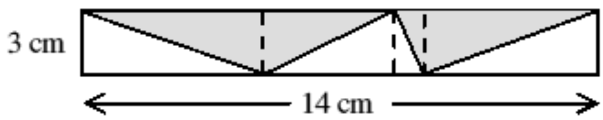
If, instead, the ant uses the white pair, it must first move to one side, then climb up to one of the white pair then across to the other square of that pair. That uses 3 moves. The fourth move can then take it to any of the three adjoining black squares. This gives 6 end squares, but these include the two already identified. So there are just 6 possible end squares which are black.

13. What is the area of the shaded region in the rectangle?



A 21 cm² B 22 cm² C 23 cm² D 24 cm² E more information needed

13. A Three vertical lines have been added to the diagram. These divide the original diagram into 4 rectangles. In each of these, a diagonal divides the rectangle into two triangles of equal area, one shaded and one unshaded. So the total shaded area in the original rectangle is equal to the total unshaded area and is therefore equal to half the area of the original rectangle. So the total shaded area is $\frac{1}{2} \times 3 \times 14\,\text{cm}^2 = 21\,\text{cm}^2$.



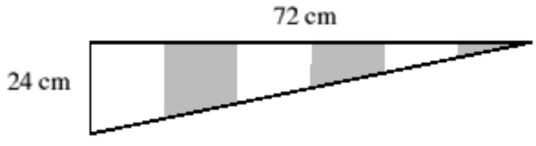
14. In a sequence, each term after the first two terms is the mean of all the terms which come before that term. The first term is 8 and the tenth term is 26. What is the second term?

A 17 B 18 C 44 D 52 E 68

14. C Suppose the first three terms of the sequence are a, b, c. Then $c = \frac{1}{2}(a+b)$ and so $a+b=2c$. The mean of the first three terms is then $\frac{1}{3}(a+b+c) = \frac{1}{3}(2c+c) = c$, so the fourth term is c. Similarly, the following terms are all equal to c. Since one of these terms is 26 and $a = 8$ then $b = 2c - a = 52 - 8 = 44$.

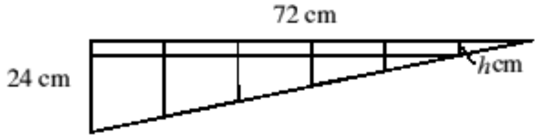
15. A flag is in the shape of a right-angled triangle, as shown, with the horizontal and vertical sides being of length 72 cm and 24 cm respectively. The flag is divided into 6 vertical stripes of equal width.

What, in cm^2, is the difference between the areas of any two adjacent stripes?



A 96 B 72 C 48 D 32 E 24

15. C The stripes are of equal width, so the width of each stripe is $(72 \div 6)$ cm = 12 cm. The diagram shows that the difference between the areas of any two adjacent stripes is equal to the area of a rectangle of width 12 cm and height h cm. By similar triangles, $\frac{h}{12} = \frac{24}{72}$. So $h = \frac{12 \times 24}{72} = 4$. So the required area is $12 \times 4\ \text{cm}^2 = 48\ \text{cm}^2$.



16. You are asked to choose two positive integers, m and n with $m > n$, so that as many as possible of the expressions $m + n$, $m - n$, $m \times n$ and $m \div n$ have values that are prime. When you do this correctly, how many of these four expressions have values that are prime?

A 0 B 1 C 2 D 3 E 4

16. D All four values cannot be prime. If this were so, both $m \times n$ and $m \div n$ would be prime which can happen only if m is prime and $n = 1$. If m is an odd prime then $m + 1$ is even and at least 4, hence not prime, while if $m = 2$ then $m - 1$ is not prime but $m + 1 = 3$ is. Thus three prime values are the most we can have.

17. The football shown is made by sewing together 12 black pentagonal panels and 20 white hexagonal panels. There is a join wherever two panels meet along an edge.

How many joins are there?

A 20 B 32 C 60 D 90 E 180

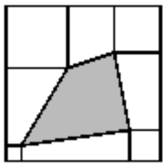
17. D The 12 pentagonal panels have a total of $12 \times 5 = 60$ edges. The 20 hexagonal panels have a total of $20 \times 6 = 120$ edges. So in total the panels have 180 edges. When the panels are sewn together, two edges form each join. So the number of joins is $180 \div 2 = 90$.

18. The total weight of a box, 20 plates and 30 cups is 4.8 kg. The total weight of the box, 40 plates and 50 cups is 8.4 kg. What is the total weight of the box, 10 plates and 20 cups?

A 3 kg B 3.2 kg C 3.6 kg D 4 kg E 4.2 kg

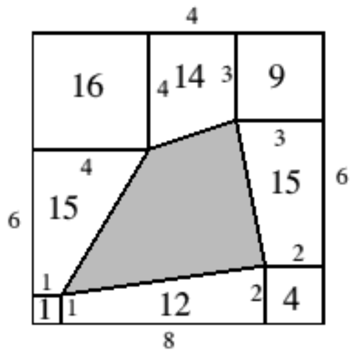
18. A Let the weights in kg of the box, 1 plate and 1 cup be b, p and c respectively.
Then: $b + 20p + 30c = 4.8$ (i); $b + 40p + 50c = 8.4$ (ii).
Subtracting (i) from (ii): $20p + 20c = 3.6$ (iii). So $10p + 10c = 1.8$ (iv).
Subtracting (iv) from (i): $b + 10p + 20c = 3$. So the required weight is 3 kg.

19. The figure shows four smaller squares in the corners of a large square. The smaller squares have sides of length 1 cm, 2 cm, 3 cm and 4 cm (in anticlockwise order) and the sides of the large square have length 11 cm.
What is the area of the shaded quadrilateral?



A 35 cm^2 B 36 cm^2 C 37 cm^2 D 38 cm^2 E 39 cm^2

19. A The small numbers in the figure show the lengths in cm of each line segment. The larger numbers inside the figure show the areas in cm^2 of each square or trapezium. (The area of a trapezium is $\frac{1}{2}(a + b)h$ where a and b are the lengths of the parallel sides and h is the perpendicular distance between them.)
So the area of the shaded portion in cm^2 is
$11 \times 11 - (1 + 12 + 4 + 15 + 9 + 14 + 16 + 15) = 35$.
(*See the extended solutions for a beautifully elegant solution of this problem.*)

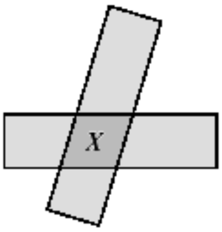


20. A voucher code is made up of four characters. The first is a letter: V, X or P. The second and third are different digits. The fourth is the units digit of the sum of the second and third digits. How many different voucher codes like this are there?

A 180 B 243 C 270 D 300 E 2700

20. C There are 3 different possibilities for the first character. The second character may be any digit from 0 to 9 inclusive, so it has 10 different possibilities. The third character differs from the second digit, so has 9 different possibilities. Once the second and third characters are determined, the fourth character is also determined since it is the units digit of the sum of the second and third characters.
So, the number of different codes is $3 \times 10 \times 9 = 270$.

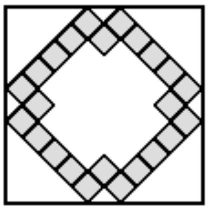
21. A rectangle is placed obliquely on top of an identical rectangle, as shown.
The area X of the overlapping region (shaded more darkly) is one eighth of the total shaded area.
What fraction of the area of one rectangle is X ?



A $\frac{1}{3}$ B $\frac{2}{7}$ C $\frac{1}{4}$ D $\frac{2}{9}$ E $\frac{1}{5}$

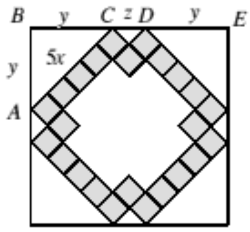
21. D Let the area of each rectangle be Y. Then the total shaded area is $2(Y - X) + X = 2Y - X$. Therefore $X = \frac{1}{8}(2Y - X)$. So $8X = 2Y - X$, that is $9X = 2Y$. Therefore $\frac{X}{Y} = \frac{2}{9}$.

22. The diagram shows a shaded region inside a large square. The shaded region is divided into small squares.
What fraction of the area of the large square is shaded?



A $\frac{3}{10}$ B $\frac{1}{3}$ C $\frac{3}{8}$ D $\frac{2}{5}$ E $\frac{3}{7}$

22. B Let the length of the sides of each small square be x. Then the shaded area is $24x^2$. Let the perimeter of the square be divided into eight line segments, each of length y, and four line segments of length z. Some of these are labelled in the diagram. By Pythagoras' Theorem in triangle ABC:
$y^2 + y^2 = (5x)^2$, that is $2y^2 = 25x^2$.



So $y = \frac{5}{\sqrt{2}}x = \frac{5\sqrt{2}}{2}x$. Similarly, in the triangle with hypotenuse CD: $x^2 + x^2 = z^2$, that is $2x^2 = z^2$. So $z = \sqrt{2}x$. Therefore the length of the side of the large square is $2y + z = 5\sqrt{2}x + \sqrt{2}x = 6\sqrt{2}x$. So the area of the large square is $(6\sqrt{2}x)^2 = 72x^2$. Hence the required fraction is $\frac{24x^2}{72x^2} = \frac{1}{3}$.

23. There are 120 different ways of arranging the letters, U, K, M, I and C. All of these arrangements are listed in dictionary order, starting with CIKMU. Which position in the list does UKIMC occupy?

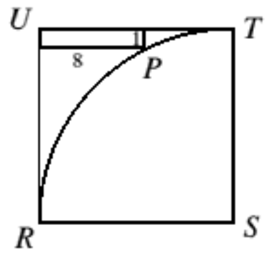
A 110 th B 112 th C 114 th D 116 th E 118 th

1533

23. B The permutations which follow UKIMC in dictionary order are UKMCI, UKMIC, UMCIK, UMCKI, UMICK, UMIKC, UMKCI, UMKIC. There are eight of these, so UKIMC is 112th in the list.

24. In square $RSTU$ a quarter-circle arc with centre S is drawn from T to R. A point P on this arc is 1 unit from TU and 8 units from RU.
What is the length of the side of square $RSTU$?

A 9 B 10 C 11 D 12 E 13



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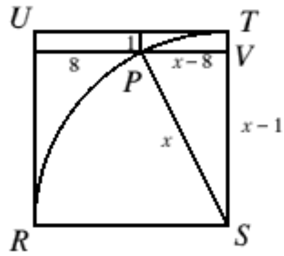
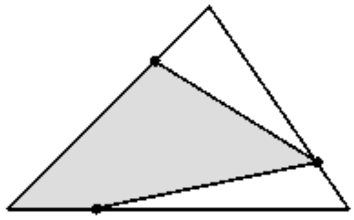
24. E In the diagram V is the point where the perpendicular from P meets TS. Let the side of the square $RSTU$ be x. So the radius of the arc from R to T is x. Therefore SP has length x, PV has length $x-8$ and VS has length $x-1$. Applying Pythagoras' Theorem to triangle PVS:
$(x-8)^2+(x-1)^2=x^2$. So $x^2-16x+64+x^2-2x+1=x^2$.
Therefore $x^2-18x+65=0$, so $(x-5)(x-13)=0$.
Hence $x=5$ or $x=13$, but $x>8$ so the length of the side of the square $RSTU$ is 13.



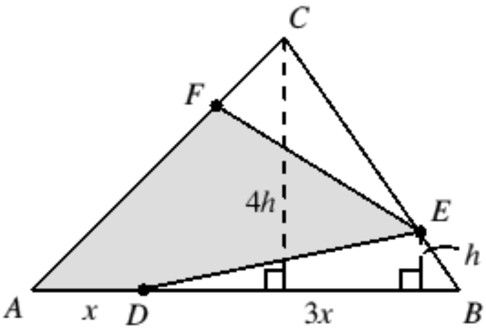
25. A point is marked one quarter of the way along each side of a triangle, as shown.
What fraction of the area of the triangle is shaded?



A $\frac{7}{16}$ B $\frac{1}{2}$ C $\frac{9}{16}$ D $\frac{5}{8}$ E $\frac{11}{16}$

1535

25. D Points A, B, C, D, E, F on the perimeter of the triangle are as shown. Let AD have length x so that DB has length $3x$. Let the perpendicular from C to AB have length $4h$. So, by similar triangles, the perpendicular from E to DB has length h. The area of triangle ABC is $\frac{1}{2} \times 4x \times 4h = 8xh$. The area of triangle DBE is $\frac{1}{2} \times 3x \times h = \frac{3}{2}xh$. So the area of triangle DBE is $\frac{3}{16}$ of the area of triangle ABC.



Similarly, by drawing perpendiculars to CB from A and from F, it may be shown that the area of triangle FEC is $\frac{3}{16}$ of the area of triangle ABC.
So the fraction of the area of the triangle that is shaded is $1 - \frac{3}{16} - \frac{3}{16} = \frac{5}{8}$.