Intermediate Mathematical Challenge 2013

1. Which of the following is divisible by 6?

A one million minus one B one million minus two C one million minus three D one million minus four E one million minus five

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1. D In order to be a multiple of 6, a number must be both even and a multiple of 3. Of the numbers given, only B 999 998 and D 999 996 are even. Using the rule for division by 3, we see that, of these two, only 999 996 is a multiple of 3.

2. A machine cracks open 180 000 eggs per hour. How many eggs is that per second?

A 5 B 50 C 500 D 5000 E 50 000

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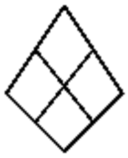
2. B 180 000 eggs per hour is equivalent to 3000 eggs per minute, i.e. to 50 eggs per second.

3. How many quadrilaterals are there in this diagram, which is constructed using 6 straight lines ?



A 4 B 5 C 7 D 8 E 9

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3. **E** The figure is itself a quadrilateral. It can be divided into four small quadrilaterals labelled A, B, C, D. There are also four quadrilaterals formed in each case by joining together two of the smaller quadrilaterals: A and B; B and C; C and D; D and A.

4. A standard pack of pumpkin seeds contains 40 seeds. A special pack contains 25% more seeds. Rachel bought a special pack and 70% of the seeds germinated. How many pumpkin plants did Rachel have?

A 20 B 25 C 28 D 35 E 50

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4. **D** The number of seeds in a special packet is $1.25 \times 40 = 50$. So the number of seeds which germinate is $0.7 \times 50 = 35$.

5. The northern wheatear is a small bird weighing less than an ounce. Some northern wheatears migrate from sub-Saharan Africa to their Arctic breeding grounds, travelling almost 15 000 km. The journey takes just over 7 weeks. Roughly how far do they travel each day, on average?

A 1 km B 9 km C 30 km D 90 km E 300 km

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5. E A wheatear travels the distance of almost 15 000 km in approximately 50 days. This is on average roughly 300 km per day.

6. Which of the following has the least value?

A $1^0 - 0^1$ B $2^1 - 1^2$ C $3^2 - 2^3$ D $4^3 - 3^4$ E $5^4 - 4^5$

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6. **E** In order, the values of the expressions given are: $1 - 0 = 1$; $2 - 1 = 1$; $9 - 8 = 1$; $64 - 81 = -17$; $625 - 1024 = -399$.

7. The faces of a regular octahedron are to be painted so that no two faces which have an edge in common are painted in the same colour. What is the smallest number of colours required?

A 2 B 3 C 4 D 6 E 8

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7. **A** Only two colours are needed for the upper four faces of the octahedron. If, for example, blue and red are used then these four faces may be painted alternately red and blue. Consider now the lower four faces: every face adjacent to an upper blue face may be painted red and every face adjacent to an upper red face may be painted blue. So only two colours are required for the whole octahedron.

8. Jim rolled some dice and was surprised that the sum of the scores on the dice was equal to the product of the scores on the dice. One of the dice showed a score of 2, one showed 3 and one showed 5. The rest showed a score of 1. How many dice did Jim roll?

A 10 B 13 C 17 D 23 E 30

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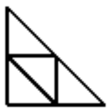
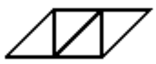
8. D Let the number of scores of 1 be n. Then the product of the scores is $1^n \times 2 \times 3 \times 5 = 30$. Therefore $1 \times n + 2 + 3 + 5 = 30$, i.e. $n = 20$. So Jim threw 23 dice.

9. Jane has 20 identical cards in the shape of an isosceles right-angled triangle. She uses the cards to make the five shapes below. Which of the shapes has the shortest perimeter?

A  B  C  D  E 

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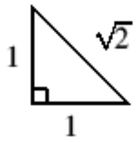
9. A Let the length of the shorter sides of the cards be 1 unit. Then, by Pythagoras' Theorem, the length of the hypotenuse of each card is $\sqrt{1^2 + 1^2} = \sqrt{2}$.



So the lengths of the perimeters of the five figures in order are: $4\sqrt{2}$; $4 + 2\sqrt{2}$; $4 + 2\sqrt{2}$; 6; $4 + 2\sqrt{2}$. Also, as $\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4} > 2$ we see that $\frac{3}{2} > \sqrt{2}$. Therefore, $4\sqrt{2} < 6 < 4 + 2\sqrt{2}$. So figure A has the shortest perimeter.

10. *ABCDE* is a regular pentagon and *BCF* is an equilateral triangle such that *F* is inside *ABCDE*.

What is the size of $\angle FAB$?

A 48° B 63° C 66° D 69° E 72°

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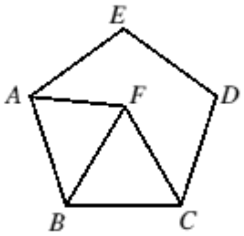
10. C The sum of the interior angles of a pentagon is 540° so $\angle ABC = 540° \div 5 = 108°$. Each interior angle of an equilateral triangle is 60°, so $\angle FBC = 60°$.
Therefore $\angle ABF = 108° - 60° = 48°$. As *ABCDE* is a regular pentagon, $BC = AB$. However, $BC = FB$ since triangle *BFC* is equilateral.
So triangle *ABF* is isosceles with $FB = AB$.
Therefore $\angle FAB = \angle AFB = (180° - 48°) \div 2 = 66°$.



11. For which of the following numbers is the sum of all its factors *not* equal to a square number?

A 3 B 22 C 40 D 66 E 70

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11. C We first look at $66 = 2 \times 3 \times 11$. Its factors involve none, one, two or all three of these primes. So the factors are 1, 2, 3, 11, 6, 22, 33, 66; and their sum is $144 = 12^2$. Similarly, we can check that the sum of the factors of 3, 22, 40 and 70 is, respectively, $4 = 2^2$, $36 = 6^2$, 90 and $144 = 12^2$. So 40 is the only alternative for which the sum of the factors is not a square number.

12. The sum one + four = seventy

becomes correct if we replace each word by the number of letters in it to give $3 + 4 = 7$. Using the same convention, which of these words could be substituted for x to make the sum

three + five = x true?

A eight B nine C twelve D seventeen E eighteen

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12. D As the words ‘three’ and ‘five’ contain 5 and 4 letters respectively, their ‘sum’ will be a 9-letter word. Of the alternatives given, only ‘seventeen’ contains 9 letters.

13. Four congruent isosceles trapeziums are placed so that their longer parallel sides form the diagonals of a square *PQRS*, as shown. The point *X* divides *PQ* in the ratio 3:1. What fraction of the square is shaded?

A $\frac{5}{16}$ B $\frac{3}{8}$ C $\frac{7}{16}$ D $\frac{5}{12}$ E $\frac{1}{2}$

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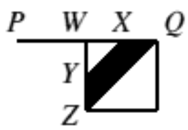
13. B The diagram shows the top-right-hand portion of the square. The shaded trapezium is labelled *QXYZ* and *W* is the point at which *ZY* produced meets *PQ*.



As *QXYZ* is an isosceles trapezium, $\angle QZY = \angle ZQX = 45°$.
Also, as *YX* is parallel to *ZQ*, $\angle XYW = \angle WXY = 45°$. So *WYX* and *WZQ* are both isosceles right-angled triangles. As $\angle ZWQ = 90°$ and *Z* is at the centre of square *PQRS*, we deduce that *W* is the midpoint of *PQ*. Hence $WX = XQ = \frac{1}{4}PQ$. So the ratio of the side-lengths of similar triangles *WYX* and *WZQ* is 1 : 2 and hence the ratio of their areas is 1 : 4.
Therefore the area of trapezium $QXYZ = \frac{3}{4} \times$ area of triangle $ZWQ = \frac{3}{32} \times$ area *PQRS* since triangle *ZWQ* is one-eighth of *PQRS*. So the fraction of the square which is shaded is $4 \times \frac{3}{32} = \frac{3}{8}$.

14. Which of the following has the greatest value?

A $\left(\frac{11}{7}\right)^3$ B $\left(\frac{5}{3}\right)^3$ C $\left(\frac{7}{4}\right)^3$ D $\left(\frac{9}{5}\right)^3$ E $\left(\frac{3}{2}\right)^3$

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14. D As all the fractions are raised to the power 3, the expression which has the largest value is that with the largest fraction in the brackets.
Each of these fractions is a little larger than $1\frac{1}{2}$. Subtracting $1\frac{1}{2}$ from each in turn, we get the fractions $\frac{1}{14}, \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, 0$, the largest of which is $\frac{3}{10}$ (because $0 < \frac{1}{14} < \frac{1}{6} < \frac{1}{4} = \frac{2\frac{1}{2}}{10} < \frac{3}{10}$). Hence $\left(\frac{9}{5}\right)^3$ is the largest.

15. I have a bag of coins. In it, one third of the coins are gold, one fifth of them are silver, two sevenths are bronze and the rest are copper. My bag can hold a maximum of 200 coins. How many coins are in my bag?

A 101 B 105 C 153 D 195 E more information is needed

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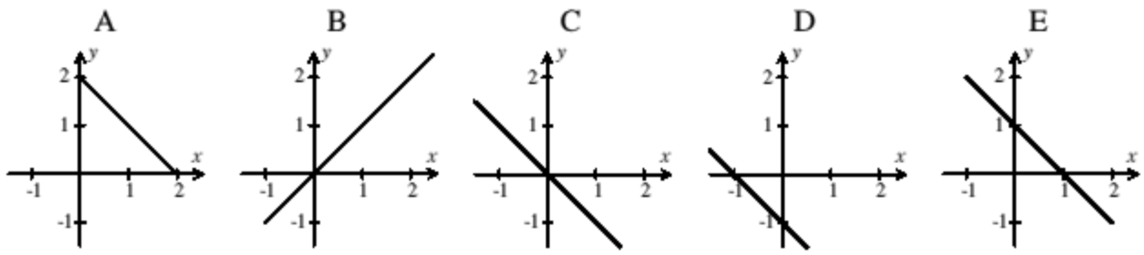
15. B From the information given, we may deduce that the number of coins is a multiple of each of 3, 5, 7. Since these are distinct primes, their lowest common multiple is $3 \times 5 \times 7 = 105$. So the number of coins in the bag is a multiple of 105. So there are 105 coins in the bag since 105 is the only positive multiple of 105 less than or equal to 200.

16. Which diagram shows the graph of $y = x$ after it has been rotated 90° clockwise about the point (1, 1)?



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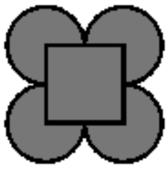
16. A The image of a straight line under a rotation is also a straight line. The centre of rotation, the point (1, 1), lies on the given line and so also lies on the image. The given line has slope 1 and so its image will have slope -1. Hence graph A shows the image.

17. The diagram shows four equal discs and a square. Each disc touches its two neighbouring discs. Each corner of the square is positioned at the centre of a disc. The side length of the square is $2/\pi$. What is the length of the perimeter of the figure?



A 3 B 4 C $\frac{3\pi}{2}$ D 6 E 2π

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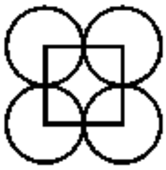
17. D The radius of each disc in the figure is equal to half the side-length of the square, i.e. $\frac{1}{\pi}$. Because the corners of a square are right-angled, the square hides exactly one quarter of each disc.



So three-quarters of the perimeter of each disc lies on the perimeter of the figure. Therefore the length of the perimeter is $4 \times \frac{3}{4} \times 2\pi \times \frac{1}{\pi} = 6$.

18. The triangle T has sides of length 6, 5, 5. The triangle U has sides of length 8, 5, 5.

What is the ratio area T : area U?

A 9 : 16 B 3 : 4 C 1 : 1 D 4 : 3 E 16 : 9

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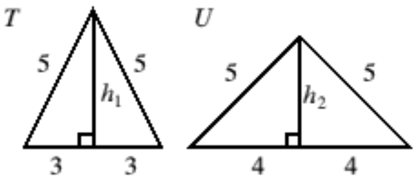
18. C The diagrams show isosceles triangles T and U. The perpendicular from the top vertex to the base divides an isosceles triangle into two congruent right-angled triangles as shown in both T and U. Evidently, by Pythagoras' Theorem, $h_1 = 4$ and $h_2 = 3$. So both triangles T and U consist of two '3, 4, 5' triangles and therefore have equal areas.



19. Which of the expressions below is equivalent to $(x \div (y \div z)) \div ((x \div y) \div z)$?

A 1 B $\frac{1}{xyz}$ C x^2 D y^2 E z^2

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19. E $(x \div (y \div z)) \div ((x \div y) \div z) = \left(x \div \frac{y}{z}\right) \div \left(\left(\frac{x}{y}\right) \div z\right) = \left(x \times \frac{z}{y}\right) \div \left(\frac{x}{y} \times \frac{1}{z}\right)$
$= \frac{xz}{y} \div \frac{x}{yz} = \frac{xz}{y} \times \frac{yz}{x} = z^2$.

20. Jack's teacher asked him to draw a triangle of area 7cm^2. Two sides are to be of length 6cm and 8cm. How many possibilities are there for the length of the third side of the triangle?

A 1 B 2 C 3 D 4 E more than 4

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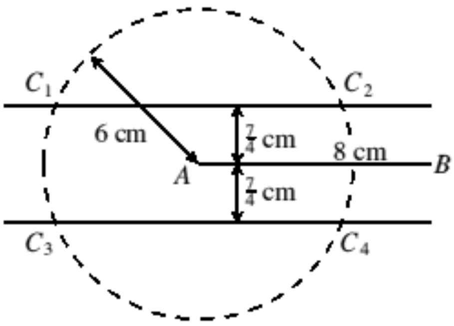
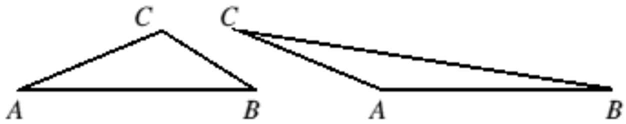
20. B Let the base AB of the triangle be the side of length 8 cm and let AC be the side of length 6 cm. So C must lie on the circle with centre A and radius 6 cm as shown. The area of the triangle is to be 7 cm^2, so the perpendicular from C to AB (or to BA produced) must be of length $\frac{7}{4}$ cm.



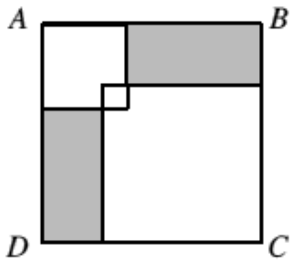
The diagram shows the four possible positions of C. However, since $\angle BAC_1 = \angle BAC_3$ and $\angle BAC_2 = \angle BAC_4$, these correspond to exactly two possibilities for the length of the third side AC. The diagrams below show the two possibilities.



21. The square $ABCD$ has an area of 196. It contains two overlapping squares; the larger of these squares has an area 4 times that of the smaller and the area of their overlap is 1. What is the total area of the shaded regions?

A 44 B 72 C 80 D 152

E more information is needed



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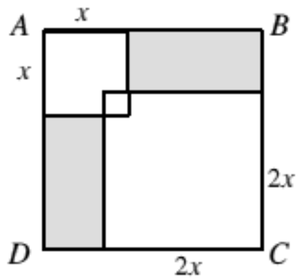
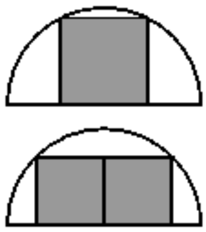
21. B The large square has area $196 = 14^2$. So it has side-length 14. The ratio of the areas of the inner squares is 4 : 1, so the ratio of their side-lengths is 2 : 1. Let the side-length of the larger inner square be $2x$, so that of the smaller is x. The figure is symmetric about the diagonal AC and so the overlap of the two inner squares is also a square which therefore has side-length 1. Thus the vertical height can be written as $x + 2x - 1$. Hence $3x - 1 = 14$ and so $x = 5$. Also, the two shaded rectangles both have side-lengths $2x - 1$ and $x - 1$; that is 9 and 4. So the total shaded area is 72.



22. The diagrams show squares placed inside two identical semicircles. In the lower diagram the two squares are identical.

What is the ratio of the areas of the two shaded regions?

A 1 : 2 B 2 : 3 C 3 : 4 D 4 : 5 E 5 : 6



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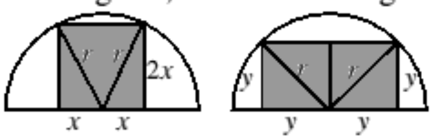
22. D Let the radius of each semicircle be r. In the left-hand diagram, let the side-length of the square be $2x$. By Pythagoras' Theorem, $x^2 + (2x)^2 = r^2$ and so $5x^2 = r^2$. So this shaded area is $4x^2 = \frac{4r^2}{5}$. In the right-hand diagram, let the side-length of each square be y. Then by Pythagoras' Theorem, $y^2 + y^2 = r^2$ and so this shaded area is r^2. Therefore the ratio of the two shaded areas is $\frac{4}{5} : 1 = 4 : 5$.



23. Four brothers are discussing the order in which they were born. Two are lying and two are telling the truth. Which two are telling the truth?

Alfred: "Bernard is the youngest."
Horatio: "Bernard is the oldest and I am the youngest."
Inigo: "I was born last."
Bernard: "I'm neither the youngest nor the oldest."

A Bernard and Inigo B Horatio and Bernard C Alfred and Horatio D Alfred and Bernard E Inigo and Horatio

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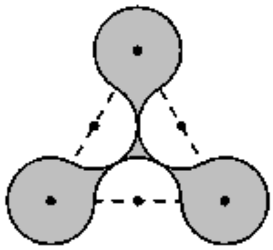
23. A If Alfred is telling the truth, the other three are lying (as their statements would then be false) and we know this is not the case. Hence Alfred is lying. Similarly, if Horatio is telling the truth, the other three are lying which again cannot be the case. So Horatio is lying. Hence the two who are telling the truth are Bernard and Inigo. (A case where this situation would be realised would be if the brothers in descending order of age were Alfred , Bernard, Horatio and Inigo.)

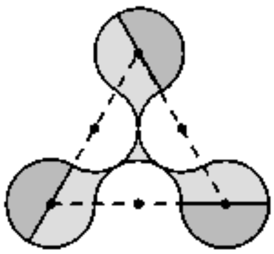
24. The diagram shows a shaded shape bounded by circular arcs with the same radius. The centres of three arcs are the vertices of an equilateral triangle; the other three centres are the midpoints of the sides of the triangle. The sides of the triangle have length 2.

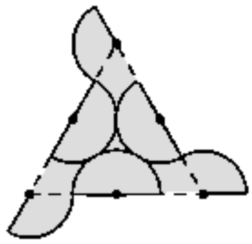
What is the difference between the area of the shaded shape and the area of the triangle?



A $\frac{\pi}{6}$ B $\frac{\pi}{4}$ C $\frac{\pi}{3}$ D $\frac{\pi}{2}$ E π

24. B The length of the side of the triangle is equal to four times the radius of the arcs. So the arcs have radius $2 \div 4 = \frac{1}{2}$. In the first diagram, three semicircles have been shaded dark grey. The second diagram shows how these semicircles may be placed inside the triangle so that the whole triangle is shaded. Therefore the difference between the area of the shaded shape and the area of the triangle is the sum of the areas of three sectors of a circle. The interior angle of an equilateral triangle is 60°, so the angle at the centre of each sector is $180° - 60° = 120°$. Therefore each sector is equal in area to one-third of the area of a circle. Their combined area is equal to the area of a circle of radius $\frac{1}{2}$. So the required area is $\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.





25. In 1984 the engineer and prolific prime-finder Harvey Dubner found the biggest known prime each of whose digits is either a one or a zero. The prime can be expressed as $\frac{10^{641} \times (10^{640} - 1)}{9} + 1$. How many digits does this prime have?

A 640 B 641 C 1280 D 1281 E 640×641

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25. D $(10^{640} - 1)$ is a 640-digit number consisting entirely of nines. So $\frac{(10^{640} - 1)}{9}$ is a 640-digit number consisting entirely of ones.

Therefore $\frac{10^{641} \times (10^{640} - 1)}{9}$ consists of 640 ones followed by 641 zeros. So $\frac{10^{641} \times (10^{640} - 1)}{9} + 1$ consists of 640 ones followed by 640 zeros followed by a single one. Therefore it has 1281 digits.