Intermediate Mathematical Challenge 2010

1. What is the value of $10 + 10 \times 10 \times (10 + 10)$?

A 21 000 B 20 100 C 2100 D 2010 E 210

1011

1. D $10 + 10 \times 10 \times (10 + 10) = 10 + 10 \times 10 \times 20 = 10 + 2000 = 2010$.

2. Three of the interior angles of a given quadrilateral are each 80°. What is the fourth angle of this quadrilateral?

A 120° B 110° C 100° D 90° E 80°

2. A The sum of the interior angles of a quadrilateral is 360°, so the fourth angle is $(360 - 3 \times 80)^\circ = 120^\circ$.

3. Exactly one of the following is a prime number. Which is it?

A 2345 B 23 456 C 234 567 D 2 345 678 E 23 456 789

3. E 2345 has units digit 5 and so is a multiple of 5; 23 456 is even; the digit sum of 234 567 is 27 so it is a multiple of 9; 2 345 678 is even. So if exactly one of the numbers is prime then it must be 23 456 789.

4. A radio advertisement claimed that using a particular brand of artificial sweetener every day would ‘save 7 000 calories in a year’. Approximately how many calories is this per day?

A 20 B 40 C 70 D 100 E 140

4. A The number of calories saved per day is $\frac{7000}{365} \approx \frac{7000}{350} = 20$.

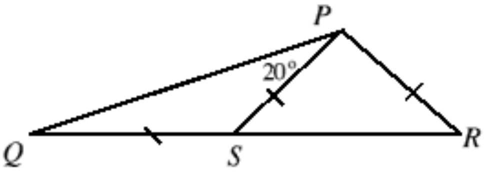
5. Which of the following has the greatest value?

A one half of $\frac{1}{25}$ B one third of $\frac{1}{20}$ C one quarter of $\frac{1}{15}$

D one fifth of $\frac{1}{10}$ E one sixth of $\frac{1}{5}$

5. E The values are A $\frac{1}{50}$, B $\frac{1}{60}$, C $\frac{1}{60}$, D $\frac{1}{50}$, E $\frac{1}{30}$.

6. In triangle PQR, S is a point on QR such that $QS = SP = PR$ and $\angle QPS = 20°$. What is the size of $\angle PRS$?



A 20° B 35° C 40° D 55° E 60°

6. C Triangle PQS is isosceles with $PS = QS$ so $\angle PQS = \angle SPQ = 20°$. Therefore $\angle PSR = 20° + 20° = 40°$ (exterior angle theorem). Triangle PSR is also isosceles, with $PS = PR$, so $\angle PRS = \angle PSR = 40°$.

7. The Three Choirs Festival is held annually. Its venue rotates in a three-year cycle among Hereford, Gloucester and Worcester. In 2009, it was held in Hereford, in 2010 it will be held in Gloucester, next year it will be held in Worcester.

Assuming that this three-year cycle continues, in which one of the following years will the Festival *not* be held in Worcester?

A 2020 B 2032 C 2047 D 2054 E 2077

7. D The Festival will next be held in Worcester in 2011. As it follows a three-year cycle, the Festival is held in Worcester when the number of the year leaves a remainder of 1 when divided by 3. So it will be held in Worcester in 2020, 2032, 2047 and 2077, but not in 2054.

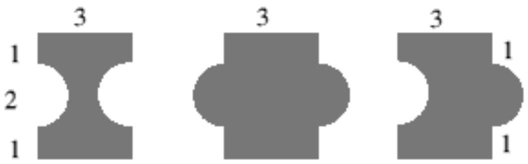
8. On my clock's display, the time has just changed to 02:31. How many minutes will it be until all the digits 0, 1, 2, 3 next appear together again?

A 1 B 41 C 50 D 60 E 61

8. B The next such display will be 03:12, that is in 41 minutes' time.

9. The perimeters of the three shapes shown are made up of straight lines and semi-circular arcs of diameter 2. They will fit snugly together as in a jigsaw.

What is the difference between the total perimeter of the three separate pieces and the perimeter of the shape formed when the three pieces fit together?



A 0 B $2 + 2\pi$ C $8 + 4\pi$ D $22 + 2\pi$ E $30 + 6\pi$

9. C The difference in perimeters is the total length of the edges which are hidden when the pieces are fitted together. These are eight straight edges of length 1 and four semicircular arcs of radius 1.
So the required difference is $8 \times 1 + 4\left(\frac{1}{2} \times 2 \times \pi \times 1\right) = 8 + 4\pi$.

10. One year in the 1990s, January 1st fell on a Monday. Eleven years later, January 1st was also a Monday. How many times did February 29th occur during those eleven years?

A 1 B 2 C 3 D 4 E 5

10. C Every year, the day of the week on which a particular date falls is one day later than it fell the previous year unless February 29th has occurred in the meantime, in which case it falls two days later. As January 1st returned to a Monday after 11 years, it must have 'moved on' 14 days during that time, so February 29th occurred three times in those 11 years.

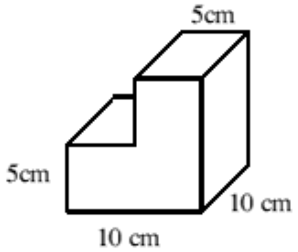
11. "You eat more than I do," said Tweedledee to Tweedledum.
"That is not true," said Tweedledum to Tweedledee.
"You are both wrong," said Alice, to them both.
"You are right," said the White Rabbit to Alice.
How many of the four statements were true?

A 0 B 1 C 2 D 3 E 4

11. B If the first statement is true, then the three other statements are all false. If the first statement is false, however, then the second statement is the only true statement. Either way, exactly one of the four statements is true.

12. A cuboid is cut away from a cube of side 10 cm as shown. By what fraction does the total surface area of the solid decrease as a result?

A $\frac{1}{4}$ B $\frac{1}{6}$ C $\frac{1}{10}$ D $\frac{1}{12}$ E $\frac{1}{18}$



12. D When the cuboid is cut away, the surface area of the solid 'loses' two rectangles measuring 10 cm × 5 cm and two squares of side 5 cm. However, it also 'gains' two rectangles measuring 10 cm × 5 cm. So the surface area decreases by an area equal to one half of the area of one of the faces of the original cube, that is one twelfth of its original surface area.

13. At Corbett's Ironmongery a fork handle and a candle cost a total of £6.10. The fork handle costs £4.60 more than the candle. What is the cost of two fork handles and four candles?

A £14.45 B £13.70 C £12.95 D £12.20 E £8.35

1023

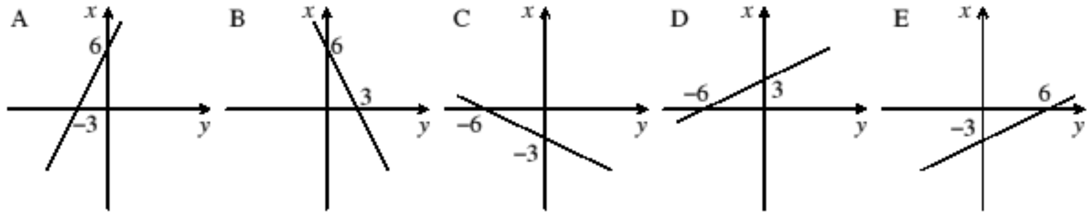
13. B Let the prices of a fork handle and a candle be £x and £y respectively.
Then $x + y = 6.1$ and $x - y = 4.6$. Adding these two equations gives $2x = 10.7$.
So a fork handle costs £5.35 and a candle costs £0.75.
Therefore the required total is £10.70 + £3.00 = £13.70.

14. Given that $4x - y = 5$, $4y - z = 7$ and $4z - x = 18$, what is the value of $x + y + z$?

A 8 B 9 C 10 D 11 E 12

14. C Adding the three equations gives $3x + 3y + 3z = 30$, so $x + y + z = 10$. (*The equations may be solved to obtain* $x = 2$, $y = 3$, $z = 5$. *However, as the above method shows, this is not necessary in order to find the value of* $x + y + z$.)

15. Bill is trying to sketch the graph of $y = 2x + 6$ but in drawing the axes he has placed the x-axis up the page and the y-axis across the page. Which of these five graphs is a correct sketch of $y = 2x + 6$ when the axes are placed in this way?



15. E The line $y = 2x + 6$ intersects the y-axis when $x = 0$ and $y = 6$. It intersects the x-axis when $x = -3$ and $y = 0$. So E is the correct line.
(*Alternatively*: $y = 2x + 6$ may be rearranged to give $x = \frac{1}{2}y - 3$. So the required line looks the same as the line $y = \frac{1}{2}x - 3$ when the axes are drawn in the traditional way.)

16. Albert Einstein is experimenting with two unusual clocks which both have 24-hour displays. One clock goes at twice the normal speed. The other clock goes backwards, but at the normal speed. Both clocks show the correct time at 13:00.
What is the correct time when the displays on the clocks next agree?

A 05:00 B 09:00 C 13:00 D 17:00 E 21:00

16. E After x hours, the first clock will have gone forward $2x$ hours and the second clock will have gone back x hours. So the next time they agree is when $2x + x = 24$, that is when $x = 8$. The correct time then is 21:00.

17. Last year Gill's cylindrical 21st birthday cake wasn't big enough to feed all her friends. This year she will double the radius and triple the height. What will be the ratio of the volume of this year's birthday cake to the volume of last year's cake?

A 12:1   B 7:1   C 6:1   D 4:1   E 3:1

17. A The volume of a cylinder of radius r and height h is $\pi r^2 h$. Replacing r by $2r$ and h by $3h$ multiplies this volume by 12.

18. Supergran walks from her chalet to the top of the mountain. She knows that if she walks at a speed of 6 mph she will arrive at 1 pm, whereas if she leaves at the same time and walks at 10 mph, she will arrive at 11 am.
At what speed should she walk if she wants to arrive at 12 noon?

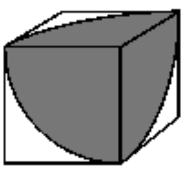
A 7.5 mph   B $7\frac{1}{7}$ mph   C 7.75 mph   D $\sqrt{60}$ mph   E 8 mph

18. A Let the distance from the chalet to the top of the mountain be x miles. Then, at 6 mph Supergran would take $\frac{x}{6}$ hours, whereas at 10 mph she would take $\frac{x}{10}$ hours. So $\frac{x}{6} - \frac{x}{10} = 2$, that is $5x - 3x = 60$, so $x = 30$. Hence Supergran's departure time is 8 am and to arrive at 12 noon she should walk at $\frac{30}{4}$ mph, that is $7\frac{1}{2}$ mph.

19. A snail is at one corner of the top face of a cube with side length 1 m. The snail can crawl at a speed of 1 m per hour. What proportion of the cube's surface is made up of points which the snail could reach within one hour?

A $\frac{\pi}{16}$ B $\frac{\pi}{8}$ C $\frac{1}{4}$ D $\frac{1}{2}$ E $\frac{\sqrt{3}}{4}$

19. B In one hour, the snail can reach points within 1 m of the corner at which it starts. So it can reach some of the points on the three faces which meet at that corner, but none of the points on the other three faces. On each of the three reachable faces, the points which the snail can reach form a quarter of a circle of radius 1 m.



So the required fraction is $\frac{3 \times \frac{1}{4}\pi \times 1 \times 1}{6 \times 1 \times 1} = \frac{\pi}{8}$.

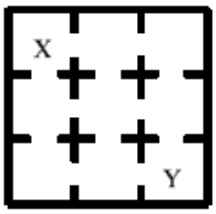
20. Shahbaz thinks of an integer, n, such that the difference between $\sqrt{n}$ and 7 is less than 1. How many different possibilities are there for n?

A 13 B 14 C 26 D 27 E 28

20. D If the difference between $\sqrt{n}$ and 7 is less than 1, then $6 < \sqrt{n} < 8$. Therefore $36 < n < 64$, so there are 27 possible values of n.

21. A square maze has 9 rooms with gaps in the walls between them. Once a person has travelled through a gap in the wall it then closes behind them. How many different ways can someone travel through the maze from X to Y?

A 8 B 10 C 12 D 14 E 16



21. E The rooms are labelled A, B, C, D, E, F, G, X, Y as shown. We look first at routes which visit no room more than once. We need consider only routes which go from X to A, since each of these routes has a corresponding route which goes from X to C. For example, the route X A D E Y corresponds to the route X C D G Y.

Routes which start X A then go to B or to D. There are three routes which start X A B, namely X A B E Y, X A B E D G Y and X A B E D C F G Y. There are also three routes which start X A D, namely X A D E Y, X A D G Y and X A D C F G Y.

The condition that a gap in a wall closes once a person has travelled through it means that it is not possible to visit a room more than once unless that room has at least four gaps leading into and out of it and the only such room is D. There are two routes which start X A and visit D twice. These are X A D G F C D E Y and X A D C F G D E Y. So there are 8 routes which start X A and there are 8 corresponding routes which start X C so there are 16 routes in all.

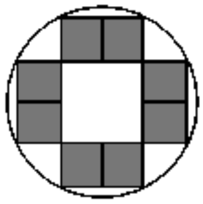
22. Curly and Larry like to have their orange squash made to the same strength. Unfortunately, Moe has put 25 ml of squash with 175 ml of water in Curly's glass and 15 ml of squash with 185 ml of water in Larry's glass. How many millilitres of the mixture in Curly's glass must be put into Larry's glass so that they end up with drinks of the same strength?

A 5 B 7 C 10 D 12 E it is not possible

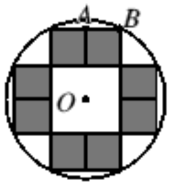
22. E Curly's drink has squash and water in the ratio 1: 7, whilst the corresponding ratio for Larry's drink is 3 : 37. This ratio is less than 1 : 7. When some of Curly's mixture is poured into Larry's, the strength will be between 1 : 7 and 3 : 37, but not equal to either.

23. The diagram shows a pattern of eight equal shaded squares inside a circle of area π square units. What is the area (in square units) of the shaded region?

A $1\frac{1}{3}$ B $1\frac{3}{5}$ C $1\frac{2}{3}$ D $1\frac{7}{9}$ E 2



23. B Let the centre of the circle be O and let A and B be corners of one of the shaded squares, as shown. As the circle has area π units², its radius is 1 unit. So OB is 1 unit long. Let the length of the side of each of the shaded squares be x units.
By Pythagoras' Theorem: $OB^2 = OA^2 + AB^2$, that is $1^2 = (2x)^2 + x^2$. So $5x^2 = 1$. Now the total shaded area is $8x^2 = 8 \times \frac{1}{5} = 1\frac{3}{5}$ units².



24. A new taxi firm needs a memorable phone number. They want a number which has a maximum of two different digits. Their phone number must start with the digit 3 and be six digits long. How many such numbers are possible?

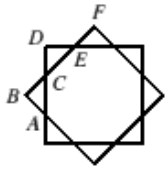
A 288 B 280 C 279 D 226 E 225

24. B There is the possibility of using only 3s giving one possible number 333333. Let's suppose a second digit is used, say x. After the initial digit 3, there are 5 positions into which we can put either 3 or x. So there are 2 choices in each of these 5 positions and so $2^5 = 32$ possible choices – except that one such choice would be five 3s. So we get 31 choices. There are 9 possible values for x, namely 0, 1, 2, 4, 5, 6, 7, 8, 9. So this gives $9 \times 31 = 279$ numbers. Together with 333333, this gives 280 numbers.

25. Two squares, each of side length $1 + \sqrt{2}$ units, overlap. The overlapping region is a regular octagon.
What is the area (in square units) of the octagon?

A $1 + \sqrt{2}$ B $1 + 2\sqrt{2}$ C $2 + \sqrt{2}$ D $2 + 2\sqrt{2}$ E $2 + 3\sqrt{2}$

25. D Let the length of the side of the regular octagon be x units and let A, B, C, D, E, F be the points shown. So $AC = CE = x$. Now $\angle ACE = 135°$ (interior angle of regular octagon), so $\angle ACB = 45°$ and hence triangle ABC is an isosceles right-angled triangle with $AB = BC$.



Also, by Pythagoras' Theorem: $AB^2 + BC^2 = AC^2 = x^2$ so $AB = BC = \frac{\sqrt{2}}{2}x$.
Similarly, $EF = \frac{\sqrt{2}}{2}x$.
Therefore $BF = \left(\frac{\sqrt{2}}{2}x + x + \frac{\sqrt{2}}{2}x\right)$ units $= x(1 + \sqrt{2})$ units.
But we are given that $BF = (1 + \sqrt{2})$ units so $x = 1$.
Now the area of the octagon formed by the overlap of the squares is equal to the area of one of these squares minus the sum of the area of four triangles, each of which is congruent to triangle CDE.
Thus, in square units, the required area is

$$(1 + \sqrt{2})^2 - 4 \times \frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = 3 + 2\sqrt{2} - 1 = 2 + 2\sqrt{2}.$$