Intermediate Mathematical Challenge 2009

1. What is the value of $1 + 2^3 + 4 \times 5$?

A 27 B 29 C 55 D 65 E 155

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1. B $1 + 2^3 + 4 \times 5 = 1 + 8 + 20 = 29$.

2. What is the sum of the first five non-prime positive integers?

A 15 B 18 C 27 D 28 E 39

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2. **D** The first five non-prime positive integers are 1, 4, 6, 8, 9.

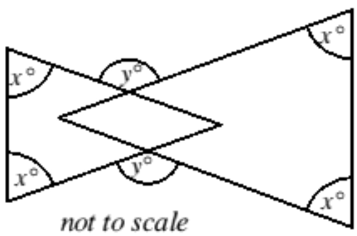
3. Which of the following has the greatest value?

A 50% of 10 B 40% of 20 C 30 % of 30 D 20% of 40 E 10% of 50

3. **C** The values of these expressions are 5, 8, 9, 8, 5 respectively.

4. The diagram shows two isosceles triangles, in which the four angles marked $x°$ are equal. The two angles marked $y°$ are also equal. Which of the following is always true?

A $y = 2x$ B $y = x + 30$ C $y = x + 60$
D $y = x + 90$ E $y = 180 - x$



not to scale

4. **A** The two acute angles in the quadrilateral in the centre of the diagram are both $(180 - 2x)°$ and the two obtuse angles are both $y°$, so $360 - 4x + 2y = 360$. So $y = 2x$.

5. The square of a positive number is twice as big as the cube of that number. What is the number?

A 8 B 4 C 2 D $\frac{1}{2}$ E $\frac{1}{4}$

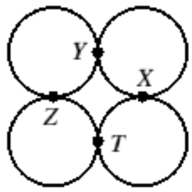
5. **D** Let the number be x. Then $x^2 = 2x^3$, that is $x^2(1 - 2x) = 0$. So $x = 0$ or $x = \frac{1}{2}$. However, x is positive, so the only solution is $x = \frac{1}{2}$.

6. Which of the following is half way between $\frac{4}{5}$ and $-\frac{2}{3}$?

A $\frac{1}{15}$ B $\frac{7}{30}$ C $\frac{7}{15}$ D $\frac{17}{30}$ E $\frac{3}{4}$

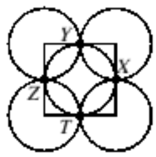
6. **A** $\frac{4}{5} = \frac{12}{15}$ and $-\frac{2}{3} = -\frac{10}{15}$, so the number half way between these is $\frac{1}{2}\left(\frac{-10}{15} + \frac{12}{15}\right)$, that is $\frac{1}{15}$.

7. Four touching circles all have radius 1 and their centres are at the corners of a square.
What is the radius of the circle through the points of contact X, Y, Z and T?

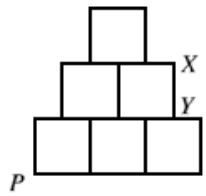


A $\frac{1}{2}$ B $\frac{1}{2}\sqrt{2}$ C 1 D $\sqrt{2}$ E 2

7. **C** As can be seen from the diagram, the square whose vertices are the centres of the original four circles has side of length 2 units and this distance is equal to the diameter of the circle through X, Y, Z and T.



8. The diagram shows a figure made from six equal, touching squares arranged with a vertical line of symmetry. A straight line is drawn through the bottom corner P in such a way that the area of the figure is halved. Where will the cut cross the edge XY?



A at X B one quarter the way down XY

C half way down XY D three-quarters the way down XY

E at Y

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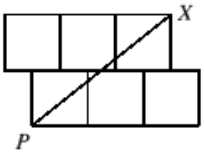
8. A The small square on top will be in the upper half of the divided figure. Now consider the figure formed by moving this square to become an extra square on the left of the second row, as shown. It may now be seen from the symmetry of the figure that the line PX splits the new figure in half – with that small square in the upper half. So the line PX does the same for the original figure.



9. Joseph's flock has 55% more sheep than goats. What is the ratio of goats to sheep in the flock?

A 9:11 B 20:31 C 11:20 D 5:9 E 9:20

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9. B The ratio of goats to sheep is 100:155 = 20:31.

10. Fussy Fiona wants to buy a new house but she doesn't like house numbers that are divisible by 3 or by 5. If all the houses numbered between 100 and 150 inclusive are for sale, how many houses can she choose from?

A 24 B 25 C 26 D 27 E 28

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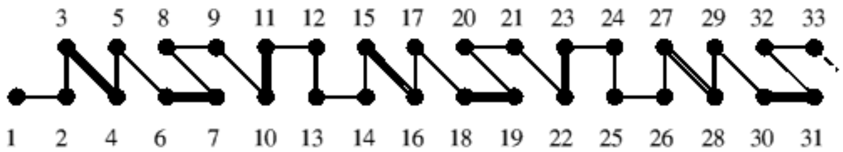
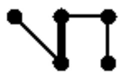
10. D There are 51 houses numbered from 100 to 150 inclusive. Of these, 17 are multiples of 3, 11 are multiples of 5 and 4 are multiples of both 3 and 5. So the number of houses Fiona can choose from is 51 − (17 + 11 − 4) = 27.

11. The diagram below shows a pattern which repeats every 12 dots.



Which of the following does the piece between 2007 and 2011 look like?

A  B  C  D  E 

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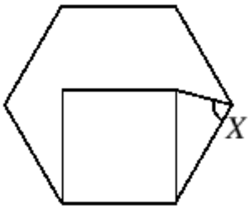
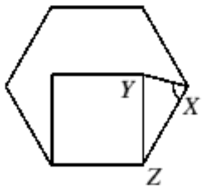
11. E Note that 2004 is a multiple of 3 (since its digit sum is a multiple of 3) and also a multiple of 4 (since its last two digits form a multiple of 4). So 2004 is a multiple of 12 and hence the part of the pattern between 2007 and 2011 is the same as the part of the pattern between 3 and 7.

12. The diagram shows a square inside a regular hexagon. What is the size of the marked angle at X?

A $45°$ B $50°$ C $60°$ D $75°$ E $80°$

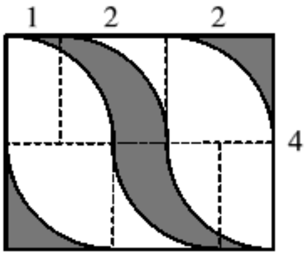


12. D Let Y and Z be the points shown. The interior angle of a regular hexagon is $120°$, so $\angle XZY = 120° - 90° = 30°$. The side of the square has the same length as the side of the regular hexagon, so $YZ = XZ$. Hence triangle XYZ is isosceles and $\angle ZXY = \angle ZYX = \frac{1}{2}(180° - 30°) = 75°$.

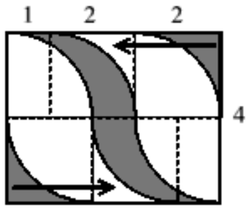


13. The diagram on the right shows a rectangle with sides of length
5 cm and 4 cm. All the arcs are quarter-circles of radius 2 cm.
What is the total shaded area in cm^2?

A $12 - 2\pi$ B 8 C $8 + 2\pi$
D 10 E $20 - 4\pi$



13. A If the shaded regions in the top-right and bottom-left corners of the diagram are moved as shown, the area of the shaded region in both the top half and bottom half of the diagram is now that of a 3×2 rectangle which has a quarter of a circle of radius 2 removed from it.
So the total shaded area is
$2\left(3 \times 2 - \frac{1}{4} \times \pi \times 2^2\right) \text{cm}^2 = (12 - 2\pi)\ \text{cm}^2$.



14. Catherine's computer correctly calculates $\frac{66^{66}}{2}$. What is the units digit of its answer?

A 1 B 2 C 3 D 6 E 8

14. E If n is a positive integer then the units digit of 66^n is 6. So when a power of 66 is divided by 2, the units digit of the quotient is either 3 or 8. Now 66^{66} is clearly a multiple of 4, so $\frac{1}{2}(66^{66})$ is even and therefore has units digit 8 rather than 3.

15. What is the value of $\frac{1}{x+2}$, given that $\frac{1}{x} = 3.5$?

A $\frac{7}{9}$ B $\frac{7}{16}$ C $\frac{9}{7}$ D $\frac{7}{4}$ E $\frac{16}{7}$

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15. B As $\frac{1}{x} = 3.5 = \frac{7}{2}, x = \frac{2}{7}$. So $x + 2 = \frac{16}{7}$. Hence $\frac{1}{x+2} = \frac{7}{16}$.

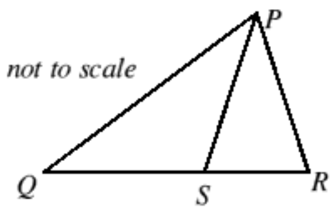
16. How many different positive integers n are there for which n and $n^3 + 3$ are both prime numbers?

A 0 B 1 C 2 D 3 E infinitely many

16. B If n is an odd prime, then $n^3 + 3$ is an even number greater than 3 and therefore not prime. The only even prime is 2 (which some would say makes it very odd!) and when $n = 2$, $n^3 + 3 = 11$ which is also prime. So there is exactly one value of n for which n and $n^3 + 3$ are both prime.

17. PQR is a triangle and S is a point on QR.
$QP = QR = 9\text{cm}$ and $PR = PS = 6\text{cm}$.
What is the length of SR?

A 1cm B 2cm C 3cm D 4cm E 5cm



17. D Triangles PRS and QPR are similar because: $\angle PSR = \angle QRP$ (since $PR = PS$) and $\angle PRS = \angle QPR$ (since $QP = QR$). Hence $\frac{SR}{RP} = \frac{RP}{PQ}$, that is $\frac{SR}{6} = \frac{6}{9}$, that is $SR = 4$.

18. If p, q are distinct primes less than 7, what is the largest possible value of the highest common factor of $2p^2q$ and $3pq^2$?

A 60 B 45 C 36 D 20 E 15

18. B For all positive integer values of p and q, $2p^2q$ and $3pq^2$ have a common factor of pq. They will also have an additional common factor of 2 if $q = 2$ and an additional common factor of 3 if $p = 3$. As the values of p and q are to be chosen from 2, 3 and 5, the largest possible value of the highest common factor will occur when $p = 3$ and $q = 5$. For these values of p and q, $2p^2q$ and $3pq^2$ have values 90 and 225 respectively, giving a highest common factor of 45.

19. Driving to Birmingham airport, Mary cruised at 55 miles per hour for the first two hours and then flew along at 70 miles per hour for the remainder of the journey. Her average speed for the entire journey was 60 miles per hour. How long did Mary's journey to Birmingham Airport take?

A 6 hours B $4\frac{1}{2}$ hours C 4 hours D $3\frac{1}{2}$ hours E 3 hours

19. E Let the time for which Mary drove at 70 mph be t hours. Then the total distance covered was $(55 \times 2 + 70 \times t)$ miles. Also, as her average speed over $(2 + t)$ hours was 60 mph, the total distance travelled was $60(2 + t)$ miles.

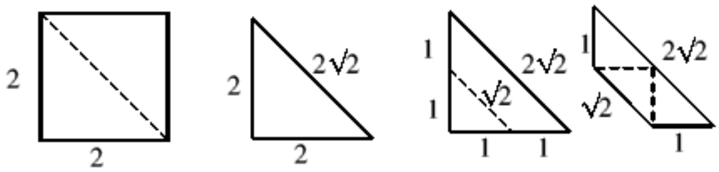
Therefore $110 + 70t = 120 + 60t$, that is $10t = 10$, that is $t = 1$.

So, in total, Mary's journey took 3 hours.

20. A square, of side two units, is folded in half to form a triangle. A second fold is made, parallel to the first, so that the apex of this triangle folds onto a point on its base, thereby forming an isosceles trapezium. What is the perimeter of this trapezium?

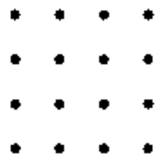
A $4 + \sqrt{2}$ B $4 + 2\sqrt{2}$ C $3 + 2\sqrt{2}$ D $2 + 3\sqrt{2}$ E 5

20. D As can be seen from the figures below, the perimeter of the trapezium is $2 + 3\sqrt{2}$.



21. There are lots of ways of choosing three dots from this 4 by 4 array. How many triples of points are there where all three lie on a straight line (not necessarily equally spaced)?

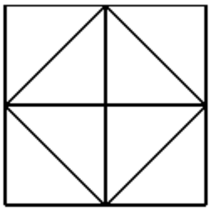
A 8 B 16 C 20 D 40 E 44



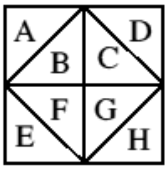
21. E Consider the top row of four dots. One can obtain a triple of dots by eliminating any one of the four – so there are four such triples. The same is true for each of the four rows, each of the four columns and the two main diagonals, giving 40 triples. In addition there are four diagonal lines consisting of exactly three dots, so there are 44 triples in total.

22. A square is divided into eight congruent triangles, as shown. Two of these triangles are selected at random and shaded black.
What is the probability that the resulting figure has at least one axis of symmetry?

A $\frac{1}{4}$ B $\frac{4}{7}$ C $\frac{1}{2}$ D $\frac{5}{7}$ E 1

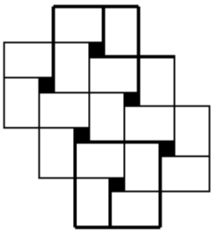


22. D If the first triangle selected to be shaded is a corner triangle, then the final figure will have at least one axis of symmetry provided that the second triangle selected is one of five triangles. For example, if A is chosen first then there will be at least one axis of symmetry in the final figure if the second triangle selected is B, D, E, G or H. The same applies if an inner triangle is selected first: for example, if B is chosen first then there will be at least one axis of symmetry in the final figure if the second triangle selected is A, C, F, G or H.
So, irrespective of which triangle is selected first, the probability that the final figure has at least one axis of symmetry is $\frac{5}{7}$.



23. The diagram shows part of a tiling pattern which is made from two types of individual tiles: 8 by 6 rectangular white tiles and square black tiles. If the pattern is extended to cover an infinite plane, what fraction is coloured black?

A $\frac{1}{12}$ B $\frac{1}{13}$ C $\frac{1}{25}$ D $\frac{1}{37}$ E $\frac{1}{40}$



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23. C Firstly, note that the black squares have side 2 units. The pattern may be considered to be a tessellation of the shape shown on the right. So the ratio of squares to rectangles is 1:2 and hence the fraction coloured black is $\frac{4}{4 + 2 \times 48} = \frac{4}{100} = \frac{1}{25}$.

24. What is the largest number of the following statements that can be true at the same time?

$0 < x^2 < 1$, $x^2 > 1$, $-1 < x < 0$, $0 < x < 1$, $0 < x - x^2 < 1$

A 1 B 2 C 3 D 4 E 5

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24. C Reading from the left, we number the statements I, II, III, IV and V.
Statement I is true if and only if $-1 < x < 1$; statement II is true if $x > 1$ or if $x < -1$.
By considering the graph of $y = x - x^2$, which intersects the x-axis at (0, 0) and (1, 0) and has a maximum at $\left(\frac{1}{2}, \frac{1}{4}\right)$, it may be seen that statement V is true if and only if $0 < x < 1$.
We see from the table below that a maximum of three statements may be true at any one time.

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
True statement(s)	II	none	I, III	none	I, IV, V	none	II

25. One coin among N identical-looking coins is a fake and is slightly heavier than the others, which all have the same weight. To compare two groups of coins you are allowed to use a set of scales with two pans which balance exactly when the weight in each pan is the same. What is the largest value of N for which the fake coin can be identified using a maximum of two such comparisons?

A 4 B 6 C 7 D 8 E 9

25. E As it is known that the fake coin is heavier than all of the others, it is possible in one comparison to identify which, if any, is the fake in a group of three coins: simply compare any two of the three coins – if they do not balance then the heavier coin is the fake, whereas if they do balance then the third coin is the fake. This means that it is possible to find the fake coin when $N = 9$ using two comparisons: the coins are divided into three groups of three and, using the same reasoning as for three individual coins, the first comparison identifies which group of three coins contains the fake. The second comparison then identifies which of these three coins is the fake. However, it is not possible to identify the fake coin in a group of four coins in one comparison only, so it is not always possible to identify the fake coin using two comparisons when $N = 10$. If less than four are put on each side for the first comparison and they balance, then there are more than three left and the fake coin amongst these cannot be identified in one further comparison. Alternatively, if more than three are put on each side for the first comparison and they do not balance, then the fake coin in the heavier group cannot be identified in one further comparison.