Intermediate Mathematical Challenge 2004

1. What is the value of 4002 – 2004?

A 2004 B 2002 C 2000 D 1998 E 1996

1. D 4004 – 2004 = 2000, so 4002 – 2004 = 2000 – 2 = 1998.

2. You are told that 30 pupils have 25 different birthdays between them. What is the largest number of these pupils who could share the same birthday?

A 2 B 3 C 4 D 5 E 6

0412

2. **E** There must be 25 pupils who all have different birthdays. If the remaining 5 pupils all have the same birthday as one of these pupils, then 6 pupils will share the same birthday.

3. Four of these numbers can make two pairs so that each pair adds up to 98 765. Which number is the odd one out?

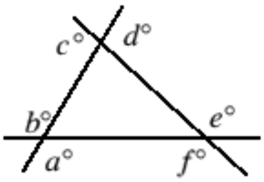
A 37 373 B 45 678 C 53 087 D 61 392 E 70 082

0413

3. **E** 37 373 + 61 392 = 98 765 and 45 678 + 53 087 = 98 765.

4. What is the value of $a + b + c + d + e + f$?

A 360 B 540 C 720 D 900
E it depends on the triangle



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4. **C** The angles marked $a°$, $c°$ and $e°$ may be considered to be the exterior angles of the triangle and therefore have a total of 360°. As $b° = a°$, $d° = c°$ and $f° = e°$ (all pairs of vertically opposite angles), $b + d + f = 360$. So $a + b + c + d + e + f = 720$.

5. The sum of two numbers is 2. The difference between them is 4. What is their product?

A -8 B -3 C 0 D 3 E 8

0415

5. B Let the numbers be x and y. Then $x + y = 2$; $x - y = 4$. Adding these equations gives $2x = 6$, so $x = 3$. Hence $y = 2 - 3 = -1$ so the numbers are 3 and -1.

6. In Niatirb they use Cibara numerals. These are the same shape as normal Arabic numerals, but with the meanings in the opposite order. So "0" means "nine", "1" means "eight" and so on. But they write their numbers from left to right and use arithmetic symbols just as we do. So, for example, they use 62 for the number we write as 37.

How do the inhabitants of Niatirb write the answer to the sum which they write as $837 + 742$?

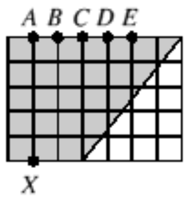
A 419 B 580 C 1579 D 5317 E 8420

0416

6. **B** The sum is what we would write as 162 + 257 and this equals 419. However, in Niatirb it would be written 580.

7. Which of the following straight lines cuts the shaded area in half?

A *XA* B *XB* C *XC* D *XD* E *XE*



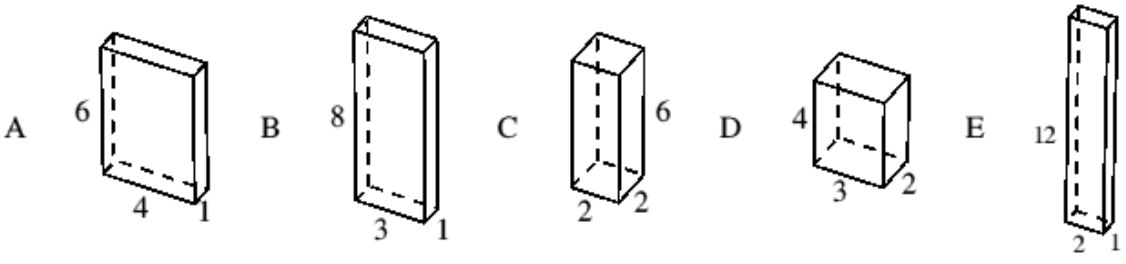
7. **D** The shaded area is a trapezium of area $\frac{1}{2}(3 + 7) \times 5 = 25$. Line *XD* forms one side of a trapezium of area 12.5, since $\frac{1}{2}(1 + 4) \times 5 = 12.5$.

8. In March 2003 Welshman Tony Evans dropped a ball from an aircraft a mile above the Mojave desert to see if it would bounce. The ball was made from 6 million rubber bands, had a circumference of 14 ft 8 in, weighed 2600 pounds and took Mr Evans five years to build. On average, roughly how many rubber bands did he add each day whilst building the ball?

A 3 B 33 C 330 D 3300 E 33 000

8. D The average number of rubber bands added each day was approximately $\frac{6\,000\,000}{5 \times 365} \approx \frac{6\,000\,000}{1800} \approx 3300.$

9. The cuboids below all have the same volume. Which of them has the greatest surface area?



9. E The surface areas of the cuboids are: A 68; B 70; C 56; D 52; E 76.

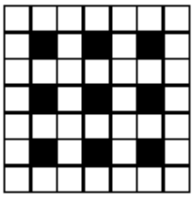
10. What is the mean of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$?

A $\frac{1}{5}$ B $\frac{1}{15}$ C $\frac{5}{12}$ D $\frac{7}{24}$ E $\frac{5}{16}$

0420

10. E The four fractions total $\frac{5}{4}$, so their mean = $\frac{5}{4} \div 4 = \frac{5}{16}$.

11. The diagram shows a square board in which strips of white squares alternate with strips of black and white squares. A larger board, constructed in the same way, has 49 black squares. How many white squares are there on the larger board?

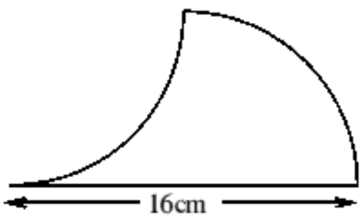


A 176 B 196 C 245 D 289 E 392

0421

11. A The 49 black squares will be in a 7×7 formation, so the board will measure 15×15 squares. Hence the number of white squares $= 225 - 49 = 176$.

12. This figure is made from a straight line 16 cm long and two quarter circles, one with its centre at the midpoint of the straight line. What is the area of the figure (in cm^2)?



A 64 B 16π C $32 + 16\pi$ D 32π E $16 + 8\pi$

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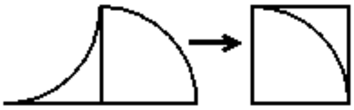
12. A As the diagram shows, the figure may be cut into two parts which fit together to form a square measuring 8cm $\times$ 8cm.



13. Four of these points lie on a single straight line. Which is the odd one out?

A $(-3, -3)$ B $(-2, -1)$ C $(2, 5)$ D $(4, 11)$ E $(5, 13)$

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13. C Points $(-3, -3)$, $(-2, -1)$, $(4, 11)$ and $(5, 13)$ all lie on the line whose equation is $y = 2x + 3$, but $(2, 5)$ does not lie on this line.

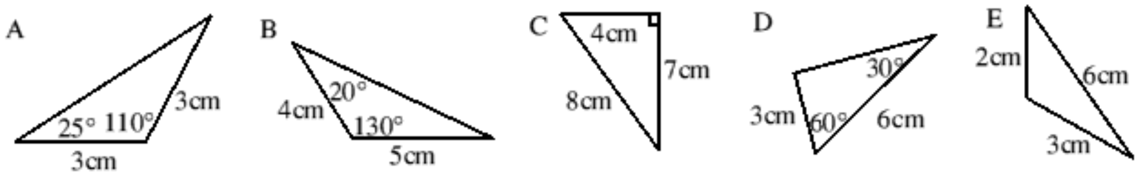
14. In this addition sum, each letter represents a different non-zero digit. What is the value of $a + w + a + y$?

$$\begin{array}{r} f\ l\ y \\ +\ f\ l\ y \\ +\ f\ l\ y \\ \hline a\ w\ a\ y \\ \hline \end{array}$$

A 13   B 15   C 16   D 17   E 18

14. B We note first that $y = 5$ since that is the only non-zero digit which, when it is multiplied by 3, has itself as the units digit. So there is a carry of 1 into the tens column. We note also that $a = 1$ or $a = 2$ as "fly" < 1000 and therefore $3 \times$ "fly" < 3000. We now need $3 \times l + 1$ to end in either 1 or 2 and the only possibility is $l = 7$, giving $a = 2$ with a carry of 2 into the hundreds column. As $a = 2$, f must be at least 6. However, if $f = 6$ then $w = 0$ which is not allowed. Also the letters represent different digits, so $f \neq 7$ and we can also deduce that $f \neq 9$ since $f = 9$ would make $w = 9$.
Hence $f = 8$, making $w = 6$ and the letters represent $875 \times 3 = 2625$.

15. Only one of these triangles can actually be made. Which is it?



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15. D As triangle A has two equal sides, it should have two equal angles, but its angles are 25°, 110° and 45° so it is impossible to make. In a triangle, the smallest angle lies opposite the shortest side which makes B impossible since 20° is the smallest angle, but 5 cm is not the shortest side. Triangle C is impossible as $4^2 + 7^2 \neq 8^2$, so it does not obey Pythagoras' Theorem. The longest side of a triangle must be shorter than the sum of the other two sides, but this is not the case in triangle E, so it cannot be made either. Triangle D, however, is obtained by cutting an equilateral triangle of side 6cm in half along an axis of symmetry and so can certainly be made.

16. If the pattern shown is continued, what number will appear directly below 400?

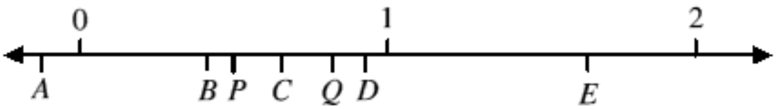
A 438 B 439 C 440 D 441 E 442

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16

0426

16. C Note that the number at the end of the n th row is n^2, so 400 will lie at the end of the 20 th row. The row below will end in 21^2, i.e. 441, so the number directly below 400 will be 440.

17.



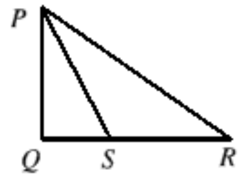
A, B, C, D, E, P and Q are points on the number line as shown. One of the points represents the product of the numbers represented by P and Q. Which is it?

A B C D E

0427

17. B As P and Q both lie between 0 and 1, their product will be greater than 0 but smaller than P and smaller than Q. Of the options available, only B satisfies these conditions. (Furthermore, its position is correct since P is approximately equal to $\frac{1}{2}$, which means that the product of P and Q lies approximately half way between 0 and Q.)

18. In the triangle PQR, there is a right angle at Q and angle QPR is 60°. The bisector of the angle QPR meets QR at S, as shown.
What is the ratio $QS : SR$?



A 1:1 B $1:\sqrt{2}$ C $1:(3-\sqrt{3})$ D $1:\sqrt{3}$ E 1:2

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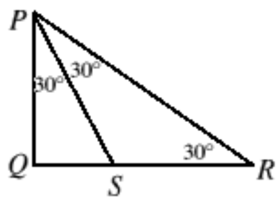
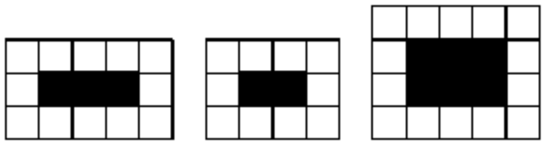
18. E Angle PRS = 30°, so triangle PRS is isosceles with $SP = SR$.
Hence $\frac{QS}{SR} = \frac{QS}{SP} = \frac{1}{2}$ as PQS is half of an equilateral triangle.
[Alternatively, we can use the angle bisector theorem: $\frac{QS}{SR} = \frac{PQ}{PR} = \frac{1}{2}$ as PQR is also half of an equilateral triangle.]



19. Three rectangular-shaped holes have been drilled passing all the way through a solid $3 \times 4 \times 5$ cuboid. The diagrams show the front, side and top views of the resulting block. What fraction of the original cuboid remains?



A $\frac{13}{30}$ B $\frac{7}{15}$ C $\frac{1}{2}$ D $\frac{8}{15}$ E $\frac{17}{30}$

19. D Consider the cuboid to be made up of 60 unit cubes. The front and side views show that top and bottom layers consist of the same number of cubes and from the top view we see that this number is 14. The front and side views indicate that only the 4 corner cubes remain in the middle layer, so the total number of cubes remaining is $2 \times 14 + 4 = 32$. The required fraction, therefore, is $\frac{32}{60} = \frac{8}{15}$.

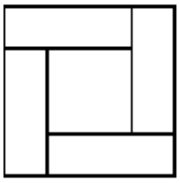
20. What is the largest power of 2 that divides $127^2 - 1$?

A 2^1 B 2^7 C 2^8 D 2^{63} E 2^{127}

0430

20. C Using the formula for the difference of two squares:
$127^2 - 1 = 127^2 - 1^2 =$
$(127 + 1)(127 - 1) = 128 \times 126 = 2^7 \times 2 \times 63 = 2^8 \times 3^2 \times 7.$

21. A square is divided into four congruent rectangles and a smaller square, as shown. (The diagram is not to scale.) The area of the small square is $\frac{1}{4}$ of the area of the whole square. What is the ratio of the length of a short side of one of the rectangles to the length of a long side?



A $1:\sqrt{2}$ B $1:\sqrt{3}$ C 1:2 D 1:3 E 1:4

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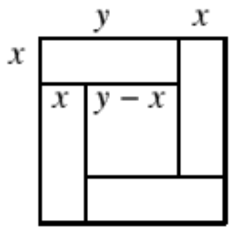
21. D Let the length of a short side of a rectangle be x and the length of a long side be y. Then the whole square has side of length $(y + x)$, whilst the small square has side of length $(y - x)$. As the area of the whole square is four times the area of the small square, the length of the side of the whole square is twice the length of the side of the small square. Therefore $y + x = 2(y - x)$ i.e. $y = 3x$ so $x : y = 1 : 3$.



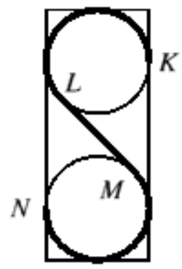
22. In a maths exam with N questions, you score m marks for a correct answer to each of the first q questions and $m + 2$ marks for a correct answer to each of the remaining questions. What is the maximum possible score?

A $(m+2)N - 2q$ B Nm C $mq + (m+2)q$ D $N(m + 1)$ E $Nm + q(m+2)$

0432 ©UKMT

22. A If you answer all questions correctly, you receive m marks for each of the N questions plus an extra 2 marks for the last $(N - q)$ questions. So the maximum possible score $= m \times N + 2 \times (N - q) = mN + 2N - 2q = (m+2)N - 2q$.

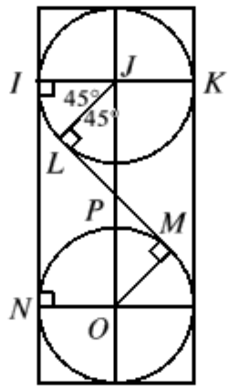
23. In the diagram, the letter S is made from two arcs, KL and MN, which are each five-eighths of the circumference of a circle of radius 1, and the line segment LM, which is tangent to both circles. At points K and N, common tangents to the two circles touch one of the circles. What is the length LM?



A $\frac{3}{2}$ B $3 - \sqrt{2}$ C 2 D $\frac{3\sqrt{2}}{2}$ E $1 + \sqrt{2}$

0433 ©UKMT

23. C Let the centres of the circles be J and O and let NI be the common tangent shown. Let P be the point of intersection of JO and LM. As arc KL is 5/8 of the circumference of the top circle, $\angle IJL = 45°$. Consider quadrilateral $IJON$: sides IJ and NO are radii, so they are both of unit length and they are both perpendicular to tangent NI. So $IJON$ is a rectangle. Hence $\angle IJO = 90°$ and $\angle LJP = 90° - 45° = 45°$. In triangle JLP, $\angle JPL = 180° - (90° + 45°) = 45°$. So triangle JLP is isosceles and $LP = LJ = 1$ unit.



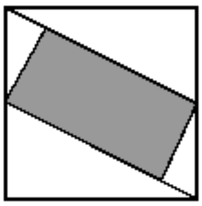
By a similar argument, it may be shown that PM is also of length 1 unit, so $LM = LP + PM = 2$ units.

24. If p, q and $p - q$ are all positive integers, which of the following is least?

A $\frac{q^2}{p^2}$ B $\frac{p^2}{q^2}$ C $\frac{q}{p}$ D $\sqrt{\frac{q}{p}}$ E $\sqrt{\frac{p}{q}}$

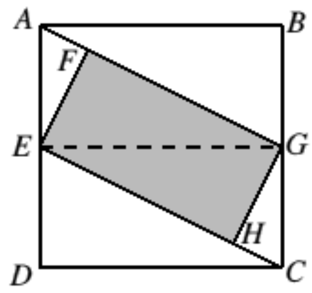
24. A We may deduce that $p > q$, so, as p and q are both positive, $\frac{p^2}{q^2} > \frac{\sqrt{p}}{\sqrt{q}} > 1$. We may also deduce that $0 < \frac{q^2}{p^2} < \frac{q}{p} < \frac{\sqrt{q}}{\sqrt{p}} < 1$ and hence $\frac{q^2}{p^2}$ is the least of the five numbers.

25. The diagram shows a square with two lines from a corner to the middle of an opposite side. The rectangle fits exactly inside these two lines and the square itself. What fraction of the square is occupied by the shaded rectangle?



A $\frac{1}{3}$ B $\frac{2}{5}$ C $\frac{3}{10}$ D $\frac{1}{2}$ E $\frac{3}{8}$

25. B Triangles ABG, GFE and EFA are similar, so $AF : FE = EF : FG = GB : BA = 1 : 2$. Thus if $AF = a$, then $FE = 2a$, $FG = 4a$, and the shaded area is $8a^2$. By Pythagoras' Theorem, $AE = \sqrt{5}a$, so $AD = 2\sqrt{5}a$ and the area of the square is $20a^2$. Thus the required fraction is $8/20$, which is $2/5$.



[Alternatively, as we have shown that $FG = 4AF$, we can divide parallelogram $AGCE$ into 10 congruent triangles, 8 of which make up rectangle $EFGH$. So the area of the rectangle is 4/5 of the area of the parallelogram, which in turn is half the area of square $ABCD$.]