Further International Selection Test, 1974

Time 3 hours

No tables and no lists of formulae are allowed

1. (i) In ΔABC $\sin^2 A + \sin^2 B + \sin^2 C = 2$. Prove that ΔABC is right-angled. (Hungary, modified)

(ii) O is the circumcentre and G the centroid of ΔABC. Prove $9OG^2 = R^2 (1 - 8\cos A \cos B \cos C)$. (Bulgaria, modified)

2. A polygonal line is a continuous line $A_1 A_2 A_3 \ldots A_{n+1}$, where, for $r = 1$ to n, $A_r A_{r+1}$ is a straight line segment.

In a square of side 50, a polygonal line L is constructed in such a way that the distance of any point inside the square from L (i.e. from the nearest point of L) is less than 1. Prove that the length of L is greater than 1248. (USSR)

3. A circular hoop of radius 1 is placed in the corner of the room. (The corner consists of a horizontal floor and two perpendicular vertical walls and the hoop touches all three planes.) Find the locus of the centre of the hoop. (France)

4. EITHER

(a) Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression. (USA)

OR

(b) $x + y + z = 3$, $x^3 + y^3 + z^3 = 15$, and $x^4 + y^4 + z^4 = 35$.

Being given that $x^2 + y^2 + z^2$ is less than 10, find $x^5 + y^5 + z^5$. (GDR, modified)