# X204/13/01

NATIONAL QUALIFICATIONS 2012

FRIDAY, 25 MAY
1.00 PM – 4.00 PM

# APPLIED MATHEMATICS
## ADVANCED HIGHER
### Mechanics

**Read carefully**

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

## Section A (Mechanics 1 and 2)

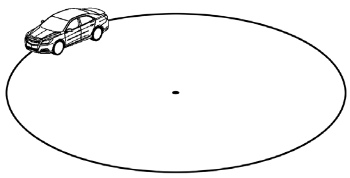
*Marks*

**Answer all the questions**

**Candidates should observe that $g$ m s$^{-2}$ denotes the magnitude of the acceleration due to gravity.**

**Where appropriate, take its magnitude to be 9·8 m s$^{-2}$.**

**A1.** A car travels at a uniform speed of 80 km h$^{-1}$ on a horizontal circular track of radius 150 metres without slipping. Calculate the coefficient of friction between the tyres and the track. **3**



**A2.** The greatest height reached by a projectile is one tenth of its range on horizontal ground. Calculate the angle of projection. **5**

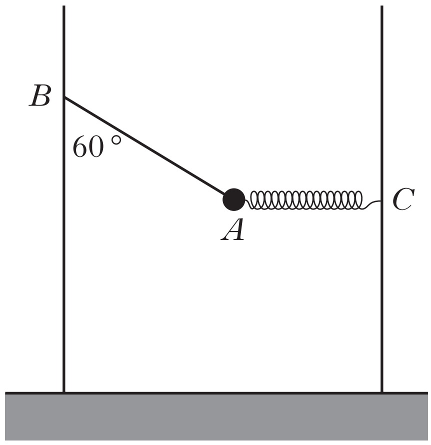
**A3.** A sprinter competes in a 100 metre race along a straight track.

He starts from rest and for the first 4 seconds he has speed $\left(\frac{13}{2}t - t^2\right)$ m s$^{-1}$. For the the next 6 seconds he maintains a constant speed of 10 m s$^{-1}$ before decelerating at 0·4 m s$^{-2}$ for the remainder of the race.

Calculate the total time taken by the sprinter to complete the race. **6**

*Marks*

**A4.** A particle of mass 1 kg is held in equilibrium by a light, inextensible string $AB$ and a light spring $AC$. $AC$ is horizontal and $AB$ is inclined at $60°$ to the vertical, as shown in the diagram.




($a$) Show that the tension in the string $AB$ is $2g$ newtons and calculate the tension in the spring $AC$. **3**

($b$) The spring has modulus of elasticity 40 newtons and natural length 10 centimetres. Calculate the distance $AC$. **2**

**A5.** As a train leaves a station, it climbs a hill inclined at an angle $\theta$ to the horizontal where $\sin\theta = 0{\cdot}1$. The top of the hill is 100 metres above the horizontal level of the bottom of the hill.

The train's engine exerts a constant force of 120 kN and the coefficient of friction between the train and the tracks is 0·2.

The speed of the train at the bottom of the hill is $4\,\mathrm{m\,s^{-1}}$. Given that at the top of the hill the speed of the train has increased to $10\,\mathrm{m\,s^{-1}}$, calculate the mass of the train. **6**

**A6.** An International Space Station is held in a circular orbit by the Earth's gravitational field and travels at a height of 390 km above the surface of the Earth.

Given that the radius of the Earth is 6380 km, calculate:

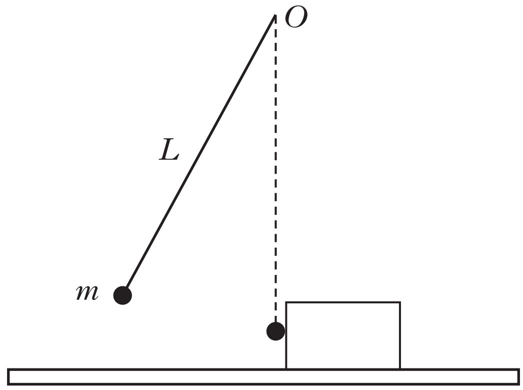
($a$) the speed of the satellite in its orbit; **4**

($b$) the number of times the satellite orbits the Earth in one day. **3**

You should assume that Newton's Inverse Square Law of Gravitation applies.

**[Turn over**

**A7.** A ball of mass $m$ kg is suspended from a fixed point $O$ by a light, inextensible string of length $L$ metres. The ball is released when the string makes an angle of 45° to the downward vertical and strikes a stationary block of mass $M$ kg horizontally. The block rests on a smooth horizontal surface, as shown in the diagram.




When the ball hits the block, its speed is $u$ m s$^{-1}$. The ball then rebounds with speed $v$ m s$^{-1}$ and comes to rest when the string makes an angle of 30° to the vertical. Show that

$$u = \sqrt{gL\left(2 - \sqrt{2}\right)} \qquad \text{and} \qquad v = \sqrt{gL\left(2 - \sqrt{3}\right)}.$$ **4**

The block moves to the right with an initial speed $V$ m s$^{-1}$. Show that

$$V = k\frac{m}{M}\sqrt{gL}$$

and state the value of $k$. **3**

**A8.** A particle $P$ is projected so that its position vector is given by $(t^2 + 3)\mathbf{i} + 4t\mathbf{j}$. The time is measured in seconds, distances are measured in metres and $\mathbf{i}$, $\mathbf{j}$ are the unit vectors in the directions of rectangular axes $Ox$ and $Oy$ respectively. A second particle $Q$ has the same acceleration as the particle $P$ and, at time $t = 0$, the particle $Q$ has velocity $(-4\mathbf{i} + \mathbf{j})$ and position vector $8\mathbf{j}$.

Find:

($a$) an expression, in terms of $t$, for the position vector of $Q$; **3**

($b$) the time taken from the start of the motion until the particles are closest to each other; **4**

($c$) the time at which the particles are moving at right angles to each other. **3**

*Marks*

**A9.** A sledge and rider of combined mass 100 kg are pulled along a straight, horizontal track by a team of huskies.

When moving with speed $v\,\mathrm{m\,s^{-1}}$, the sledge and rider experience a resistance whose magnitude is given by $(100 + 5v)$ newtons.

Given that the huskies are working at the rate of 1·5 kW, show that

$$\frac{dv}{dt} = \frac{300 - 20v - v^2}{20v}.$$ **3**

Hence find the time taken for the sledge and rider to accelerate from rest to $8\,\mathrm{m\,s^{-1}}$. **5**

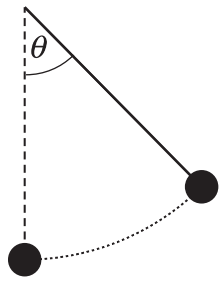
Show that the maximum speed of the sledge and rider is $10\,\mathrm{m\,s^{-1}}$. **1**

$$\left(\text{You may assume that } \frac{v}{300 - 20v - v^2} = \frac{1}{4}\left[\frac{-3}{30 + v} + \frac{1}{10 - v}\right].\right)$$

**[Turn over**

*Marks*

**A10.** (*a*) A simple pendulum consists of a mass $m$ kg suspended from a fixed point by a light, inextensible string of length $L$ metres. The mass is pulled to the side so that the taut string makes an angle $\theta$ with the downwards vertical through the fixed point, as shown in the diagram. The mass is then released from rest.




By considering the forces acting on the mass along the tangent to the circle that the mass describes, show that, for small values of $\theta$,

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta .$$ **3**

Assuming that $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$, find an expression, in terms of $L$ and $g$, for the period of oscillation of the pendulum and calculate the length of string required for the period to be 2 seconds. **3**

(*b*) A particle moving in a straight line, whose motion is also simple harmonic, oscillates with period 2 seconds about a point $O$. The particle is moving towards $O$ with speed $4\pi \, \text{m s}^{-1}$ when it passes through a point $P$ which is 3 metres from $O$.

Show that the amplitude of the motion is 5 metres. **2**

Calculate the time which elapses from the instant when the particle leaves $P$ to when it next passes through $O$. **2**

[*END OF SECTION A*]

**[Turn over for Section B on *Pages eight and nine***

## Section B (Mathematics for Applied Mathematics)

*Marks*

**Answer all the questions**

**B1.** Write down and simplify the general term in the expansion of

$$(x^2 + 3x)^8.$$ **3**

Hence, or otherwise, obtain the coefficient of $x^{13}$. **2**

**B2.** (*a*) Given the curve $y = \dfrac{x}{x^2 + 4}$, calculate the gradient when $x = 2$. **3**

(*b*) Determine $\int e^{-2t}\,dt$. **2**

**B3.** Given $M = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

(*a*) Calculate $M^2$. **2**

(*b*) Calculate $M + M^2 + M^3$. **2**

(*c*) For what values of $\lambda$ does $M$ have an inverse? **2**

**B4.** Express $\dfrac{1}{x^2 + x}$ in partial fractions, where $x$ is neither 0 nor $-1$. **3**

A region is enclosed by the curve with equation $y = \dfrac{1}{\sqrt{x^2 + x}}$, the $x$-axis and the lines $x = 1$ and $x = 3$.

Calculate the volume of the solid of revolution formed by rotating this region through 360° about the $x$-axis. **4**

*Marks*

**B5.** A turkey is taken from a refrigerator to be cooked. Its temperature is 4 °C when it is placed in an oven preheated to 180 °C.

Its temperature, $T$ °C, after a time of $x$ hours in the oven satisfies the equation

$$\frac{dT}{dx} = k(180 - T).$$

(*a*) Show that $T = 180 - 176e^{-kx}$. **4**

(*b*) After an hour in the oven the temperature of the turkey is 30 °C.

Calculate the value of $k$ correct to 2 decimal places. **2**

(*c*) The turkey will be cooked when it reaches a temperature of 80 °C.

After how long (to the nearest minute) will the turkey be cooked? **3**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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