X204/701

NATIONAL
QUALIFICATIONS
2011

FRIDAY, 20 MAY
1.00 PM – 4.00 PM

APPLIED MATHEMATICS
ADVANCED HIGHER
Mechanics

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer all questions.

   Section A assesses the Units Mechanics 1 and 2
   Section B assesses the Unit Mathematics for Applied Mathematics

3. **Full credit will be given only where the solution contains appropriate working.**

Section A (Mechanics 1 and 2)

Marks

Answer all the questions.

Candidates should observe that $g\,\mathrm{m\,s^{-2}}$ denotes the magnitude of the acceleration due to gravity.

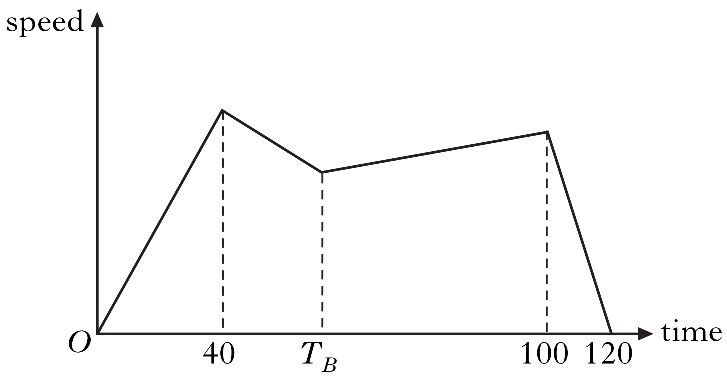
Where appropriate, take its magnitude to be $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

A1. A smooth horizontal surface contains the perpendicular unit vectors $\mathbf{i}$ and $\mathbf{j}$. A body of mass 1 kg has velocity $-2\mathbf{i} + 4\mathbf{j}\,\mathrm{m\,s^{-1}}$ and collides with a second body of mass 2 kg moving in the plane.

The bodies coalesce and move with velocity $\mathbf{i} + 4\mathbf{j}\,\mathrm{m\,s^{-1}}$.

Calculate the speed of the larger mass before the collision. **3**

A2. The speed-time graph of the motion of a car as it travels along a straight road is shown below. The car accelerates from O and passes markers on the road at A, B, C before stopping at D after 120 seconds. The car passes A after 40 seconds, B after T_B seconds, and C after 100 seconds.



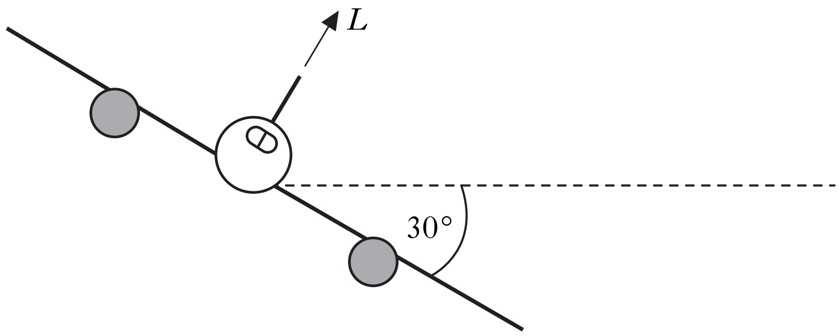
The speed of the car between A and B is given by $v_1(t) = -\frac{1}{2}t + 45 \quad (40 \leq t \leq T_B)$ and between B and C by $v_2(t) = \frac{1}{8}t + \frac{15}{2} \quad (T_B \leq t \leq 100)$, where the speed is measured in metres per second and time t is measured in seconds from the beginning of the motion.

(*a*) Calculate the speed of the car at B. **3**

(*b*) Calculate the distance between B and D. **3**

Marks

A3. An aircraft flies at constant speed U metres per second in a horizontal circular orbit of radius R metres.



The wings of the aircraft are banked at 30° to the horizontal and generate a lift force of L newtons. This force acts perpendicular to the wing surface, as shown in the diagram.

Show that the radius of the circular orbit is given by

$$R = \frac{\sqrt{3}U^2}{g}.$$ **4**

Hence find an expression for the orbital period in terms of U and g. **2**

A4. Relative to a rectangular coordinate system with origin O the position vector of a passenger aircraft is $-100\mathbf{i} + 250\mathbf{j}$, at 09.00 hours, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the Ox and Oy directions. The aircraft is travelling with uniform velocity $300\mathbf{i} + 400\mathbf{j}$.

Relative to the same coordinate system, a military aircraft travelling with uniform velocity $600\mathbf{i} + 500\mathbf{j}$, has position vector $-100\mathbf{i} + 400\mathbf{j}$ at 09.30 hours. In these expressions, the distances are measured in kilometres and speeds in kilometres per hour.

Show that the two aircraft are on a collision course. **5**

A5. A charged particle oscillates with simple harmonic motion of amplitude 0·05 metres between two parallel plates in an electric field. The speed of the particle, v metres per second, satisfies the differential equation

$$v\frac{dv}{dx} = -\omega^2 x,$$

where x metres is the displacement from a fixed point and ω is a constant.

Obtain an expression for v^2 in terms of x and ω. **3**

Given that the period of the oscillation is $1{\cdot}5 \times 10^{-4}$ seconds, calculate the maximum speed of the particle. **3**

[Turn over

A6. A particle with mass 0·25 kg moves along a straight line. Its velocity $\mathbf{v}$ is given by

$$\mathbf{v} = 8\,(1 - e^{-2t})\mathbf{i}$$

where $\mathbf{i}$ is a unit vector in the Ox direction of a rectangular coordinate system with origin O. The time t is measured in seconds and the speed is measured in metres per second.

Obtain an expression for the force acting on the particle at time t. **2**

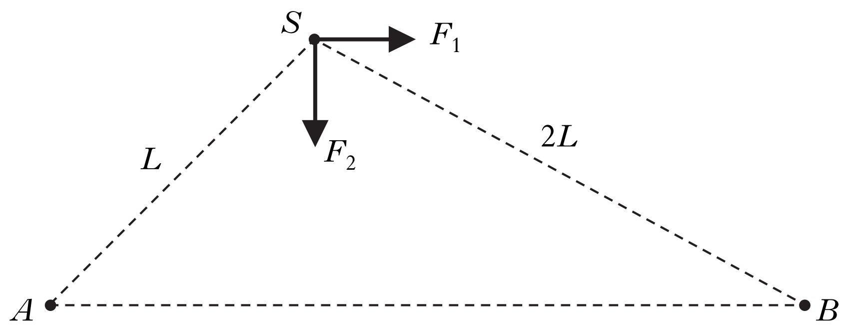
Show that the work done by this force during the time interval $0 \leq t \leq 1$ is given by

$$32\int_0^1 (e^{-2t} - e^{-4t})\,dt \text{ joules,}$$

and hence calculate this value. **4**

A7. A space probe S experiences a gravitational attraction from two asteroids, A and B. The mass of asteroid B is four times the mass of asteroid A.

At one instant, the probe is a distance L metres from asteroid A and distance $2L$ metres from asteroid B. The lines AS and BS are perpendicular. The force on the space probe has components F_1 and F_2 newtons, parallel and perpendicular to the line AB respectively as shown in the diagram.

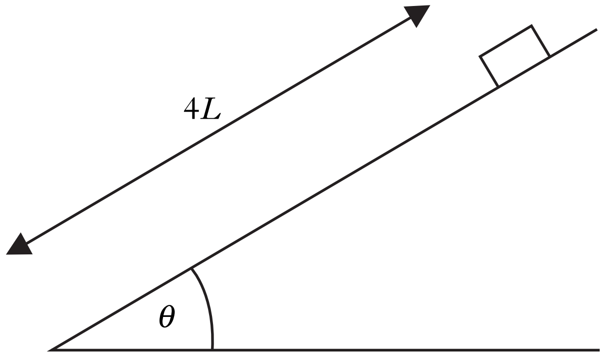


Assuming that the gravitational attraction on the probe due to each asteroid obeys the inverse square law, show that

$$F_2 = 3F_1.$$ **6**

Marks

A8. A small box accelerates from rest down a rough plane inclined at an angle θ to the horizontal. After travelling a distance $4L$ metres, the box reaches the bottom of the plane with speed U metres per second. The coefficient of friction between the box and the inclined plane is μ.



Show that

$$\frac{U^2}{gL} = 8(\sin\theta - \mu\cos\theta).$$ **3**

The box is now projected up the same inclined plane with initial speed U metres per second, coming to rest instantaneously after travelling $3L$ metres. Show that

$$\mu = \frac{1}{7}\tan\theta.$$ **5**

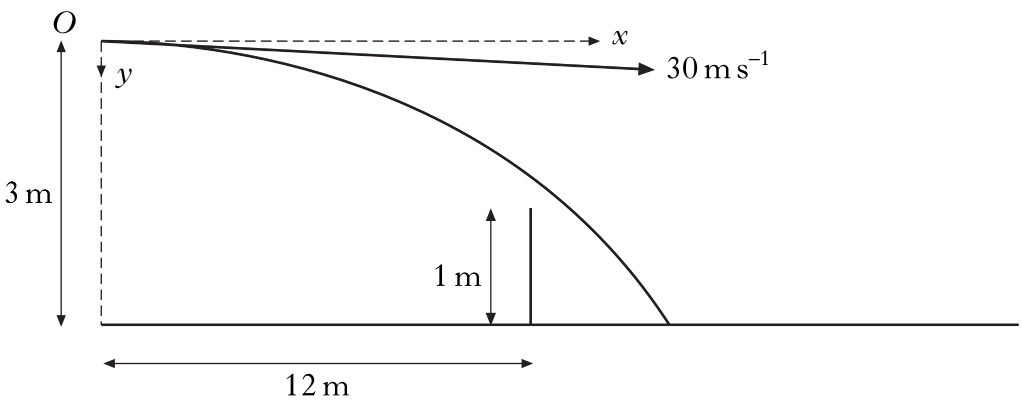
When $\tan\theta = \frac{3}{4}$, find an expression for L in terms of U and g. **2**

[Turn over

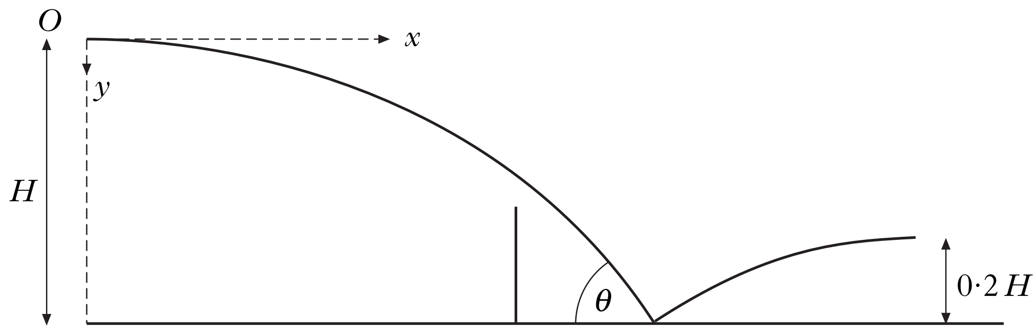
Marks

A9. (*a*) A tennis ball is projected with speed $30\,\mathrm{m\,s^{-1}}$ at angle of $5°$ below the horizontal, from a height of 3 metres and horizontal distance of 12 metres from the net which has height one metre.

By how much does the ball clear the net? **3**



(*b*) A second ball is projected from height H metres with initial velocity vector $\mathbf{u} = \sqrt{6gH}\,\mathbf{i}$, where $\mathbf{i}$ is a unit vector along the Ox direction. At the instant the ball hits the tennis court, the velocity vector makes an acute angle θ with the surface of the court.



Obtain an expression for the speed of impact of the ball with the court in terms of g and H, and determine angle θ. **4**

During the bounce, the ball loses 50% of its kinetic energy and continues, reaching a maximum height of $0{\cdot}2\,H$ metres.

Obtain an expression, in terms of g and H, for the speed of the ball when it reaches its maximum height. **3**

Marks

A10. A small ball of mass m kilograms is projected vertically from ground level with initial speed U metres per second. In addition to gravity, the ball experiences a resistive force of magnitude mkv^2 newtons, where v metres per second is the speed of the ball and k is a positive constant.

Show that the speed of the ball at height x metres above ground level satisfies the equation

$$v^2 = \left(\frac{g}{k} + U^2\right)e^{-2kx} - \frac{g}{k}.$$ **6**

Given that $U = \sqrt{\frac{g}{k}}$, show that the kinetic energy of the ball, after it has travelled from ground level to half its maximum height, is

$$\frac{1}{2}mU^2(\sqrt{2} - 1).$$ **4**

[*END OF SECTION A*]

[Turn over for Section B on *Pages eight* and *nine*

Section B (Mathematics for Applied Mathematics)

Marks

Answer all the questions.

B1. Differentiate the following functions, simplifying where possible:

(a) $f(x) = \dfrac{1+\sin x}{1+2\sin x}$, $0 \le x \le \pi$; **3**

(b) $g(x) = \ln(1+e^{2x})$. **2**

B2. Given $A = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$, obtain A^{-1}.

Given $AB = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix}$, find the matrix B. **5**

B3. A curve is defined by the equations

$$x = 5\cos t \qquad \text{and} \qquad y = 3\sin t, \qquad (0 \le t < 2\pi).$$

Find the gradient of the curve when $t = \dfrac{\pi}{6}$. **4**

B4. Find the value of N for which $\displaystyle\sum_{r=1}^{N} r = 210$. **3**

Evaluate $\displaystyle\sum_{r=1}^{N} r^2$ for this value of N. **2**

Marks

B5. Use the substitution $u = \ln x$ to obtain $\int \frac{2}{x \ln x} dx$, where $x > 1$. **4**

B6. At any point (x, y) on a curve C, where $x \neq 0$, the gradient of the tangent is $4 - \frac{3y}{x}$.
Given that the point $(1, 3)$ lies on C, obtain an equation for C in the form $y = f(x)$. **9**

[*END OF SECTION B*]

[*END OF QUESTION PAPER*]

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