THE ROYAL STATISTICAL SOCIETY

2004 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

PAPER I – STATISTICAL THEORY

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<u>Higher Certificate, Paper I, 2004. Question 1</u>

(i) (a) P(all four favour the complex) = $(0.6)^4$.
P(all four oppose the complex) = $(0.3)^4$.
P(all four are indifferent) = $(0.1)^4$.

So P(all four think alike) = $(0.6)^4 + (0.3)^4 + (0.1)^4 = 0.1378$.

(b) P(an individual is not opposed) = $0.6 + 0.1 = 0.7$.

So P(none of the four is opposed) = $(0.7)^4 = 0.2401$.

(c) Possible favourable results are *FFOI*, *FOOI*, *FOII*, in any order.

$P(FFOI) = (0.6)^2(0.3)(0.1) = 0.0108$
$P(FOOI) = (0.6)(0.3)^2(0.1) = 0.0054$
$P(FOII) = (0.6)(0.3)(0.1)^2 = 0.0018$

Each result can be arranged in $\dfrac{4!}{2!1!1!} = 12$ ways.

So overall probability is $12(0.0108 + 0.0054 + 0.0018) = 0.216$.

(d) From (a), P(all four in favour) = $(0.6)^4 = 0.1296$. From (b), P(none opposed) = 0.2401. So the required conditional probability is

$$0.1296/0.2401 = 0.5398.$$

(ii) The number in favour is binomially distributed with $n = 4$ and $p = 0.6$. So the expectation (mean) is $4 \times 0.6 = 2.4$ and the variance is $4 \times 0.6 \times 0.4 = 0.96$.

(iii) $P(\text{opposed}) = P(\text{opposed} \mid \text{young})P(\text{young}) + P(\text{opposed} \mid \text{older})P(\text{older})$

$$= (0.12 \times 0.25) + (p \times 0.75)$$

where $p = P(\text{opposed} \mid \text{older})$. But we are given that P(opposed) = 0.3. Hence $p = 0.36$.

(iv) In samples of one "young" and three "olders",

P(exactly one opposes) = P("young" opposes, "olders" do not)

+ P("young" does not oppose, <u>one</u> "older" opposes)

$= \{(0.12)(0.64)^3\} + \{3(0.88)(0.36)(0.64)^2\} = 0.03146 + 0.38928 = 0.4207.$

Higher Certificate, Paper I, 2004. Question 2

(i) (a) The moment generating function is

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \frac{\left(\lambda e^t\right)^x}{x!} = \exp\left(\lambda e^t - \lambda\right) = \exp\left(\lambda\left(e^t - 1\right)\right)$$

(b) $E(X) = M_X'(0) = \left[\lambda e^t e^{\lambda\left(e^t-1\right)}\right]_{t=0} = \lambda$.

$$E\left(X^2\right) = M_X''(0) = \left[\frac{d}{dt}\left(\lambda e^t e^{\lambda\left(e^t-1\right)}\right)\right]_{t=0}$$

$$= \left[\lambda e^t . e^{\lambda\left(e^t-1\right)} \lambda e^t + e^{\lambda\left(e^t-1\right)} . \lambda e^t\right]_{t=0} = \lambda^2 + \lambda .$$

Hence $\operatorname{Var}(X) = E\left(X^2\right) - \left[E(X)\right]^2 = \lambda$.

(Alternatively, the moments could be obtained from the power series expansion of $M_X(t)$.)

(Alternatively, though with comparatively lengthy algebra, the moments could be obtained directly by $E(X) = \Sigma x P(X = x)$ and $E(X^2) = \Sigma x^2 P(X = x)$; or, somewhat easier, use $E[X(X - 1)] = \Sigma x(x - 1)P(X = x)$ (this is λ^2) and then $\operatorname{Var}(X) = E[X(X - 1)] + E(X) - \{E(X)\}^2$.)

(c) The binomial distribution with parameters n and p may be approximated by the Poisson distribution with parameter np if n is large and p is small. As a "rule of thumb", $\frac{1}{2} \le np \le 10$ gives an indication of how large n should be and how small p should be. (If $np > 10$, a Normal approximation to the binomial may be better.)

(ii) Let X = number of wrong calculations. We have $X \sim B(200, 0.0075)$.

$$P(X = 1) = \binom{200}{1}(0.0075)(0.9925)^{199} = 200 \times 0.0075 \times 0.9925^{199} = 0.3353(2).$$

$$P(X = 4) = \binom{200}{4}(0.0075)^4 (0.9925)^{196}$$

$$= \frac{200 \times 199 \times 198 \times 197}{4 \times 3 \times 2 \times 1} \times 0.0075^4 \times 0.9925^{196} = 0.0468(0).$$

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(iii) We approximate using $X \sim$ Poisson($200 \times 0.0075 = 1.5$). With this,

$$P(X=1)=1.5e^{-1.5}=0.3347(0)$$

giving a percentage error of $\dfrac{100(0.33532-0.33470)}{0.33532}=0.18\%$, and

$$P(X=4)=\frac{e^{-1.5}(1.5)^4}{4!}=0.0470(7)$$

giving a percentage error of $\dfrac{100(0.04707-0.04680)}{0.04680}=0.58\%$.

[Note. These percentage errors might come out *slightly* differently if more accuracy is kept in the binomial and Poisson probabilities.]

Both approximations are remarkably accurate, with percentage errors well below 1%. The approximation for $X = 1$ (one wrong calculation) is the more accurate of the two. That approximation is an underestimate; the other is an overestimate.

<u>Higher Certificate, Paper I, 2004. Question 3</u>

Actual volume $X \sim$ N(1010, 8^2). Let $Z \sim$ N(0,1).

(i) $P(X < 1000) = P\left(Z < \frac{1000-1010}{8}\right) = P(Z < -1.25) = 0.1056$.

(ii) Let Y be the total volume in a 6-pack.

We have $Y \sim$ N(6×1010, 64 + 64 + 64 + 64 + 64 + 64), i.e. $Y \sim$ N(6060, 384).

$P(Y < 6000) = P\left(Z < \frac{6000-6060}{\sqrt{384}}\right) = P(Z < -3.06) = 0.0011$.

(Alternatively, could use $\bar{X} \sim$ N(1010, 64/6) and calculate $P(\bar{X} < 1000)$.)

This probability is considerably smaller than that in part (i). In practical terms, this is because there will be a tendency for heavier and lighter cartons in a 6-pack to balance each other out. Alternatively, in terms of probability distributions, consider X and $\bar{X}$: $\bar{X}$ has the same mean as X but only one-sixth of the variance, so less of the lower tail of the distribution of $\bar{X}$ is below the nominal volume of 1000.

(iii) The new volume $W \sim$ N(μ, 4^2), where μ is the new mean. So we have that $P(W < 1000) = P\left(Z < \frac{1000-\mu}{4}\right)$. We require that this probability must be no greater than 0.1056. Thus the cut-off point for Z is to be z = –1.25 (as before). Hence $\frac{1000-\mu}{4} = -1.25$, giving μ = 1005.

This means that 5 ml per carton could be saved, i.e. a cost saving per carton of $\frac{5}{1000} \times £1$. To recover the £200, the number of cartons required is $\frac{200}{5/1000}$ = 40000.

Higher Certificate, Paper I, 2004. Question 4

(i) The binomial distribution with parameters n, p can be approximated by $N(np, np(1-p))$ when n is large and p is not too near to 0 or 1. As a "rule of thumb", the approximation is likely to be good if both np and $np(1-p)$ are > 10.

Let $X \sim B(n, p)$ and let Φ denote the c.d.f. of $N(0, 1)$. Using a continuity correction, $P(X \le x) \approx \Phi\left(\dfrac{x+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right)$ and $P(X < x) \approx \Phi\left(\dfrac{x-\frac{1}{2}-np}{\sqrt{np(1-p)}}\right)$.

The 95% confidence interval for p uses the estimated variance $\hat{p}(1-\hat{p})/n$, giving the approximate interval

$$\hat{p}-1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} < p < \hat{p}+1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

The estimate of p is $\hat{p}=\dfrac{30}{50}=0.6$, so $\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\dfrac{(0.6)(0.4)}{50}} = 0.0693$. Thus the approximate interval is

$$0.6-(1.96\times 0.0693)\ ,\ 0.6+(1.96\times 0.0693)$$

i.e. (0.464, 0.736).

(ii) P(neither hits) $= (1-p)^2$. Therefore P(at least 1 hit) $= 1-(1-p)^2 = p(2-p)$. We estimate this by $(0.6)(2-0.6) = 0.84$.

(iii) When $p = 0.464$ (lower limit of interval in part (i)), we have $p(2-p) = 0.713$. Similarly, when $p = 0.736$, we have $p(2-p) = 0.930$. Thus (0.713, 0.930) is the required interval.

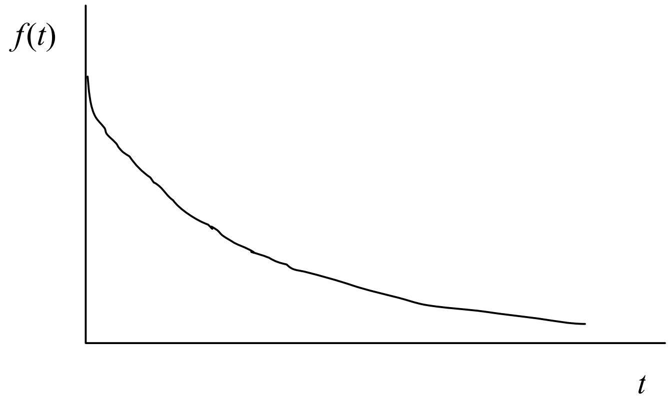
(iv) When n pairs are fired, P(all miss) $= [(1-p)^2]^n$, estimated by $(0.16)^n$. Hence $(0.16)^n < 0.0005$ is required. Solving this by taking logarithms to base 10, we have $n\log_{10}(0.16) < \log_{10}(0.0005)$, i.e. $-0.79588n < -3.30103$ which gives $n > 4.148$. so n must be at least 5.

Higher Certificate, Paper I, 2004. Question 5

(i) $$f(t) = \lambda e^{-\lambda t}, \qquad t > 0; \quad \lambda > 0$$

(a) Sketch of $f(t)$.

[***NOTE.*** *The curve should of course appear as a smooth decaying exponential; it might not do so, due to the limits of electronic reproduction.*]



(b) C.d.f. is $F(t) = P(T \le t) = \int_0^t \lambda e^{-\lambda v} dv = \lambda \left[-\frac{1}{\lambda} e^{-\lambda v} \right]_0^t = 1 - e^{-\lambda t}$.

(c) $P(a < T \le b) = F(b) - F(a) = e^{-\lambda a} - e^{\lambda b}$.

(ii) Assume all settlements of invoices are independent.

P(50 in first week) = $\{F(1)\}^{50} = \left(1 - e^{-\lambda}\right)^{50}$, because $T \le 1$ for all these 50.

Likewise, $1 < T \le 2$ for the 35 in the second week, so we have P(35 in second week) = $\{F(2) - F(1)\}^{35}$ = $\left(e^{-\lambda} - e^{-2\lambda}\right)^{35}$.

The remaining 15 have $T > 2$, which has probability $1 - P(T \le 2) = e^{-2\lambda}$, and thus P(15 after week 2) = $\left(e^{-2\lambda}\right)^{15}$.

The likelihood is therefore the product

$$L(\lambda) = k\left(1 - e^{-\lambda}\right)^{50} \left(e^{-\lambda} - e^{-2\lambda}\right)^{35} \left(e^{-2\lambda}\right)^{15}$$

where k is a constant of proportionality.

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Taking logarithms (base *e*),

$$\log L(\lambda) = \log k + 50\log\left(1-e^{-\lambda}\right) + 35\log\left\{e^{-\lambda}\left(1-e^{-\lambda}\right)\right\} + 15\log\left(e^{-2\lambda}\right)$$

$$= \log k + 85\log\left(1-e^{-\lambda}\right) - (35+30)\lambda \;=\; \log k + 85\log\left(1-e^{-\lambda}\right) - 65\lambda\,.$$

$$\therefore \frac{d}{d\lambda}\log L \;=\; \frac{85e^{-\lambda}}{1-e^{-\lambda}} - 65 \;=\; \frac{85}{e^{\lambda}-1} - 65\,.$$

Equating to zero, $85 = 65\left(e^{\lambda}-1\right)$ or $e^{\lambda} = 150/65$, so that $\hat{\lambda} = \log(150/65) = 0.836$.

[It is easy to check that this is indeed a maximum; e.g. $\dfrac{d^2}{d\lambda^2}\log L = -\dfrac{85}{\left(e^{\lambda}-1\right)^2} \;<\; 0$.]

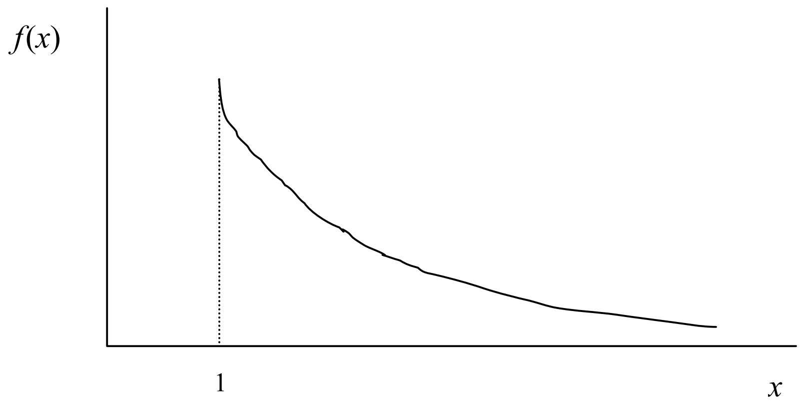
(iii) $1-e^{-0.836} = 0.5666$; $e^{-0.836} - e^{-1.672} = 0.43344 - 0.18787 = 0.2456$. Hence, out of 100 invoices, 56.66, 24.56 and 18.78 would be expected to be paid, on this model, in weeks 1, 2 and later. The actual numbers were 50, 35 and 15. The prediction for the second week is a long way from what happened, balanced by smaller discrepancies in the other two periods. This does not seem very satisfactory.

<u>Higher Certificate, Paper I, 2004. Question 6</u>

$$f(x) = \frac{k}{x^{k+1}}, \qquad x \geq 1; \quad k > 0$$

(i) Sketch of $f(x)$.

[***NOTE.*** *The curve should of course appear as a smooth curve; it might not do so, due to the limits of electronic reproduction.*]



C.d.f. is $F(x) = P(X \leq x) = \int_1^x \frac{k}{u^{k+1}}\,du = \left[-\frac{1}{u^k}\right]_1^x = 1 - \frac{1}{x^k}$ (for $x \geq 1$).

(ii) Median M has $\frac{1}{2} = F(m) = 1 - M^{-k}$, so $\frac{1}{2} = M^{-k}$ and hence $M = 2^{1/k}$.

Lower quartile Q_1 has $\frac{1}{4} = F(Q_1) = 1 - Q_1^{-k}$, so $\frac{3}{4} = Q_1^{-k}$, i.e. $Q_1 = (4/3)^{1/k}$.

Upper quartile Q_3 has $\frac{3}{4} = F(Q_3) = 1 - Q_3^{-k}$, so $Q_3 = (4)^{1/k}$.

Hence the semi-interquartile range is $\frac{1}{2}\left\{4^{1/k} - \left(\frac{4}{3}\right)^{1/k}\right\}$.

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(iii) $E(X)=\int_1^\infty xf(x)\,dx=\int_1^\infty \frac{k}{x^k}\,dx=\left[\frac{-k}{(k-1)x^{k-1}}\right]_1^\infty=\frac{k}{k-1}.$

$$E(X^2)=\int_1^\infty x^2 f(x)\,dx=\int_1^\infty \frac{k}{x^{k-1}}\,dx=\left[\frac{-k}{(k-2)x^{k-2}}\right]_1^\infty=\frac{k}{k-2}.$$

$$\therefore \mathrm{Var}(X)=E(X^2)-\{E(X)\}^2=\frac{k}{k-2}-\frac{k^2}{(k-1)^2}$$

$$=\frac{k}{(k-2)(k-1)^2}\left\{(k-1)^2-k(k-2)\right\}=\frac{k}{(k-1)^2(k-2)}.$$

$P(X>E(X))=\int_{k/(k-1)}^\infty \frac{k}{x^{k+1}}\,dx=\left[-\frac{1}{x^k}\right]_{k/(k-1)}^\infty=\left(\frac{k-1}{k}\right)^k$, or this can be written down directly from the c.d.f. found in part (i).

(iv) For the case $k=3$,

(a) $M=2^{1/3}$ in the units given, or £12599,

(b) mean = 3/2 in the units given, or £15000,

(c) inserting $X=10$, $P(X\le 10)=1-\frac{1}{10^3}$, so $P(X>10)=\frac{1}{10^3}$, i.e. 0.1%.

Higher Certificate, Paper I, 2004. Question 7

(i) $$E(X)=\int_0^\theta \frac{x}{\theta}dx=\frac{1}{\theta}\left[\frac{1}{2}x^2\right]_0^\theta=\frac{1}{2}\theta.$$

$$E(X^2)=\int_0^\theta \frac{x^2}{\theta}dx=\frac{1}{\theta}\left[\frac{1}{3}x^3\right]_0^\theta=\frac{1}{3}\theta^2.$$

$$\therefore \text{Var}(X)=E(X^2)-\{E(X)\}^2=\frac{1}{3}\theta^2-\left(\frac{1}{2}\theta\right)^2=\frac{1}{12}\theta^2.$$

(ii) P(longest offcut is $\le x$) = P(all n offcuts are $\le x$).

The c.d.f. for each X_i is $F(x)=P(X\le x)=\int_0^x \frac{du}{\theta}=\left[\frac{u}{\theta}\right]_0^x=\frac{x}{\theta}$, and the X_i are all independent. Therefore P(all n offcuts are $\le x$) = $\{F(x)\}^n=\left(\frac{x}{\theta}\right)^n$, and this is also P(longest offcut is $\le x$), i.e. the c.d.f. of the sample maximum $X_{(n)}$. Thus the p.d.f. of $X_{(n)}$ is the derivative of this, i.e. nx^{n-1}/θ^n. This is for the interval $(0, \theta)$.

$$\therefore E\left(X_{(n)}\right)=\int_0^\theta \frac{nx^n}{\theta^n}dx=\frac{n}{\theta^n}\left[\frac{x^{n+1}}{n+1}\right]_0^\theta=\frac{n\theta}{n+1}.$$

$$E\left(X_{(n)}{}^2\right)=\int_0^\theta \frac{nx^{n+1}}{\theta^n}dx=\frac{n}{\theta^n}\left[\frac{x^{n+2}}{n+2}\right]_0^\theta=\frac{n\theta^2}{n+2}.$$

$$\therefore \text{Var}\left(X_{(n)}\right)=E\left(X_{(n)}{}^2\right)-\left\{E\left(X_{(n)}\right)\right\}^2=\frac{n\theta^2}{n+2}-\frac{n^2\theta^2}{(n+1)^2}$$

$$=n\theta^2\left(\frac{(n+1)^2-n(n+2)}{(n+2)(n+1)^2}\right)=\frac{n\theta^2}{(n+1)^2(n+2)}.$$

Immediately we have $E\left(\frac{n+1}{n}X_{(n)}\right)=\theta$, so $\frac{n+1}{n}X_{(n)}$ is an unbiased estimator of θ.

$$\text{Var}\left(\frac{n+1}{n}X(n)\right)=\frac{(n+1)^2}{n^2}\text{Var}\left(X_{(n)}\right)=\frac{(n+1)^2}{n^2}\frac{n\theta^2}{(n+1)^2(n+2)}=\frac{\theta^2}{n(n+2)}.$$

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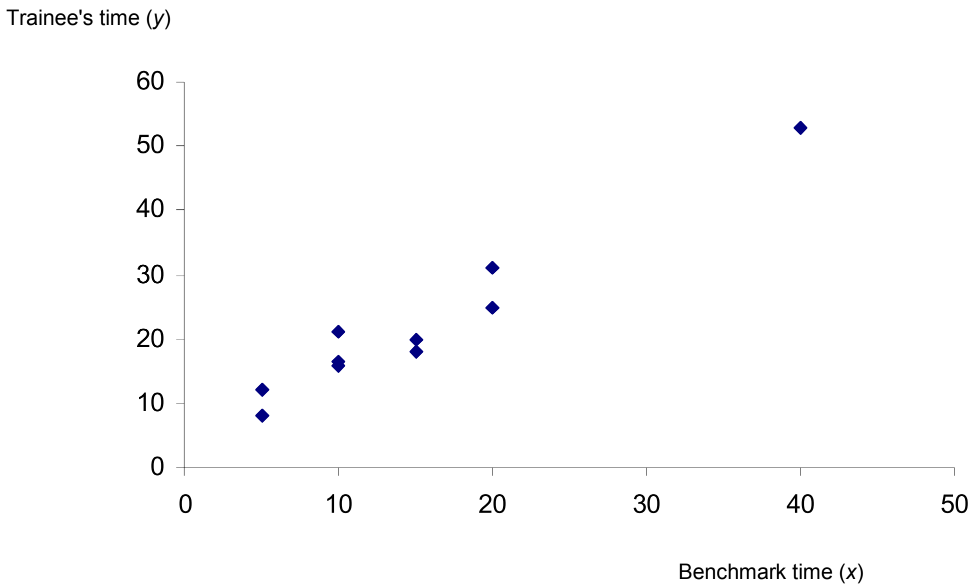
(iii) We have (see part (i)) that $E(X) = \theta/2$. Thus the method of moments estimator of $\theta/2$ is $\bar{X}$, and so the method of moments estimator of θ is $2\bar{X}$ or $\frac{2}{n}\sum X_i$ as required.

$$\mathrm{Var}\left(\frac{2}{n}\sum X_i\right) = \mathrm{Var}\left(2\bar{X}\right) = 4\mathrm{Var}\left(\bar{X}\right) = \frac{4}{n}\mathrm{Var}\left(X\right) = \frac{4}{n}.\frac{\theta^2}{12} = \frac{\theta^2}{3n}.$$

(i)



Simple linear regression analysis seems quite suitable.

(ii) The model is $y_i = \alpha + \beta x_i + e_i$, where $\{e_i\}$ are uncorrelated with zero mean and (constant) variance σ^2 (independent identically distributed $N(0, \sigma^2)$ for the purpose of undertaking statistical tests, as in part (iii)). Estimating by the method of least squares gives

$$\hat{\beta} = \frac{S_{xy}}{S_{xx}}, \qquad \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x},$$

where (standard notation)

$$S_{xy} = \Sigma(x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{\sum x_i \sum y_i}{n},$$

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n}.$$

We have

$$\hat{\beta} = \frac{S_{xy}}{S_{xx}} = \frac{4440 - (150 \times 220/10)}{3200 - (150^2/10)} = \frac{1140}{950} = 1.20 \quad \text{and} \quad \hat{\alpha} = 22 - (1.20 \times 15) = 4,$$

so the line is

$y = 4 + 1.2x$.

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The total sum of squares is $S_{yy} = \sum (y_i - \overline{y})^2 = \sum y_i^{\,2} - \dfrac{\left(\sum y_i\right)^2}{10} = 1440$.

The sum of squares for regression is $\hat{\beta} S_{xy}$ (or $S_{xy}^{\,2} / S_{xx}$) = 1368.

Therefore the residual sum of squares is 1440 – 1368 = 72.

This has 8 degrees of freedom, so the residual mean square ($\hat{\sigma}^2$) is 72/8 = 9.

The coefficient of determination R^2 = 1368/1440 = 0.95 (usually given as 95%).

(iii) The estimated variance of $\hat{\beta}$ is 9/950 = 0.009474. So the test statistic for testing the null hypothesis $\beta = 1$ is $\dfrac{1.2 - 1}{\sqrt{0.009474}}$ = 2.05, which we refer to t_8.

This is not significant at the 5% level, so the null hypothesis $\beta = 1$ cannot be rejected.

(iv) The model here is $y_i = bx_i + e_i$.

Estimating b by least squares, we minimise $\Omega = \sum_{i=1}^{n} (y_i - bx_i)^2$.

Differentiating with respect to b, we have $\dfrac{d\Omega}{db} = -2\sum (y_i - bx_i) x_i$.

Setting this equal to zero gives $\Sigma x_i y_i = \hat{b} \Sigma x_i^{\,2}$, i.e. $\hat{b} = \Sigma x_i y_i / \Sigma x_i^{\,2}$.

(Note that $\dfrac{d^2\Omega}{db^2} = 2\sum x_i^{\,2} > 0$, so this is a minimum.)

Thus we have $\hat{b}$ = 4440/3200 = 1.3875.