THE ROYAL STATISTICAL SOCIETY

2003 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

PAPER I – STATISTICAL THEORY

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Higher Certificate, Paper I, 2003. Question 1

(i) (a) Any of 0, 1, …, 9 can occur in each of the six positions, so the number is $10^6 = 1000000$.

(b) $10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151200$, since no repetition is allowed.

[Alternatively, $\dfrac{10!}{(10-6)!} = \dfrac{10!}{4!}$ as above.]

(c) There are $\binom{10}{6}$ choices of six different digits from ten, each of which can be used in only one of its possible orders. So the number is

$$\binom{10}{6} = \frac{10!}{6!4!} = 210 .$$

(d) Here there are $\binom{10}{3}$ choices of digits, for each of which there are $\dfrac{6!}{2!2!2!} = 90$ orders, so the number is $\binom{10}{3} \times 90 = 10800$.

(ii) (a) There are $3! = 6$ possible orders for the first three digits, and 1 order (the reverse order of the first three digits) for the last three. So there are 6 codes.

(b) If one digit is used 4 times in a palindromic code, another must be used twice. These digits may be chosen in 3 and 2 ways respectively, i.e. in 6 ways for the pair. Once the digits have been chosen, only 3 patterns are possible; for example, say the digits are 1 and 2, then the possible patterns are 1 1 2 2 1 1, 1 2 1 1 2 1 and 2 1 1 1 1 2. So the total number of codes is $6 \times 3 = 18$.

(c) There are 3 choices of digit and only one possible pattern for each; so there are 3 codes.

(iii) Using the patterns from part (ii), there are $\binom{10}{3}$ choices of digits for (a), each giving 6 codes, i.e. $\binom{10}{3} \times 6 = 720$ altogether. For (b), there are $\binom{10}{2}$ choices of digits, i.e. 45, with 3 patterns as above, in which the two digits can be used in 2 ways (4 of 1 and 2 of 2 *or* vice versa), giving $45 \times 3 \times 2 = 270$ ways. And (c) can occur in 10 ways (because any digit can be used 6 times). Thus the total is $720 + 270 + 10 = 1000$ ways.

ALTERNATIVELY, the first three positions may each be filled in 10 ways, and then the whole sequence is determined, so there are $10^3 = 1000$ ways.

Higher Certificate, Paper I, 2003. Question 2

$$P(A)=\frac{2}{3} \qquad P(B)=\frac{1}{2} \qquad P(C)=\frac{1}{4}$$

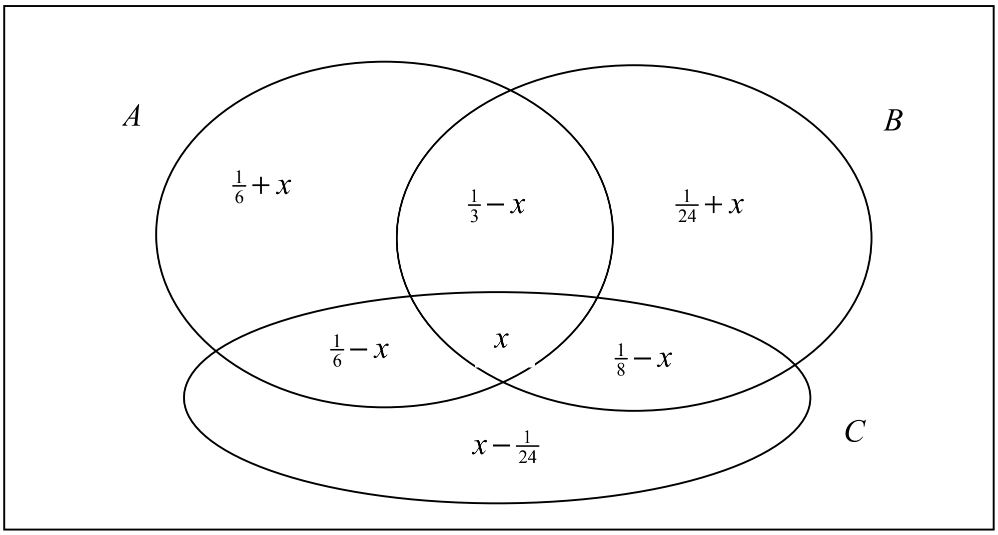
(i) (a) By the given independence,

$$P\left(A\cap\overline{B}\cap\overline{C}\right)=P(A)P\left(\overline{B}\right)P\left(\overline{C}\right)=\frac{2}{3}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)=\frac{1}{4}\ .$$

(b) $$P\left(A\cap\overline{C}\,\middle|\,A\cap\overline{B}\right)=\frac{P\left(\left(A\cap\overline{C}\right)\cap\left(A\cap\overline{B}\right)\right)}{P\left(A\cap\overline{B}\right)}=\frac{P\left(A\cap\overline{C}\cap\overline{B}\right)}{P\left(A\cap\overline{B}\right)}$$

$$=\frac{\frac{1}{4}}{\frac{2}{3}\left(1-\frac{1}{2}\right)}=\frac{3}{4}\ .$$

(ii)



Using pairwise independence, the value of $P(A\cap B)$ is $P(A)P(B)$, etc, and hence the values $\frac{1}{3}-x$, $\frac{1}{8}-x$ and $\frac{1}{6}-x$ are found. The others follow using $P(A)$, $P(B)$ and $P(C)$.

(a) $P\left(A\cap\overline{B}\cap\overline{C}\right)=\frac{1}{6}+x$ from the diagram.

(b) $$P\left(A\cap\overline{C}\,\middle|\,A\cap\overline{B}\right)=\frac{P\left(\left(A\cap\overline{C}\right)\cap\left(A\cap\overline{B}\right)\right)}{P\left(A\cap\overline{B}\right)}=\frac{P\left(A\cap\overline{C}\cap\overline{B}\right)}{P\left(A\cap\overline{B}\right)}$$

$$=\frac{\frac{1}{6}+x}{\left(\frac{1}{6}+x\right)+\left(\frac{1}{6}-x\right)}=3x+\frac{1}{2}\ .$$

(c) $$P(A\cup B\cup C)=\frac{19}{24}+x\ .$$

Since all probabilities must lie in [0,1], we have $x\geq\frac{1}{24}$ and $x\leq\frac{1}{8}$, i.e.

$$\frac{1}{24}\leq x\leq\frac{1}{8}\ .$$

Higher Certificate, Paper I, 2003. Question 3

(i) (a) $X + Y_A$ is N(10 + 15 , 12 + 16) i.e. N(25, 28).

(b) $X + Y_B$ is N(10 + 12 , 12 + 9) i.e. N(22, 21).

(ii) If X is the same for both, we require $P(Y_A < Y_B)$, i.e. $P(Y_A - Y_B < 0)$.

$Y_A - Y_B$ is N(15 – 12 , 16 + 9) i.e. N(3, 25).

$P(Y_A - Y_B < 0) = \Phi\left(\frac{0-3}{5}\right)$ where (as usual) Φ denotes the cdf of the N(0, 1) distribution. From tables $\Phi(-0.6) = 1 - \Phi(0.6) = 0.2743$.

(iii) Writing $W_A = X + Y_A$ and $W_B = X + Y_B$, we require $P(W_A < W_B)$, i.e. $P(W_A - W_B < 0)$.

$W_A - W_B$ is N(25 – 22 , 28 + 21) i.e. N(3, 49).

$$P(W_A - W_B < 0) = \Phi\left(\frac{0-3}{7}\right) = \Phi\left(-\frac{3}{7}\right) = 0.3341.$$

(iv) $\bar{W}_A$ is $N\left(25, \frac{28}{16}\right)$ and $\bar{W}_B$ is $N\left(22, \frac{21}{16}\right)$.

Let $U = \bar{W}_A - \bar{W}_B$; then U is $N\left(3, \frac{49}{16}\right)$, and we require

$$P(U < 0) = \Phi\left(\frac{0-3}{\frac{7}{4}}\right) = \Phi\left(-\frac{12}{7}\right) = \Phi(-1.7143) = 0.0432.$$

Higher Certificate, Paper I, 2003. Question 4

$$f(x) = kx^2(1-x), \qquad 0 \le x \le 1$$

(i) $$\int_0^1 k\left(x^2 - x^3\right)dx = k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = k\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{k}{12},$$

which must be equal to 1. So $k = 12$.

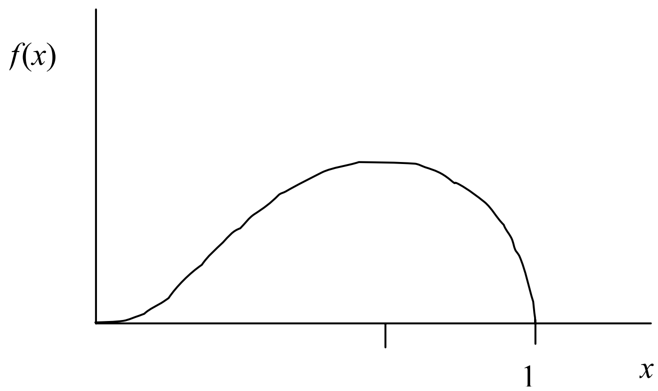
(ii) $$\frac{df(x)}{dx} = \frac{d}{dx}\left(12x^2 - 12x^3\right) = 24x - 36x^2 = 12x(2-3x)$$

which is zero for $2 - 3x = 0$ [and for $x = 0$, but this is clearly not the mode (i.e. not the maximum of $f(x)$)], i.e. $x = 2/3$. To check that this *is* the maximum (i.e. the mode), we can consider the second derivative:-

$$\frac{d^2 f(x)}{dx^2} = 24 - 72x, \text{ which is clearly} < 0 \text{ at } x = 2/3.$$

Hence the mode is at $x = 2/3$, and the graph of $f(x)$ is as shown. [***NOTE.*** *The curve should of course appear smooth; it might not do so, due to the limits of electronic reproduction.*]

[At the mode, $f(x) = 12(2/3)^2(1/3) = 16/9$.]



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(iii) $$E(X) = \int_0^1 xf(x)\,dx = \int_0^1 12\left(x^3 - x^4\right)dx = 12\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 12\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{12}{20} = \frac{3}{5}\ .$$

$$E\left(X^2\right) = \int_0^1 x^2 f(x)\,dx = \int_0^1 12\left(x^4 - x^5\right)dx = 12\left[\frac{x^5}{5} - \frac{x^6}{6}\right]_0^1$$

$$= 12\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{12}{30} = \frac{2}{5}\ .$$

So $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \dfrac{2}{5} - \dfrac{9}{25} = \dfrac{1}{25}$.

(iv) The cumulative distribution function is $F(x) = \int_0^x 12\left(u^2 - u^3\right)du$

$$= \left[12\left(\frac{u^3}{3} - \frac{u^4}{4}\right)\right]_0^x = 4x^3 - 3x^4 = x^3(4 - 3x), \quad \text{for } 0 \le x \le 1\ .$$

The mean is $\frac{3}{5}$ and the standard deviation is $\frac{1}{5}$. We require $P\left(\frac{2}{5} < X < \frac{4}{5}\right)$. This can be found by integrating the pdf between $\frac{2}{5}$ and $\frac{4}{5}$ or, directly, as

$$F\left(\frac{4}{5}\right) - F\left(\frac{2}{5}\right) = \left(\frac{4}{5}\right)^3\left(\frac{8}{5}\right) - \left(\frac{2}{5}\right)^3\left(\frac{14}{5}\right) = \frac{64\times 8\ -\ 8\times 14}{625} = \frac{16}{25}\ .$$

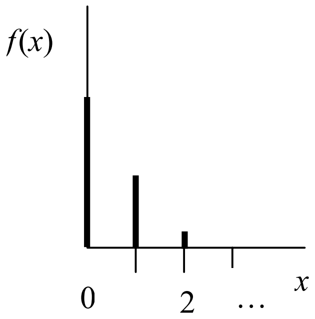
Higher Certificate, Paper I, 2003. Question 5

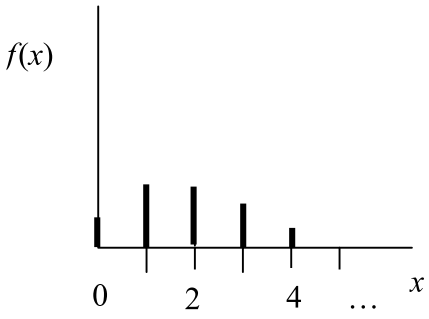
Poisson distribution: $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$. Expectation = variance = λ.

(i) $\lambda = 0.5$: $f(0) = e^{-0.5} = 0.6065$, $f(1) = 0.3033$, $f(2) = 0.0758$, … . Expectation = variance = 0.5.

$\lambda = 2$: $f(0) = 0.1353$, $f(1) = 0.2707$, $f(2) = 0.2707$, $f(3) = 0.1804$, $f(4) = 0.0902$, … . Expectation = variance = 2.

Sketches are as shown.





(ii) Likelihood $L = \prod_{i=1}^{n} \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \dfrac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\Pi x_i!}$.

Taking logarithms to base e,

$$\log L = -n\lambda + (\Sigma x_i)\log\lambda - \log(\Pi x_i!) .$$

Differentiating, $\dfrac{d\log L}{d\lambda} = -n + \dfrac{\Sigma x_i}{\lambda}$; setting this equal to 0 gives the solution $\hat{\lambda}_{ML} = \dfrac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$. We have $\dfrac{d^2\log L}{d\lambda^2} = -\dfrac{\Sigma x_i}{\lambda^2} < 0$, confirming that this is a maximum.

The central limit theorem gives $\bar{X} \sim$ (approx) $\mathrm{N}(\lambda, \lambda/n)$, so we have (approximately)

$$P\left(\lambda - 1.96\sqrt{\frac{\lambda}{n}} \le \bar{X} \le \lambda + 1.96\sqrt{\frac{\lambda}{n}}\right) = 0.95$$

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or

$$P\left(\bar{X} - 1.96\sqrt{\frac{\lambda}{n}} \le \lambda \le \bar{X} + 1.96\sqrt{\frac{\lambda}{n}} \right) = 0.95 .$$

Hence, inserting the observed value $\bar{x}$ and, further, using $\hat{\lambda}_{ML} = \bar{x}$ as an estimate for the underlying variance, an approximate 95% confidence interval for λ is

$$\bar{x} - 1.96\sqrt{\frac{\bar{x}}{n}} \ , \ \bar{x} + 1.96\sqrt{\frac{\bar{x}}{n}} \ .$$

(iii) $n = 400, \quad \Sigma x_i = 2500; \quad \bar{x} = 6.25$.

So the approximate 95% confidence interval is

$$6.25 - 1.96\sqrt{\frac{6.25}{400}} \ , \ 6.25 + 1.96\sqrt{\frac{6.25}{400}}$$

i.e. 6.005 , 6.495 .

Now using $\Sigma x_i^2 = 25600$, we have that the sample variance s^2 is

$$s^2 = \frac{1}{399}\left(25600 - \frac{(2500)^2}{400} \right) = \frac{9975}{399} = 25.00 \ .$$

Using s^2 in the confidence interval gives the interval as

$$6.25 - 1.96\sqrt{\frac{25.00}{400}} \ , \ 6.25 + 1.96\sqrt{\frac{25.00}{400}}$$

i.e. 5.76 , 6.74 .

This interval is twice as wide – because s^2 is four times the size of $\bar{x}$ – which suggests that a Poisson assumption is <u>not</u> valid.

Higher Certificate, Paper I, 2003. Question 6

$E(Y) = np \qquad \text{Var}(Y) = np(1-p)$

(i) Binomial with $n = 48$, $p = ¼$.

(ii) Score is distributed $36 + \text{B}(12, ¼)$.

(a) Hence mean correct is $36 + (12/4) = 39$ and variance is $12 \times ¼ \times ¾ = 9/4$.

(b) Number wrong is distributed B(12, ¾).

(c) The required probability is $1 - P(0) - P(1) - P(2)$ based on the B(12, ¼) distribution. This is

$$1-\left(\frac{3}{4}\right)^{12}-12\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{11}-\frac{12\times 11}{2}\left(\frac{1}{4}\right)^{2}\left(\frac{3}{4}\right)^{10}$$

$$= 1-0.031676-0.126705-0.232293 = 0.6093 .$$

(iii) Number of correct answers for A is distributed as $27 + \text{B}(21, ¼)$.
Number of correct answers for B is distributed as $28 + \text{B}(20, ¼)$.
Number of correct answers for C is distributed as $30 + \text{B}(18, ¼)$.

Means are $27 + (21/4) = 32¼$, $28 + (20/4) = 33$, $30 + (18/4) = 34½$ respectively.

Variances are $(21)(¼)(¾) = 63/16$, $(20)(¼)(¾) = 60/16 = 15/4$, $(18)(¼)(¾) = 54/16 = 27/8$ respectively.

So overall mean is $\frac{1}{3}(32.25+33+34.5) = 33.25$,

and variance of overall mean is $\frac{1}{9}\left(\frac{63}{16}+\frac{15}{4}+\frac{27}{8}\right) = 1.2292$.

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$$P(A|29)=\frac{P(29|A)P(A)}{\sum_{i=A,B,C} P(29|i)P(i)}; \qquad P(i)=\frac{1}{3} \text{ for } i=A,B,C \ .$$

$$P(29|A)=P\left[\mathrm{B}\left(21,\tfrac{1}{4}\right)=2\right]=\frac{21\times 20}{2}\left(\frac{3}{4}\right)^{19}\left(\frac{1}{4}\right)^{2}$$

$$P(29|B)=P\left[\mathrm{B}\left(20,\tfrac{1}{4}\right)=1\right]=20\left(\frac{3}{4}\right)^{19}\left(\frac{1}{4}\right)$$

$$P(29|C)=P\left[\mathrm{B}\left(18,\tfrac{1}{4}\right)=-1\right]=0$$

[Note. C must get at least the 30 he knows, so it must follow that $P(C|29)=0$, which is true if $P(29|C)=0$.]

So $$P(A|29) = \frac{\frac{1}{2}.21.20\left(\frac{3}{4}\right)^{19}\left(\frac{1}{4}\right)^{2}}{\frac{1}{2}.21.20\left(\frac{3}{4}\right)^{19}\left(\frac{1}{4}\right)^{2} + 20\left(\frac{3}{4}\right)^{19}\left(\frac{1}{4}\right)}$$

$$= \frac{\frac{21}{32}}{\frac{21}{32}+\frac{1}{4}} = \frac{21}{29} = 0.7241$$

and similarly

$$P(B|29) = \frac{\frac{1}{4}}{\frac{21}{32}+\frac{1}{4}} = \frac{8}{29} = 0.2759$$

(and $P(C|29) = 0$, see above).

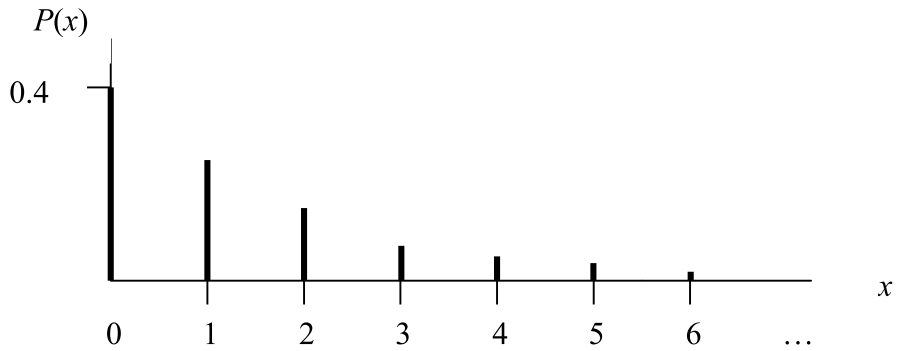
Higher Certificate, Paper I, 2003. Question 7

(i) $P(X = x) = (1 - p)(1 - p)\ldots(1 - p)p$, for $x = 0, 1, 2, \ldots$.

$F \quad F \quad\quad F \quad S$

----- x times -----

When $p = 0.4$, $P(0) = 0.4$, $P(1) = 0.24$, $P(2) = 0.144$, $P(3) = 0.0864$, $P(4) = 0.0518$, $P(5) = 0.0311$, $P(6) = 0.0187$, … .



(ii) The probability generating function of X is

$$G(s) = E\left(s^X\right) = \sum_{i=1}^{\infty} s^{x_i} p_i = \sum_{i=1}^{\infty} s^{x_i} p\left(1-p\right)^{x_i} = p\sum_{i=1}^{\infty} t^{x_i} \quad \text{where } t = (1-p)s$$

$$= p\left(1+t+t^2+t^3+\ldots\right) = p\left(1-t\right)^{-1}$$

$$= \frac{p}{1-\left(1-p\right)s} .$$

The mean is given by $G'(1)$ and the variance by $G''(1) + G'(1) - [G'(1)]^2$, where the differentiation is with respect to s.

$$G'(s) = \frac{p\left(1-p\right)}{\left\{1-\left(1-p\right)s\right\}^2} , \quad \text{so} \quad \text{mean } = G'(1) = \frac{p\left(1-p\right)}{p^2} = \frac{1-p}{p} .$$

$$G''(s) = \frac{2p\left(1-p\right)^2}{\left\{1-\left(1-p\right)s\right\}^3} , \quad \text{so} \quad G''(1) = \frac{2p\left(1-p\right)^2}{p^3} = \frac{2\left(1-p\right)^2}{p^2} .$$

Hence the variance is $\dfrac{2\left(1-p\right)^2}{p^2} + \dfrac{1-p}{p} - \dfrac{\left(1-p\right)^2}{p^2} = \dfrac{\left(1-p\right)^2}{p^2} - \dfrac{1-p}{p} = \dfrac{1-p}{p^2}$.

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(iii) We have $Y = X + 1$.

So $P(Y = y) = p(1-p)^{y-1}$, for $y = 1, 2, 3, \ldots$.

The probability generating function of Y can be obtained by a similar method to that used for X above, or it can be written down using the "linear transformation" result for probability generating functions:

Pgf of Y is $s^b G(as)$ with $a = 1$ and $b = 1$, i.e. $\dfrac{ps}{1-(1-p)s}$.

Mean of Y = (mean of X) + 1 = $1 + \dfrac{1-p}{p} = \dfrac{1}{p}$.

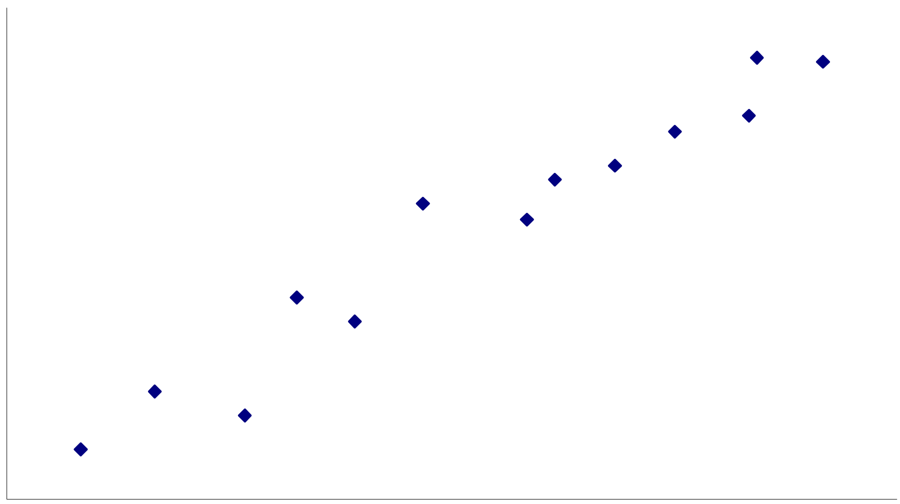
Variance of Y = variance of X.

Higher Certificate, Paper I, 2003. Question 8

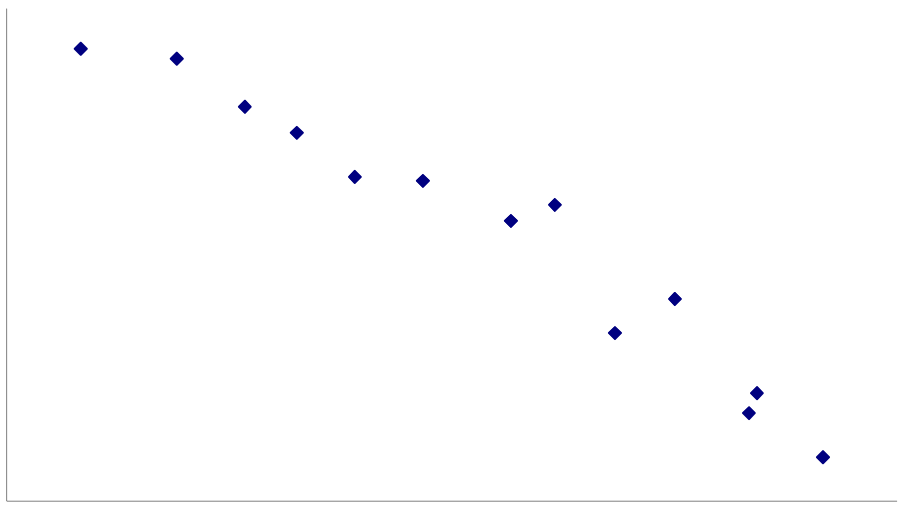
(i) $$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}.$$

This explains the strength of linear relationship between the x_i and y_i, with $r = \pm 1$ showing linearity and $r = 0$ showing no linear relationship. The underlying X and Y are both random variables.

(a) r near to $+1$, small amount of scatter about an (increasing) linear relationship



(b) r near to -1, y decreases as x increases, otherwise as in (a)



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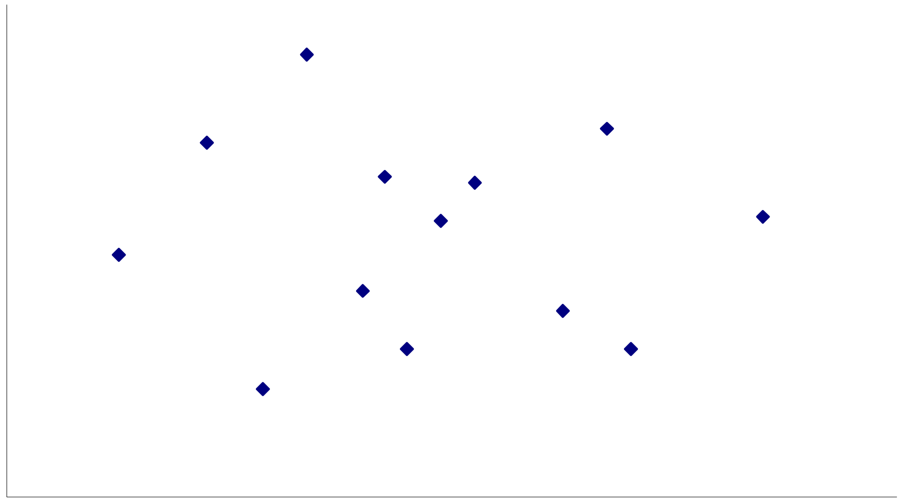
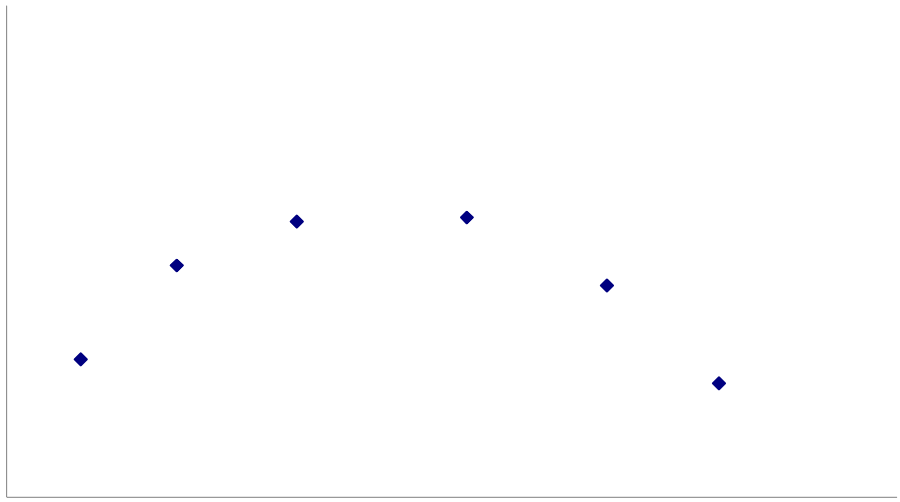
(c) Independent data ($r \approx 0$)



(d) Non-linear relationship, e.g. $y = x^2$



(ii) (a) Simple linear regression of y = cholesterol on x = age. y is the dependent variable, x the independent. Assume a linear relationship underlying the data, $Y_i = a + bx_i + \varepsilon_i$, where the $\{\varepsilon_i\}$ are independent identically distributed $N(0, \sigma^2)$ random variables with σ^2 constant for all i.

(b) $r = \sqrt{(0.323)} = 0.568$ for 'chol' and 'age'.
$r = \sqrt{(0.940)} = 0.970$ for 'newchol' and 'newage'.

The latter consists of the 8 data points omitting the observation at $x = 27$ which seems very far from the roughly linear pattern of the rest. Omitting it has made a linear relationship seem much more plausible. Subject number 2 has very high cholesterol for his age.

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(c) Using the "constant" row in either set of output, the constant term is not significantly different from 0. A model omitting *a* could perhaps be used.

This would imply cholesterol 0 at age 0, which might not be very sensible – but we do not actually have data in that region, so we cannot claim that a linear relationship still holds.

(d) There is a tendency towards a curved relationship even when the very "unusual" observation at age 27 is omitted. The fit of a line without that observation is however much better than with it, and the diagnostic plots, of residuals and Normal probability, seem acceptable.