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AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA

LEAVING CERTIFICATE EXAMINATION, 2000

MATHEMATICS — HIGHER LEVEL — PAPER 1 (300 marks)

THURSDAY, 8 JUNE — MORNING, 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

Marks may be lost if all necessary work is not clearly shown.

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1. **(a)** Show that the following simplifies to a constant when $x \neq 2$

$$\frac{3x-5}{x-2}+\frac{1}{2-x}.$$

(b) $f(x) = ax^3 + bx^2 + cx + d$ where $a,\ b,\ c,\ d \in \mathbf{R}$.

If k is a real number such that $f(k) = 0$, prove that $x - k$ is a factor of $f(x)$.

(c) $(x-t)^2$ is a factor of $x^3 + 3px + c$.

Show that

(i) $p = -t^2$

(ii) $c = 2t^3$.

2. **(a)** Solve for x, y, z

$$3x - y + 3z = 1$$
$$x + 2y - 2z = -1$$
$$4x - y + 5z = 4.$$

(b) Solve $x^2 - 2x - 24 = 0$.

Hence, find the values of x for which

$$\left(x+\frac{4}{x}\right)^2 - 2\left(x+\frac{4}{x}\right) - 24 = 0, \qquad x \in \mathbf{R},\ x \neq 0.$$

(c) **(i)** Express $a^4 - b^4$ as a product of three factors.

(ii) Factorise $a^5 - a^4b - ab^4 + b^5$.

Use your results from **(i)** and **(ii)** to show that

$$a^5 + b^5 > a^4b + ab^4$$

where a and b are positive unequal real numbers.

3. **(a)** Given that A = $\begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}$, find $B^{-1}A$.

(b) **(i)** Simplify $\left(\dfrac{-2+3i}{3+2i}\right)$ and hence, find the value of $\left(\dfrac{-2+3i}{3+2i}\right)^9$ where $i^2 = -1$.

(ii) Find the two complex numbers $a+ib$ such that

$$(a+ib)^2 = 15-8i\,.$$

(c) Use De Moivre's theorem

(i) to prove that $\cos 3q = 4\cos^3 q - 3\cos q$

(ii) to express $\left(-\sqrt{3}-i\right)^{10}$ in the form $2^n\left(1-i\sqrt{k}\right)$ where $n,\ k \in \mathbf{N}$.

4. **(a)** The first three terms of a geometric sequence are

$$2x-4,\quad x+1,\quad x-3\,.$$

Find the two possible values of x.

(b) Given that

$$u_n = \frac{1}{2}\left(4^n - 2^n\right)$$

for all integers n, show that

$$u_{n+1} = 2u_n + 4^n\,.$$

(c) **(i)** Given that $g(x) = 1 + 2x + 3x^2 + 4x^3 \ldots$ where $-1 < x < 1$, show that

$$g(x) = \frac{1}{(1-x)^2}\,.$$

(ii) $P(n) = u_1 u_2 u_3 u_4 \ldots\ u_n$ where

$u_k = ar^{k-1}$ for $k = 1, 2, 3, \ldots, n$ and $a, r \in \mathbf{R}$.

Write $P(n)$ in the form $a^n r^{f(n)}$ where $f(n)$ is a quadratic expression in n.

5. **(a)** Express the recurring decimal $1.\dot{2}$ in the form $\frac{a}{b}$ where $a,\ b \in \mathbf{N}$.

(b) Prove by induction that $n! > 2^n,\ n \in \mathbf{N},\ n \geq 4$.

(c) **(i)** Solve for x

$$2\log_9 x = \frac{1}{2} + \log_9(5x+18), \quad x > 0.$$

(ii) Solve for x

$$3e^x - 7 + 2e^{-x} = 0.$$

6. **(a)** Differentiate with respect to x

(i) $(1+5x)^3$

(ii) $\frac{7x}{x-3}, \quad x \neq 3.$

(b) **(i)** Prove, from first principles, the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $u = u(x)$ and $v = v(x)$.

(ii) Given $y = \sin^{-1}(2x-1)$, find $\frac{dy}{dx}$ and calculate its value at $x = \frac{1}{2}$.

(c) $f(x) = \frac{1}{x+1}$ where $x \in \mathbf{R},\ x \neq -1.$

(i) Find the equations of the asymptotes of the graph of $f(x)$.

(ii) Prove that the graph of $f(x)$ has no turning points or points of inflection.

(iii) If the tangents to the curve at $x = x_1$ and $x = x_2$ are parallel and if $x_1 \neq x_2$, show that

$$x_1 + x_2 + 2 = 0.$$

7. **(a)** Find the slope of the tangent to the curve

$$x^2 - xy + y^2 = 1 \text{ at the point } (1,0).$$

(b) The parametric equations of a curve are

$$x = \cos^3 t \quad \text{and} \quad y = \sin^3 t, \quad 0 \le t \le \frac{p}{2}.$$

(i) Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ in terms of t.

(ii) Hence, find integers a and b such that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a}{b}(\sin 2t)^2.$$

(c) $f(x) = \dfrac{\ln x}{x}$ where $x > 0$.

(i) Show that the maximum of $f(x)$ occurs at the point $\left(e, \dfrac{1}{e}\right)$.

(ii) Hence, show that $x^e \le e^x$ for all $x > 0$.

8. **(a)** Find **(i)** $\int (x^2 + 2)\,dx$ **(ii)** $\int e^{3x}\,dx.$

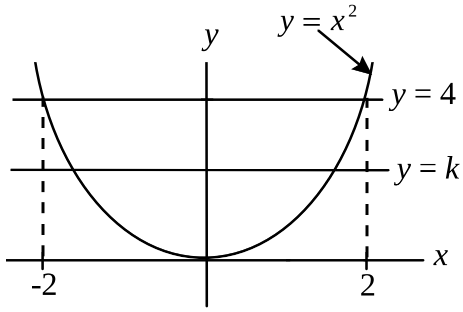
(b) Evaluate **(i)** $\displaystyle\int_0^{\frac{p}{2}} \sin^2 3q \; dq$ **(ii)** $\displaystyle\int_0^1 \frac{x}{x^2+4}\,dx.$

(c) **(i)** Find the value of the real number p given that

$$\int_2^p \frac{dx}{x^2 - 4x + 5} = \frac{p}{4}.$$

(ii) The region bounded by the curve $y = x^2$ and the line $y = 4$ is divided into two regions of equal area by the line $y = k$.

Show that $k^3 = 16$.



AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA

LEAVING CERTIFICATE EXAMINATION, 2000

MATHEMATICS – HIGHER LEVEL

PAPER 2 (300 marks)

FRIDAY, 9 JUNE — MORNING, 9.30 to 12.00

Attempt **FIVE** Questions from Section **A** and **ONE** Question from Section **B**.
Each question carries 50 marks.

Marks may be lost if all necessary work is not clearly shown.

SECTION A

Answer FIVE questions from this section.

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1. **(*a*)** The equation of a circle is $x^2 + y^2 = 130$.

Find the slope of the tangent to the circle at the point $(-7, 9)$.

(*b*) $x^2 + y^2 - 6x + 4y - 12 = 0$ is the equation of a circle.

Write down the coordinates of its centre and the length of its radius.

$x^2 + y^2 + 12x - 20y + k = 0$ is another circle, where $k \in \mathbf{R}$.

The two circles touch externally. Find the value of k.

(*c*) A circle intersects a line at the points $a(-3, 0)$ and $b(5, -4)$.

The midpoint of $[ab]$ is m. Find the coordinates of m.

The distance from the centre of the circle to m is $\sqrt{5}$.

Find the equations of the two circles that satisfy these conditions.

2. **(*a*)** $\vec{v} = t\,\vec{i} - 8\vec{j}$, where $t \in \mathbf{R}$.

Find the two possible values of t for which $|\vec{v}| = 17$.

(*b*) $\vec{a} = 2\,\vec{i} + (2k+3)\,\vec{j}$ and $\vec{b} = k^2\,\vec{i} + 6\,\vec{j}$, where $k \in \mathbf{Z}$.

$\vec{a}$ is perpendicular to $\vec{b}$.

(i) Find the value of k.

(ii) Using your value for k, write $\vec{a} + \vec{b}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) Hence, find the measure of the angle between $\vec{a}$ and $\vec{a} + \vec{b}$ correct to the nearest degree.

(*c*) **(i)** $\vec{p} + \vec{q} = 5\vec{i} - 5\vec{j}$ and $\vec{p}\vec{q} = 3\vec{i} + \vec{j}$.

Express $\vec{p}$ and $\vec{q}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Given that $\vec{r} = \dfrac{\vec{p}.\vec{q}}{\vec{q}.\vec{q}}\,\vec{q}$, express $\vec{r}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) Given that $\vec{s} = -\dfrac{7}{2}\,\vec{i} + m\,\vec{j}$, $m \in \mathbf{Q}$, find the value of m for which the origin, r and s are collinear.

3. **(*a*)** The equation of the line L is $14x + 6y + 1 = 0$.

Find the equation of the line perpendicular to L that contains the point $(3, -2)$.

(*b*) $a(1, -2)$ and $c(-4, 8)$ are two points.

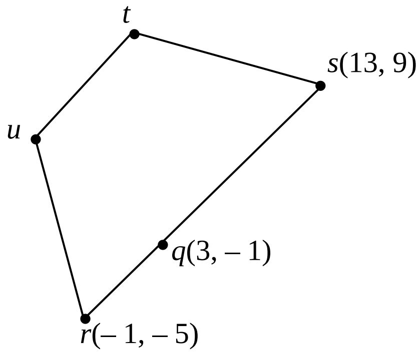
f is the transformation $(x, y) \rightarrow (x', y')$ where $x' = 2x - 3y$ and $y' = 6x + y$.

(i) b divides $[ac]$ in the ratio $3 : 2$. Find the coordinates of b.

(ii) Find $f(a)$, $f(b)$ and $f(c)$.

(iii) Verify that $|f(a)f(b)| : |f(b)f(c)| = |ab| : |bc|$.

(*c*) $rstu$ is a quadrilateral where r is $(-1, -5)$ and s is $(13, 9)$.
$q(3, -1)$ lies between r and s.



(i) The coordinates of u are $(-2k, 3k)$ where $k \in \mathbf{R}$ and $k > 0$.

The area of triangle rqu is 28 square units.

Find the value of k.

(ii) The slope of ts is $-\dfrac{3}{11}$.
sr is parallel to tu.

Find the coordinates of t.

4. **(*a*)** The area of a sector of a circle is 27 cm^2. The length of the radius of the circle is 6 cm.

Find, in radians, the measure of the angle in the sector.

(*b*) Find all the solutions of the equation

$$15\sin^2 x - 4\cos x - 11 = 0$$

in the domain $0° \leq x \leq 360°$. Give your answers correct to the nearest degree.

(*c*) **(i)** Derive the formula $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

(ii) Show that $\cos(A+B)\cos B + \sin(A+B)\sin B = \cos A$.

5. **(*a*)** Evaluate **(i)** $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$ **(ii)** $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin x}$.

(*b*) **(i)** Show that $\dfrac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A$.

(ii) Hence, or otherwise, find the values of the integers l and k such that

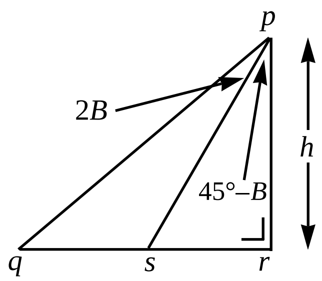
$$\frac{1-\tan^2(135^\circ - A)}{1+\tan^2(135^\circ - A)} = l\sin kA$$

for all values of A for which $\tan(135^\circ - A)$ is defined.

(*c*) In the triangle pqr, $|\angle qrp| = 90^\circ$ and $|rp| = h$.

s is a point on $[qr]$ such that $|\angle spq| = 2B$ and

$|\angle rps| = 45^\circ - B$, $0^\circ < B < 45^\circ$.



(i) Show that $|sr| = h\tan(45^\circ - B)$.

(ii) Hence, or otherwise, show that $|qs| = 2h\tan 2B$.

6. **(*a*)** A bank gives each of its customers a four digit personal identification number which is formed from the digits 0 to 9 inclusive. Examples are 2475, 0865 and 3422.

(i) How many different personal identification numbers can the bank use?

(ii) If the bank decides not to use personal identification numbers that begin with 0, how many different numbers can it then use?

(*b*) **(i)** Solve the difference equation $12u_{n+2} - 8u_{n+1} + u_n = 0$, where $n \geq 0$,

given that $u_0 = \dfrac{1}{15}$ and $u_1 = \dfrac{7}{30}$.

(ii) Hence, express u_3 in the form $\dfrac{p}{q}$ where $p, q \in \mathbf{N}$.

(*c*) Six red discs, numbered from 1 to 6, and four green discs, numbered from 7 to 10, are placed in box A.

Ten blue discs, numbered from 1 to 10, are placed in box B.

Two discs are drawn from box A and two discs are drawn from box B. The four discs are drawn at random and without replacement.

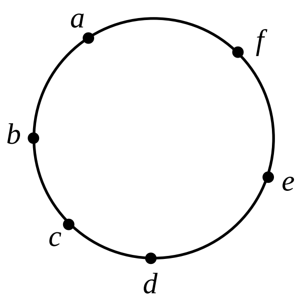
Find the probability that the discs drawn are

(i) two red discs and two even numbered blue discs

(ii) one red disc, one green disc and two blue discs with all four discs odd numbered

(iii) one red disc, one green disc and two blue discs with the total on the red and green discs equal to 10 and the total on the blue discs also equal to 10.

7. **(*a*)** The points a, b, c, d, e and f lie on a circle.



(i) If these points are used as vertices, how many different quadrilaterals can be formed?

(ii) How many of these quadrilaterals will have [ab] as one side?

(*b*) Three cards are drawn, at random and without replacement, from a pack of 52 playing cards.

Find the probability that

(i) the three cards drawn are the Jack of clubs, the Queen of clubs and the King of clubs

(ii) the three cards are aces

(iii) two cards are black and one card is a diamond

(iv) the three cards are of the same colour.

(*c*) The mean of the real numbers q, r, s and t is $\bar{x}$ and the standard deviation is σ.

Consider the numbers

$$\beta q + \alpha,\ \beta r + \alpha,\ \beta s + \alpha \text{ and } \beta t + \alpha$$

where $\beta, \alpha \in \mathbf{R}$ and $\beta > 0$.

(i) Show that the mean of these numbers is $\beta\bar{x} + \alpha$.

(ii) Show that the standard deviation of these numbers is $\beta\sigma$.

SECTION B

Answer ONE question from this section.

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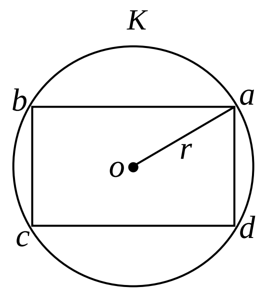
8. **(*a*)** Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{(n+2)!}{2^{n+2}}$ is divergent.

(*b*) **(i)** Use integration by parts to find $\int e^{2x} \cos x dx$.

(ii) Given that $\int_0^{\frac{\pi}{2}} e^{2x} \cos x dx = \frac{1}{n}(e^{\pi} - 2)$, find the value of n where $n \in \mathbf{N}$.

(*c*) K is a circle with centre o.



a, b, c and d are points on K such that $abcd$ is a rectangle.

$|oa| = r$ cm; $|ab| = 2x$ cm and $|ad| = 2y$ cm.

(i) Express y in terms of x and r.

(ii) Hence, or otherwise, show that the maximum area of $abcd$ is $2r^2$ cm^2.

9. **(*a*)** There are thirteen tickets in a draw. Six of the tickets are blue, four are red and three are green.

Three tickets are drawn at random, one at a time, without replacement.

Find the probability that the first ticket drawn is blue, the second is red and the third is red or green.

(*b*) The heights of students in a certain college are normally distributed with mean 165 cm and standard deviation 10 cm. If a student is chosen at random find the probability that the student's height is

(i) less than 170 cm

(ii) between 160 cm and 180 cm.

(*c*) The speeds of 150 randomly selected cars were recorded as they passed a check-point on a motorway. The mean speed of the cars was 115 km/h and the standard deviation was 24 km/h. The speed limit on the motorway is 112 km/h.

(i) At the 5% level of significance, does this indicate that the mean speed of cars passing the check-point is greater than the speed limit?

(ii) Find the 95% confidence interval for the mean speed of cars passing the check-point correct to one decimal place.

10. **(*a*)** G = $\{R_{0°}, R_{120°}, R_{240°}\}$ is the set of rotational symmetries of an equilateral triangle.

Show that G is a group under composition. You may assume that composition is associative.

(*b*) H = $\{e, f, g, h, m, p, s, t\}$ where

$$e=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix},\ f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix},\ g=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\ h=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix},$$

$$m=\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix},\ p=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix},\ s=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix},\ t=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}.$$

H,$\circ$ is a group where $\circ$ denotes composition of permutations.

(i) Investigate if $f \circ m = m \circ f$.

(ii) The elements g, m, p, s and t have the same order. What is their order?

(iii) The elements f and h have the same order. What is their order?

(iv) Find the subgroup of H that is generated by the element f.

(v) Show that this subgroup is isomorphic to the group $\{1, -1, i, -i\}$, x where $i^2 = -1$ and x denotes multiplication.

11. **(*a*)** An ellipse has centre (0,0). One of its foci is $(\sqrt{57}, 0)$. The length of its minor axis is 16. Find the equation of the ellipse.

(*b*) The point $p(x, y)$ is such that its distance from the point $(-ae, 0)$ is e times its distance from the line $ex + a = 0$ where $0 < e < 1$ and $a \in \mathbf{R}$.

Show that p satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

(*c*) f is a similarity transformation. The image of the line segment $[pq]$ under f is the line segment $[p'q']$.

If the line M is the perpendicular bisector of $[pq]$, prove that $f(M)$ is the perpendicular bisector of $[p'q']$.

Hence, prove that the circumcentre of the triangle pqr is mapped onto the circumcentre of the triangle $p'q'r'$ under f.