AN ROINN OIDEACHAIS AGUS EOLAÍOCHTA | DEPARTMENT OF EDUCATION AND SCIENCE

Scéim Mharcála — *Scrúduithe Ardteistiméireachta, 2002*

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Marking Scheme — *Leaving Certificate Examination, 2002*

Applied Mathematics — *Higher Level*

General Guidelines

1 Penalties of three types are applied to candidates' work as follows:

Slips	- numerical slips	S(-1)
Blunders	- mathematical errors	B(-3)
Misreading	- if not serious	M(-1)

Serious blunder or omission or misreading which oversimplifies:
- award the attempt mark only.

Attempt marks are awarded as follows: 5 (att 2).

2 Mark all answers, including excess answers and repeated answers whether cancelled or not, and award the marks for the best answers.

3 Mark scripts in red unless candidate uses red. If a candidate uses red, mark the script in blue or black.

4 Number the grid on each script 1 to 10 in numerical order, not the order of answering.

5 Scrutinise **all** pages of the answer book.

6 The marking scheme shows one correct solution to each question. In many cases there are other equally valid methods. Some common alternative methods are given at the end of the scheme.

1. **(a)** A stone is thrown vertically upwards under gravity with a speed of *u* m/s from a point 30 metres above the horizontal ground. The stone hits the ground 5 seconds later.

(i) Find the value of *u*.

(ii) Find the speed with which the stone hits the ground.

(i)	$s = ut + \frac{1}{2}at^2$	
	$-30 = u(5) + \frac{1}{2}(-9.8)5^2$	5,5
	$\Rightarrow \quad 5u = 4.9(25) - 30$	
	$\Rightarrow \quad u = 18.5\ \text{ms}^{-1}$	5
(ii)	$v = u + at$	
	$v = 18.5 + (-9.8)5$	
	$\Rightarrow \quad v = -30.5\ \text{ms}^{-1}$	5
	$\Rightarrow \quad \text{speed} = 30.5\ \text{ms}^{-1}$	
		20

(b) A particle, with initial speed u, moves in a straight line with constant acceleration.
During the time interval from 0 to t, the particle travels a distance p.
During the time interval from t to $2t$, the particle travels a distance q.
During the time interval from $2t$ to $3t$, the particle travels a distance r.

(i) Show that $2q = p + r$.

(ii) Show that the particle travels a further distance $2r - q$ in the time interval from $3t$ to $4t$.

(i)

$$p = ut + \tfrac{1}{2}at^2$$ 5

$$p + q = u(2t) + \tfrac{1}{2}a(2t)^2 \quad \text{or} \quad q = ut + \tfrac{3}{2}at^2$$ 5

$$p + q + r = u(3t) + \tfrac{1}{2}a(3t)^2 \quad \text{or} \quad r = ut + \tfrac{5}{2}at^2$$ 5

$$q = ut + \tfrac{3}{2}at^2$$
$$r = ut + \tfrac{5}{2}at^2$$

$$2q = 2ut + 3at^2$$
$$= p + r$$ 5

(ii)

$$p + q + r + x = u(4t) + \tfrac{1}{2}a(4t)^2 \quad \text{or} \quad x = ut + \tfrac{7}{2}at^2$$ 5

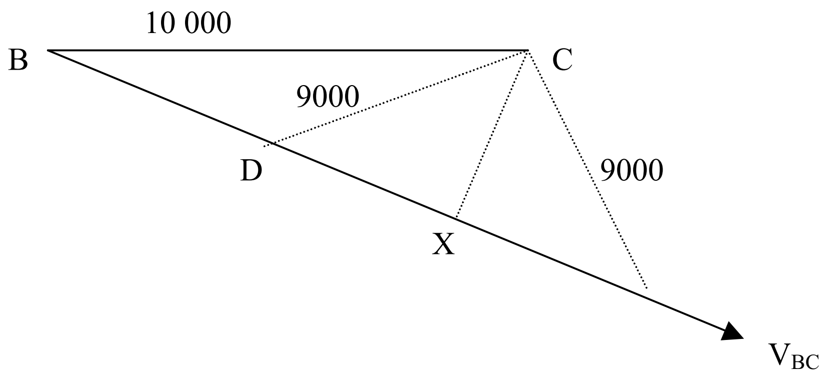
Also $2r - q = ut + \tfrac{7}{2}at^2$
$= x$ 5

30

2. **(a)** Two boats, B and C, are each moving with constant velocity.
At a certain instant, boat B is 10 km due west of boat C.
The speed and direction of boat B relative to boat C is 2.5 m/s in the direction 60° south of east.

(i) Calculate the shortest distance between the boats, to the nearest metre.

(ii) Calculate the length of time, to the nearest second, for which the boats are less than or equal to 9 km apart.



(i) shortest distance $= |CX|$

$= 10\,000 \sin 60$

$= 5000\sqrt{3}$ 5

$= 8660$ m 5

(ii) $|DX| = \sqrt{9000^2 - (5000\sqrt{3})^2}$ or 2449.49 5

$\text{time} = \dfrac{2|DX|}{V_{BC}}$ 5

$= \dfrac{4898.98}{2.5}$

$= 1960$ s 5 | 25

(b) The velocity of ship P relative to a steady wind is 20 km/hr in the direction 80° north of east.
The velocity of ship Q relative to the same steady wind is 10 km/hr in the direction 20° south of west.

Calculate the magnitude and direction of the velocity of ship P relative to ship Q.

Give your answers to the nearest km and the nearest degree, respectively.

$$V_{PQ} = V_P - V_Q \qquad 5$$

$$= \{V_{PW} + V_W\} - \{V_{QW} + V_W\}$$

$$= \{V_{PW}\} - \{V_{QW}\}$$

$$= \{20\cos 80\,\vec{i} + 20\sin 80\,\vec{j}\} - \{-10\cos 20\,\vec{i} - 10\sin 20\,\vec{j}\} \qquad 5$$

$$= \{3.473 + 9.397\}\vec{i} + \{19.696 + 3.420\}\vec{j}$$

$$= \{12.870\}\vec{i} + \{23.116\}\vec{j} \quad \text{or} \quad 12.9\,\vec{i} + 23.1\,\vec{j} \qquad 5$$

$$|V_{PQ}| = \sqrt{12.870^2 + 23.116^2} = 26.457$$

$$\Rightarrow \quad \text{magnitude} = 26 \text{ km/hr} \qquad 5$$

$$\tan\alpha = \frac{23.116}{12.870} = 1.7961$$

$$\Rightarrow \quad \text{direction} = 61° \quad \text{north of east} \qquad 5 \qquad \boxed{25}$$

3. **(a)** A particle is projected from a point on the horizontal ground with a speed of 39.2 m/s inclined at an angle α to the horizontal ground. The particle is at a height of 14.7 m above the horizontal ground at times t_1 and t_2 seconds, respectively.

(i) Show that $t_2 - t_1 = \sqrt{64\sin^2\alpha - 12}$.

(ii) Find the value of α for which $t_2 - t_1 = \sqrt{20}$.

(i)	$r_j = 14.7$	5	
	$39.2\sin\alpha.\mathrm{t} - \frac{1}{2}g\mathrm{t}^2 = 14.7$	5	
	$\mathrm{t}^2 - 8\sin\alpha.\mathrm{t} + 3 = 0$		
	$t = \dfrac{8\sin\alpha \pm \sqrt{64\sin^2\alpha - 12}}{2}$	5	
	$\mathrm{t}_1 = \dfrac{8\sin\alpha - \sqrt{64\sin^2\alpha - 12}}{2}$		
	$\mathrm{t}_2 = \dfrac{8\sin\alpha + \sqrt{64\sin^2\alpha - 12}}{2}$		
	$\mathrm{t}_2 - \mathrm{t}_1 = \sqrt{64\sin^2\alpha - 12}$	5	
(ii)	$\mathrm{t}_2 - \mathrm{t}_1 = \sqrt{20}$		
	$\sqrt{64\sin^2\alpha - 12} = \sqrt{20}$		
	$\sin\alpha = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$		
	$\alpha = 45°$	5	25

(b) A particle is projected with velocity u m/s at an angle θ **to the horizontal**, up a plane inclined at an angle β to the horizontal. (The plane of projection is vertical and contains the line of greatest slope). The particle strikes the plane at right angles.

(i) Show that $2\tan\beta\tan(\theta-\beta) = 1$.

(ii) Hence, or otherwise, show that if $\theta = 2\beta$, the range of the particle up the inclined plane is $\dfrac{u^2}{g\sqrt{3}}$.

(i)

$$r_j = 0$$

$$u\sin(\theta-\beta).t - \tfrac{1}{2}g\cos\beta.t^2 = 0$$ 5

$$\Rightarrow \quad t = \frac{2u\sin(\theta-\beta)}{g\cos\beta}$$

$$v_i = 0$$

$$u\cos(\theta-\beta) - g\sin\beta.t = 0$$ 5

$$\Rightarrow \quad t = \frac{u\cos(\theta-\beta)}{g\sin\beta}$$

$$\frac{2u\sin(\theta-\beta)}{g\cos\beta} = \frac{u\cos(\theta-\beta)}{g\sin\beta}$$

$$2\tan\beta.\tan(\theta-\beta) = 1$$ 5

(ii)

$$\theta = 2\beta \quad \Rightarrow \quad \tan\beta = \tfrac{1}{\sqrt{2}}$$

$$\Rightarrow \quad t = \frac{u\sqrt{2}}{g}$$ 5

$$\text{Range} = u\cos(2\beta-\beta).t - \tfrac{1}{2}g\sin\beta.t^2$$

$$= u\left\{\frac{\sqrt{2}}{\sqrt{3}}\right\}.\left\{\frac{u\sqrt{2}}{g}\right\} - \tfrac{1}{2}g\left\{\frac{1}{\sqrt{3}}\right\}.\left\{\frac{2u^2}{g^2}\right\}$$

$$= \frac{u^2}{g\sqrt{3}}$$ 5 25

4. **(a)** A particle is describing simple harmonic motion with period $\frac{\pi}{4}$ seconds about a point *o*. When the particle is 6 cm from the point *o*, its speed is $8\sqrt{13}$ cm/s.

Find the amplitude of the motion.

$$\text{Period} = \frac{2\pi}{\omega}$$

$$\frac{\pi}{4} = \frac{2\pi}{\omega}$$

$$\Rightarrow \quad \omega = 8 \qquad 5$$

$$v = \omega\sqrt{a^2 - x^2} \qquad 5$$

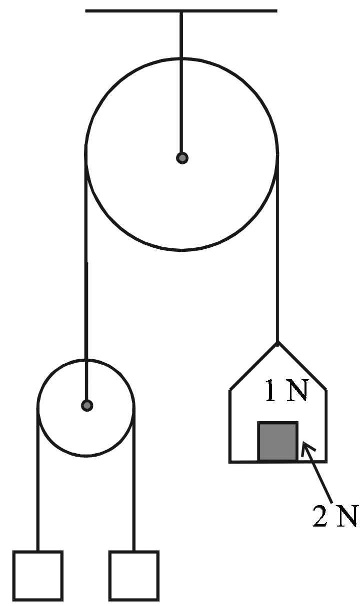
$$8\sqrt{13} = 8\sqrt{a^2 - 6^2}$$

$$13 = a^2 - 36$$

$$\Rightarrow \quad a = 7\text{ cm} \qquad 5 \qquad 15$$

(b) A smooth light pulley is connected by a light inextensible string passing over a smooth light fixed pulley to a scale pan of weight 1 N.
A particle of weight 2 N is placed symmetrically on the centre of the scale pan.
Two particles of weight 3 N and 9 N are connected by a light inextensible string passing over the smooth light pulley.
The system is released from rest.

The acceleration of the scale pan is $\frac{g}{2}$ m/s^2 vertically upwards.



(i) Find the acceleration, in terms of g, of the particle of weight 9 N.

(ii) Find the normal reaction (force) between the particle of weight 2 N and the scale pan.

(i) Mass of two particles : $\frac{3}{g}$ and $\frac{9}{g}$ kg — 5

$$9 - T_1 = \frac{9}{g}\left\{a + \frac{g}{2}\right\}$$ 5

$$T_1 - 3 = \frac{3}{g}\left\{a - \frac{g}{2}\right\}$$ 5

$$6 = \frac{12a}{g} + 3$$ 5

$$\Rightarrow \quad a = \frac{3g}{12} \text{ or } \frac{g}{4}$$ 5

$$\Rightarrow \quad \left\{a + \frac{g}{2}\right\} = \frac{3g}{4}$$

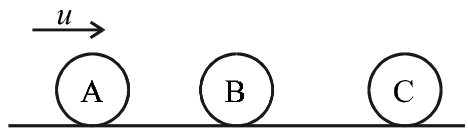
(ii) Force = mass × acceleration

$$R - 2 = \frac{2}{g}\left\{\frac{g}{2}\right\}$$ 5

$$\Rightarrow \quad R = 3 \text{ N}$$ 5

35

5. **(a)** Three identical smooth spheres, A, B and C, lie at rest on a smooth horizontal table with their centres in a straight line.



Sphere A is projected towards B with speed u. Sphere A collides directly with B and then B collides directly with C. Sphere C moves, after the collision, with a speed of $\frac{5u}{8}$.

The coefficient of restitution for each of the two collisions is e.

Find e, correct to two places of decimals.

PCM $\quad m(u)+m(0) = mv_1 + mv_2$	5	
NEL $\quad v_1 - v_2 = -e(u-0)$	5	
$\Rightarrow \quad v_2 = \frac{u}{2}(1+e)$	5	
PCM $\quad m.\left\{\frac{u}{2}(1+e)\right\} + m(0) = mv_3 + m.\left\{\frac{5u}{8}\right\}$ NEL $\quad v_3 - \left\{\frac{5u}{8}\right\} = -e\left\{\frac{u}{2}(1+e) - 0\right\}$	5	
$2e^2 + 4e - 3 = 0$		
$e = \frac{\sqrt{40}-4}{4}$		
$= 0.58$	5	25

(b) A smooth sphere P collides with an identical smooth sphere Q which is at rest. The velocity of P before impact makes an angle α with the line of centres at impact, where $0° \le \alpha < 90°$.
The velocity of P is deflected through an angle ϑ by the collision, so that its velocity after impact makes an angle $\vartheta + \alpha$ with the line of centres at impact.
The coefficient of restitution between the spheres is $\frac{1}{4}$.
Show that $\tan\vartheta = \dfrac{5\tan\alpha}{3 + 8\tan^2\alpha}$.

PCM $\quad mu\cos\alpha + m(0) = mv_1 + mv_2$ **5**

NEL $\quad v_1 - v_2 = -\frac{1}{4}(u\cos\alpha - 0)$ **5**

$$\Rightarrow \quad v_1 = \tfrac{3}{8}u\cos\alpha$$ **5**

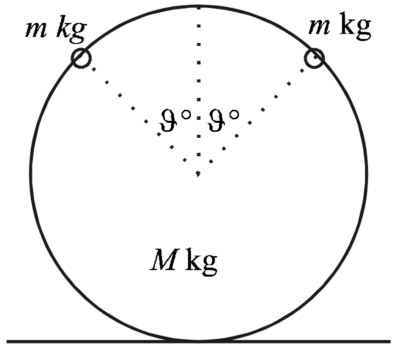
$$\tan(\vartheta + \alpha) = \frac{u\sin\alpha}{\frac{3}{8}u\cos\alpha}$$ **5**

$$\frac{\tan\vartheta + \tan\alpha}{1 - \tan\vartheta\tan\alpha} = \frac{8\tan\alpha}{3}$$

$$3\tan\vartheta + 3\tan\alpha = 8\tan\alpha - 8\tan\vartheta\tan^2\alpha$$

$$\Rightarrow \quad \tan\vartheta = \frac{5\tan\alpha}{3 + 8\tan^2\alpha}$$ **5** **25**

6. A smooth uniform vertical hoop of radius r and mass M kg stands in a vertical plane on a horizontal surface. The hoop threads two small rings, each of mass m kg. The rings are released from rest at the top of the hoop.



(i) When the two rings have each fallen through an angle of $\vartheta°$ on opposite sides of the hoop, show that the normal force of reaction exerted by the hoop on each ring is

$$mg(3\cos\vartheta - 2)\text{ N},$$

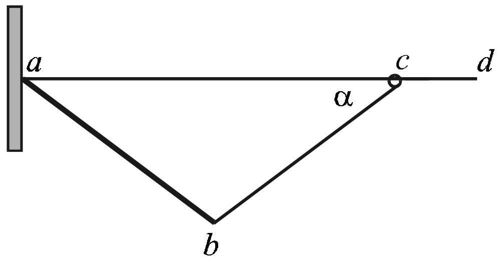
where this force is taken to act in the outward direction from the centre of the hoop.

(ii) Show that the hoop will rise from the table if $m > \dfrac{3M}{2}$.

(i)	$\frac{1}{2}mv^2 = mgr\{1-\cos\vartheta\}$	5, 5
	$mg\cos\vartheta - N = \dfrac{mv^2}{r}$	5, 5
	$mg\cos\vartheta - N = 2mg\{1-\cos\vartheta\}$	
	$\Rightarrow N = mg\{3\cos\vartheta - 2\}$	5
(ii)	$R - 2N\cos\vartheta = Mg$	5
	$R = 0$	5
	$2\{mg\{2-3\cos\vartheta\}\}\{\cos\vartheta\} = Mg$	
	$0 = 6m\cos^2\vartheta - 4m\cos\vartheta + M$	
	$\cos\vartheta = \dfrac{4m \pm \sqrt{16m^2 - 24mM}}{12m}$	5
	Hoop rises if $16m^2 > 24mM$	5
	$\Rightarrow m > \dfrac{3M}{2}$	5

50

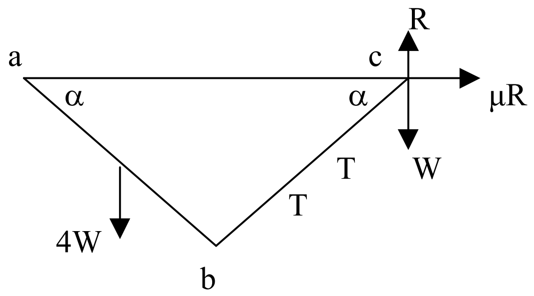
7. A uniform rod, [ab], of weight $4W$ and length $2l$, is free to rotate smoothly about the fixed point a. A fixed wire, [ad], extends horizontally from a. The end b of the rod is attached by a light inelastic string, [bc], of length $2l$, to a ring, of weight W and negligible diameter, which can slide on the wire. The coefficient of friction between the ring and the wire is μ.



The string makes an angle α with the horizontal when the system is in limiting equilibrium (that is, just on the point of slipping).

(i) Show that $\tan\alpha = \dfrac{1}{2\mu}$.

(ii) Show that the tension in the string is $W\sqrt{1+4\mu^2}$.



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(i) Take moments about a for system:

$4W \times \ell\cos\alpha + W \times 4\ell\cos\alpha = R \times 4\ell\cos\alpha$ 5,5

$\Rightarrow \quad R = 2W$ 5

Resolve forces at c:

vert: $T\sin\alpha + W = R \quad \Rightarrow \quad T\sin\alpha = W$ 5

horiz: $T\cos\alpha = \mu R \quad \Rightarrow \quad T\cos\alpha = \mu 2W$ 5

$\Rightarrow \quad \tan\alpha = \dfrac{W}{\mu 2W} = \dfrac{1}{2\mu}$ 5

OR

Take moments about b for c (or bc):

$W \times 2\ell\cos\alpha + \mu R \times 2\ell\sin\alpha = R \times 2\ell\cos\alpha$ 5,5

$W + \mu(2W) \times \tan\alpha = 2W$

$\Rightarrow \quad \tan\alpha = \dfrac{W}{\mu 2W} = \dfrac{1}{2\mu}$ 5

40

(ii) $\tan\alpha = \dfrac{1}{2\mu}$

$\Rightarrow \quad \sin\alpha = \dfrac{1}{\sqrt{1+4\mu^2}}$ or $\cos\alpha = \dfrac{2\mu}{\sqrt{1+4\mu^2}}$ — 5

Resolve forces at c :

vert.	horiz.
$T\sin\alpha + W = R$	$T\cos\alpha = \mu R$
$T = \dfrac{2W - W}{\sin\alpha}$	$T = \dfrac{\mu(2W)}{\cos\alpha}$
$T = W\sqrt{1+4\mu^2}$	$T = W\sqrt{1+4\mu^2}$

5 — 10

OR

(ii) Resolve forces at c :

vert.	horiz.
$T\sin\alpha + W = R$	$T\cos\alpha = \mu R$
$T\sin\alpha = W$	$T\cos\alpha = 2\mu W$

$$\{T\sin\alpha\}^2 + \{T\cos\alpha\}^2 = W^2 + \{2\mu W\}^2$$ 5

$$T^2 = W^2 + 4\mu^2 W^2$$

$$T = W\sqrt{1+4\mu^2}$$ 5 — 10

8. **(a)** Prove that the moment of inertia of a uniform rod of mass m and length $2l$ about an axis through its centre perpendicular to the rod is $\frac{1}{3}ml^2$.

$$\text{Let } M = \text{mass per unit length}$$

$$\text{mass of element} = M\{dx\}$$

$$\text{moment of inertia of the element} = M\{dx\}x^2 \quad 5$$

$$\text{moment of inertia of the rod} = M\int_{-\ell}^{\ell} x^2\, dx \quad 5$$

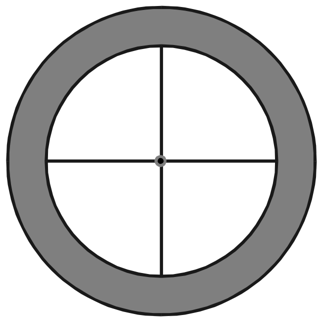
$$= M\left[\frac{x^3}{3}\right]_{-\ell}^{\ell} \quad 5$$

$$= \tfrac{2}{3}M\ell^3$$

$$= \tfrac{1}{3}m\ell^2 \quad 5$$

20

(b) The diagram shows a two-dimensional wheel (shaded area) and four spokes arranged inside the wheel as shown. The inner and outer radii of the wheel are $6a$ and $8a$, respectively. Each spoke is of mass m and length $6a$. The total mass of the wheel and four spokes is $18m$.



(i) Show that the mass per unit area of the wheel (shaded area) is $\dfrac{m}{2\pi a^2}$.

(ii) Show that the total moment of inertia of the wheel and four spokes about an axis through the centre and perpendicular to the plane of the wheel is $748ma^2$.

(iii) If $m = 100$ grammes and $a = 10$ cm, how much work is done in bringing this wheel and spokes to rest from 6000 revolutions per minute?

(i) Area of wheel $= \pi(8a)^2 - \pi(6a)^2$

$= 28\pi a^2$ | 5

$\dfrac{\text{mass}}{\text{area}} = \dfrac{14m}{28\pi a^2} = \dfrac{m}{2\pi a^2}$ | 5

(ii) mass of outer disc $= \left\{\dfrac{m}{2\pi a^2}\right\}\left\{\pi(8a)^2\right\} = 32m$

mass of inner disc $= \left\{\dfrac{m}{2\pi a^2}\right\}\left\{\pi(6a)^2\right\} = 18m$ | 5

$I = \frac{1}{2}\{32m\}\{64a^2\} - \frac{1}{2}\{18m\}\{36a^2\}$ | 5

$+ 4\left\{\frac{4}{3} m (3a)^2\right\}$

$= 748m a^2$ | 5

(iii) $\dfrac{\omega}{2\pi} = \dfrac{6000}{60}$

$\omega = 200\pi$

Work done $= \frac{1}{2}\{I\}\omega^2$

$= \frac{1}{2}\{748m a^2\}\{200\pi\}^2$

$= 374\left\{(0.1)(0.1)^2\right\}\{200\pi\}^2$

$= 147649.28$ or $14960\pi^2$ J | 5 | 30

9. **(a)** When placed in liquid A, a uniform solid cylinder floats upright with $\frac{2}{5}$ of its axis immersed in the liquid.

When placed in liquid B, the uniform solid cylinder floats upright with $\frac{4}{7}$ of its axis immersed in the liquid.

What fraction of the cylinder's axis is immersed when the cylinder floats upright in a uniform mixture of equal volumes of liquid A and liquid B?

Liquid A :

$$B_1 = W$$

$$\frac{\frac{2}{5} W s_1}{s} = W$$

$$s_1 = \frac{5s}{2} \quad \text{or} \quad \rho_1 = \frac{5\rho}{2}$$ 5

Liquid B :

$$B_2 = W$$

$$\frac{\frac{4}{7} W s_2}{s} = W$$

$$s_2 = \frac{7s}{4} \quad \text{or} \quad \rho_2 = \frac{7\rho}{4}$$ 5

Mixture :

$$B_m = W$$

$$\frac{xW s_m}{s} = W$$

$$s_m = \frac{s}{x} \quad \text{or} \quad \rho_m = \frac{\rho}{x}$$ 5

$$\rho_1 V + \rho_2 V = \rho_m \{2V\} \quad \text{or}$$

$$s_1 + s_2 = 2s_m \quad \text{or} \quad \rho_1 + \rho_2 = 2\rho_m$$ 5

$$\frac{5s}{2} + \frac{7s}{4} = \frac{2s}{x} \quad \text{or} \quad \frac{5\rho}{2} + \frac{7\rho}{4} = \frac{2\rho}{x}$$

$$\Rightarrow \quad x = \frac{8}{17}$$ 5 | 25

(b) A hollow spherical shell of external radius 0.5 m and uniform thickness 0.1 m floats in a liquid of relative density 0.9.

The relative density of the material of the shell is $\frac{36}{61}$.

What fraction of the volume enclosed by the external surface of the shell is immersed in the liquid?

Let fraction of shell immersed $= x$

$$\begin{aligned}
\text{Buoyancy } B &= \text{weight of liquid displaced} \\
&= \rho V g \\
&= 900\left\{x\tfrac{4}{3}\pi(0.5)^3\right\}g \text{ or } 900Vg && 5 \\
&= 150\pi\, x\, g
\end{aligned}$$

$$\begin{aligned}
\text{Volume of shell material} &= \left\{\tfrac{4}{3}\pi(0.5)^3 - \tfrac{4}{3}\pi(0.4)^3\right\} && 5 \\
&= \tfrac{4}{3}\pi\,(0.061)
\end{aligned}$$

$$\begin{aligned}
\text{Weight of shell } W &= \rho V g \\
&= \tfrac{36}{61}\times 1000 \times \left\{\tfrac{4}{3}\pi\,(0.061)\right\}g && 5 \\
&= 48\pi\ g
\end{aligned}$$

$$\begin{aligned}
B &= W && 5 \\
150\pi\, x\, g &= 48\pi\ g
\end{aligned}$$

$$x = \frac{48}{150} \text{ or } \frac{8}{25} \text{ or } 0.32 \qquad 5 \qquad \boxed{25}$$

10. **(a)** Solve the differential equation

$$\frac{dy}{dx} = e^{x-y}$$

given that $y = \ln 4$ when $x = 0$.

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{e^x}{e^y}$$

$$\int e^y dy = \int e^x \mathrm{dx}$$ 5

$$e^y = e^x + \mathrm{C}$$ 5

$$y = \ell n4, x = 0 \Rightarrow \mathrm{C} = 3$$ 5

$$e^y = e^x + 3$$ 5

$$\mathrm{y} = \ell\mathrm{n}\left(e^x + 3\right)$$

20

(b) A particle starts from rest and moves in a horizontal line. Its speed v at time t is given by the equation

$$\frac{dv}{dt} = 100 - v.$$

(i) Find the time taken for the speed of the particle to increase from 25 m/s to 75 m/s.

(ii) How far does the particle travel in going from rest to a speed of 75 m/s?

(iii) Determine the limiting speed, v_1, of the particle.
(that is, $v \to v_1$ as $t \to \infty$).

(i)

$$\int \frac{dv}{100 - v} = \int dt$$ 5

$$\left[-\ell\mathrm{n}(100 - v)\right]_{25}^{75} = [\mathrm{t}]_0^t$$ 5

$$-\ell\mathrm{n}25 + \ell\mathrm{n}75 = \mathrm{t}$$

$$\Rightarrow \mathrm{t} = \ell\mathrm{n}3 \quad \text{or} \quad 1.1\,\mathrm{s}$$ 5

15

10 (b) cont.

(ii)

$$v\frac{dv}{dx} = 100 - v$$

$$\int \frac{vdv}{100 - v} = \int dx$$

$$\int \left\{ -1 + \frac{100}{100 - v} \right\} \text{dv} = \int dx$$

$$\left[-v - 100\ell n(100 - v) \right]_0^{75} = x$$

$$-75 - 100\ell n25 + 100\ell n100 = x$$

$$x = 100\ell n4 - 75 \text{ or}$$

63.63 m

5	
5	10

(iii)

$$\frac{dv}{dt} = 0 \quad \Rightarrow \quad v_1 = 100 \text{ m/s}$$

5

OR

$$\int \frac{dv}{100 - v} = \int dt$$

$$\left[-\ell \text{n}(100 - v) \right]_0^v = [\text{t}]_0^t$$

$$-\ell \text{n}(100 - v) + \ell \text{n}100 = \text{t}$$

$$\Rightarrow \text{t} = \ell \text{n} \left\{ \frac{100}{100 - v} \right\}$$

$$100 - v = \frac{100}{e^{\text{t}}}$$

$$\Rightarrow v = 100 - \frac{100}{e^{\text{t}}}$$

$$\Rightarrow \text{limiting speed} = 100 \text{ m/s}$$

Alternative Solutions

1. **(a)** A stone is thrown vertically upwards under gravity with a speed of *u* m/s from a point 30 metres above the horizontal ground. The stone hits the ground 5 seconds later.

(i) Find the value of *u*.

(ii) Find the speed with which the stone hits the ground.

(i)

Stage 1:

$$v = u + at \qquad v^2 = u^2 + 2as$$

$$0 = u - gt_1 \qquad 0 = u^2 - 2gs_1$$

$$t_1 = \frac{u}{g} \quad \text{or} \quad s_1 = \frac{u^2}{2g}$$ 5

Stage 2:

$$s = ut + \tfrac{1}{2}at^2$$

$$\frac{u^2}{2g} + 30 = 0 + \tfrac{1}{2}g\left\{5 - \frac{u}{g}\right\}^2$$ 5

$$\Rightarrow \qquad u = 18.5\ \text{ms}^{-1}$$ 5

(ii)

Stage 2:

$$v = u + at$$

$$= 0 + g\left\{5 - \frac{18.5}{g}\right\}$$

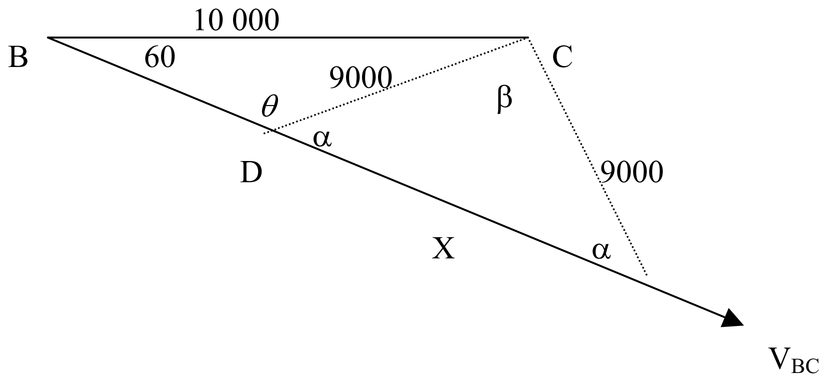
$$\Rightarrow \qquad v = 30.5\ \text{ms}^{-1}$$ 5

20

2. **(a)** Two boats, B and C, are each moving with constant velocity.
At a certain instant, boat B is 10 km due west of boat C.
The speed and direction of boat B relative to boat C is 2.5 m/s in the direction 60° south of east.

(i) Calculate the shortest distance between the boats, to the nearest metre.

(ii) Calculate the length of time, to the nearest second, for which the boats are less than or equal to 9 km apart.



(ii)

$$\frac{\sin\theta}{10\,000} = \frac{\sin 60}{9\,000}$$
$$\theta = \sin^{-1}(0.9623)$$
$$= 105.8°$$
$$\Rightarrow \quad \alpha = 74.2°$$
$$\Rightarrow \quad \beta = 31.6°$$

$$\frac{X}{\sin\beta} = \frac{9\,000}{\sin\alpha}$$
$$\frac{X}{\sin 31.6°} = \frac{9\,000}{\sin 74.2°}$$ 5
$$X = 4901.04$$

$$\text{time} = \frac{X}{V_{BC}}$$ 5
$$= \frac{4901.04}{2.5}$$
$$= 1960\,\text{s}$$ 5 | 15

2(b) The velocity of ship P relative to a steady wind is 20 km/hr in the direction 80° north of east.
The velocity of ship Q relative to the same steady wind is 10 km/hr in the direction 20° south of west.

Calculate the magnitude and direction of the velocity of ship P relative to ship Q.

Give your answers to the nearest km and the nearest degree, respectively.

$V_{PQ} = V_P - V_Q$	5	
$= \{V_{PW} + V_W\} - \{V_{QW} + V_W\}$		
$= \{V_{PW}\} - \{V_{QW}\}$		
$\lvert V_{PQ} \rvert = \sqrt{10^2 + 20^2 - 2(10)(20)\cos 120}$	5	
$= \sqrt{700}$	5	
$\Rightarrow$ magnitude = 26 km/hr	5	
$\dfrac{\sin\alpha}{20} = \dfrac{\sin 120}{\sqrt{700}}$		
$\Rightarrow \alpha = 41°$		
$\Rightarrow$ direction = 61° north of east	5	25

3. **(a)** A particle is projected from a point on the horizontal ground with a speed of 39.2 m/s inclined at an angle α to the horizontal ground. The particle is at a height of 14.7 m above the horizontal ground at times t_1 and t_2 seconds, respectively.

(i) Show that $t_2 - t_1 = \sqrt{64\sin^2\alpha - 12}$.

(ii) Find the value of α for which $t_2 - t_1 = \sqrt{20}$.

(i) $r_j = 14.7$ 5

$39.2\sin\alpha.t - \frac{1}{2}gt^2 = 14.7$ 5

$t^2 - 8\sin\alpha.t + 3 = 0$

$t_1 + t_2 = 8\sin\alpha$

$t_1 t_2 = 3$

$\{t_2 - t_1\}^2 = \{t_2 + t_1\}^2 - 4t_1t_2$ 5

$\{t_2 - t_1\}^2 = 64\sin^2\alpha - 12$

$t_2 - t_1 = \sqrt{64\sin^2\alpha - 12}$ 5

(ii) $t_2 - t_1 = \sqrt{20}$

$\sqrt{64\sin^2\alpha - 12} = \sqrt{20}$

$\sin\alpha = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\alpha = 45°$ 5 25

8 (b)

(i) Area of wheel $= \pi(8a)^2 - \pi(6a)^2$ 5

$= 28\pi a^2$

$$\frac{\text{mass}}{\text{area}} = \frac{14m}{28\pi a^2} = \frac{m}{2\pi a^2}$$ 5

(ii) $I_{wheel} = \int_{6a}^{8a} 2\pi x.dx \left\{\frac{m}{2\pi a^2}\right\} x^2$ 5

$= \left\{\frac{m}{a^2}\right\}\left[\frac{x^4}{4}\right]_{6a}^{8a}$

$= \left\{\frac{m}{4a^2}\right\}\left\{2800a^4\right\} = 700ma^2$ 5

$I = 700ma^2 + 4\left\{\frac{4}{3} m (3a)^2\right\}$

$= 748m a^2$ 5

(iii) $\frac{\omega}{2\pi} = \frac{6000}{60}$

$\omega = 200\pi$

Work done $= \frac{1}{2}\{I\}\omega^2$

$= \frac{1}{2}\left\{748m a^2\right\}\{200\pi\}^2$

$= 374\left\{(0.1)(0.1)^2\right\}\{200\pi\}^2$

$= 147649.28$ or $14960\pi^2$ J 5 30

10 (b)

(i)

$$\int \frac{dv}{100-v} = \int dt \qquad 5$$

$$\left[-\ell n(100-v)\right]_{25}^{75} = [t]_0^t \qquad 5$$

$$-\ell n25+\ell n75 = t$$

$$\Rightarrow \quad t=\ell n3 \quad \text{or} \quad 1.1\,\text{s} \qquad 5 \qquad 15$$

(ii)

$$\left[-\ell n(100-v)\right]_{0}^{75} = [t]_0^t$$

$$-\ell n25+\ell n100 = t$$

$$\Rightarrow \quad t=\ell n4$$

$$\left[-\ell n(100-v)\right]_{0}^{V} = [t]_0^t$$

$$-\ell n(100-v)+\ell n100 = t$$

$$\ell n\left\{\frac{100}{100-v}\right\} = t$$

$$v = 100-100\,e^{-t} \qquad 5$$

$$\frac{ds}{dt} = 100-100\,e^{-t}$$

$$s = \left[100t+100\,e^{-t}\right]_0^{\ell n4}$$

$$= 138.63-25+100$$

$$= 63.63 \text{ m} \qquad 5 \qquad 10$$

(iii)

$$\frac{dv}{dt}=0 \quad \Rightarrow \quad v_1 = 100 \text{ m/s} \qquad 5$$