SECOND INTERNATIONAL SELECTION TEST

Trinity College, Cambridge, 14th April 1991

Time allowed: Three-and-a-half hours.

1. Let f be the function defined on the set of positive integers by

$$f(1) = 1,\ f(2n+1) = f(2n) + 1,\ f(2n) = 3f(n).$$

Prove that there is no positive integer n for which $f(n) = 1991$. How close to 1991 can $f(n)$ get?

For each positive integer k, let f_k be the function defined on the set of positive integers by

$$f_k(1) = 1,\ f_k(2n+1) = f_k(2n) + 1,\ f_k(2n) = kf_k(n).$$

Find all values of k for which $f_k(n) = 1991$ has a solution.

2. Vertex A of triangle ABC is equidistant from the circumcentre O and the orthocentre H. Find *all* possible values of angle A.

3. Whereas the dwarfs of the FIST community wear red and blue cloaks, the SIST community distinguishes itself by its complicated pecking order. This pecking order is complicated in the sense that if X pecks Y (ie. X dominates Y) and Y pecks Z, then it does not necessarily follow that X pecks Z. However, the pecking order does satisfy three simple rules:

   (a) For any pair of dwarfs X and Y, either X pecks Y or Y pecks X.

   (b) For any pair of dwarfs X and Y, there is exactly one dwarf Z who pecks both X and Y.

   (c) For any pair of dwarfs X and Y, there is exactly one dwarf W who is pecked by both X and Y.

   How many dwarfs are there in the SIST community?