Oundle Test 3

29 May 2007

1. Let ABC be a triangle with $\angle B \neq \angle C$. The incircle I of ABC touches the sides BC, CA, AB at the points D, E, F, respectively. Let AD intersect I at D and P.

   Let Q be the intersection of the line EF and the line passing through P and perpendicular to AD, and let X, Y be intersections of the line AQ and DE, DF, respectively. Show that the point A is the midpoint of XY.

2. For three mutually distinct real numbers a_1, a_2, a_3, define three real numbers b_1, b_2, b_3 as follows:

$$b_j = \left(1 + \frac{a_j a_i}{a_j - a_i}\right)\left(1 + \frac{a_j a_k}{a_j - a_k}\right), \qquad \{i, j, k\} = \{1, 2, 3\}.$$

   Prove the inequality

$$1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq (1 + |a_1|)(1 + |a_2|)(1 + |a_3|).$$

   When does equality hold?

3. Suppose that $(x+1)^n - 1$ is divisible by a polynomial $P(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_1 x + c_0$ of an even degree k such that $c_0, c_1, \ldots, c_{k-1}$ are odd integers. Prove that n is divisible by $k+1$.