V-A4-II-Hf-Ex-12-B-38

Paper IV: Complex Analysis 22/10/12

Con. 6173-12. Paper IV **YG-8335**

Scheme A (External) (3 Hours) **[Total Marks : 100**

Scheme B (Internal) (2 Hours) **[Total Marks : 40**

N.B. : (1) **Scheme A (External)** Students should attempt any **five** questions.
(2) **Scheme B (Internal)** Students should attempt any **three** questions.
(3) Write on the top of your answer book the scheme under which you are appearing.
(4) **All** questions carry **equal** marks.

1. (a) Prove that a complex differentiable function is continuous.
   (b) Give an example of a continuous function which is not complex differentiable.
   (c) Compute values of $arg(z)$ in the interval $(-\pi, \pi)$ for $Z = -1 - i$.

2. (a) Construct the Stereographic Projection map.
   (b) Verify the continuity of the following function f of the extended complex plane $\mathbb{C} \cup \{\infty\}$ at the point $a = -\frac{3}{4}$:—

$$f(z) = \begin{cases} \infty & \text{if } Z = -\frac{3}{4}, \\ \frac{Z+1}{4Z+3} & \text{if } Z \neq -\frac{3}{4} \end{cases}$$

3. (a) State and prove the Ratio-Test for a series of complex numbers.
   (b) Prove that the series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converges.

4. (a) State and prove the Cauchy Integral formula.
   (b) Evaluate $\int_C \frac{\sin(\pi z)}{z - \frac{1}{2}}$

   where C is the unit circle : $|z| = 1$ oriented clock-wise.

5. (a) Let f be an analytic function defined on a domain U. Let $a \in U$ and $r > 0$ such that $\overline{B}(a, r) \subseteq U$. Prove that f has a power series expansion in $B(a, r)$
   (b) Let f be a non-constant analytic function difined on a domain U. Let

$$S = \{z \in U \,/\, f(z) = 0\}.$$

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Prove that S is a discrete subset of U.

6. (a) State and prove the maximum modulus principle for an analytic function.
   (b) Find the maximum of $\left| \exp(-z) \right|$ on the open ball $|z| < 4$.

7. (a) Prove that any non-constant analytic map is an open map.
   (b) Let $f(z) = z^2$ for all complex numbers z. Find the image of the open set

$$\left\{ x + iy \,\middle|\, x < 0 \text{ and } y > 0 \right\}$$

   under the map f.

8. (a) Find the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in the annular region $1 < |z| < 2$.
   (b) Compute the residue of $f(z) = \dfrac{\sin(z)}{z^4}$ at the pole $z = 0$.

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ws-Sept, 2012 (a) 27 Paper I: Algebra - I 15/10/12

Con. 6157–12. YG–8314

Paper I

External (Scheme A)] (3 Hours) [Total Marks : 100

Internal (Scheme B)] (2 Hours) [Total Marks : 40

N. B. : (1) **Scheme–A Students** answer any **five** questions.
(2) **Scheme–B Students** answer any **three** questions.
(3) **All** questions carry **equal** marks.
(4) Write on the **top** of your answer book the **scheme** under which you are **appearing**.

1. (a) Let G be a group and Z(G) be the center of G. If G/Z(G) is cyclic, then show that G is abelian.
   (b) Let G be a group and H, K subgroups of G. Show that the product set HK is a subgroup of G if and only if KH = HK.

2. (a) Let G be a finite abelian group and p be a positive prime divides order of G. Show that G has an element of order p.
   (b) Prove that any abelian group of order 45 has an element of order 15. Does every abelian group of order 45 have an element of order 9 ?

3. (a) Define prime and irreducible elements in an integral domain. Show that in a Principal Ideal Domain (PID) an element is prime if and only if it is irreducible.
   (b) Let R be a commutative ring with identity and P be a prime ideal of R. Prove that P[x] is a prime ideal in R[x], where
   $$P[x] = \{a_0 + a_1 x + \ldots\ldots + a_n x^n \,;\, a_i \in P \text{ for } i \le i \le n,\ n \in \mathbb{N}\}$$

4. (a) Show that every Euclidean Domain is a Principal Ideal Domain.
   (b) Let f(x) be a polynomial in $\mathbb{Z}[x]$. If f(x) is reducible over $\mathbb{Q}$, then show that it is reducible over $\mathbb{Z}$.

5. (a) Let A be a matrix over $\mathbb{C}$. Show that row rank of A is same as column rank of A.
   (b) Let V be a finite dimensional vector space over a field F. Show that if u_1, u_2 are subspaces of V, then $(u_1 + u_2)^0 = u_1^0 \cap u_2^0$.

6. (a) Let V be a finite dimensional vector space over a field F and V** be the double dual. Show that V** is isomorphic to V.
   (b) Show that if for a 2 × 2 matrix A over a field F, $A^2 = 0$, then for any scalar c, $\det(cI - A) = c^2$.

7. (a) Let A be an n × n matrix over a field F and m(t) be the minimal polynomial of A. Show that the charactersitic polynomial of A divides $(m(t))^n$.
   (b) Show that every square matrix is similar to an upper triangular matrix over $\mathbb{C}$.

8. (a) Let V be a finite dimensional inner product space and T a linear operator on V. Show that range of T* is the orthogonal complement of the Kernel of T.
   (b) Let T be a self adjoint operator on a finite dimensional inner product space V. Show that V has an orthonormal basis consisting of characteristic vectors of T.

Paper V: Combinatorics Oct. 2012

AGJ 2nd half (c) 55

Con. 6652-12. **YG–5833**

Scheme A (External) (3 Hours) **[Total Marks : 100**

Scheme B (Internal) (2 Hours) **[Total Marks : 40**

N.B.: 1) **Scheme A students** answer any **five** questions.

2) **Scheme B students** answer any **three** questions.

3) All questions carry **equal** marks.

4) **Write on the top of your answer book the scheme under which you are appearing.**

1. (a) How many positive integers between 100 and 999 both inclusive are not divisible by either 3 or 4?

   (b) Find the number of 3-element subsets $\{a, b, c\}$ of $\{1, 2, \ldots, 2008\}$ such that 3 divides $a+b+c$.

2. (a) Define Stirling number $S(n,k)$ of second kind for $1 \le k \le n$. Show that $S(n,1) = 1 = S(n,n)$ and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $2 \le k \le n-1$.

   (b) Find the coefficient of $x^5y^4z^3$ in the expansion of $(-2x+y-z)^{12}$.

3. (a) Define derangement of finite objects. Let D_n denote the number of derangements of n objects. Show that

   1) $D_n = (n-1)(D_{n-1} + D_{n-2})$ for all $n \ge 2$.

   2) $D_n - nD_{n-1} = (-1)^n$ for all $n \ge 1$.

   (b) For each $n \in \mathbb{N}$, show that the number of partitions of n into parts each of which appears at most twice, is equal to the number of partitions of n into parts the sizes of which are not divisible by 3.

4. (a) Find number of non-negative integer solutions of the equation $x+y+z+w = 10$, where $1 \le x \le 5, 2 \le y \le 6, z \le 2, w \le 3$.

   (b) Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

5. (a) Define matching in the bipartite graph $G = (X \cup Y, E)$. Show that the bipartite graph $G = (X \cup Y, E)$ has a complete matching if and only if $|J(A)| \ge |A|$ for all $A \subseteq X$, where $J(A) = \{y \in Y / xy \in E \text{ for some } x \in A\}$.

   (b) Find the largest number of sets in the family $A_1, A_2, \ldots, A_{10}$ which together have a system of distinct representatives, where $A_1 = \{1,8,10,13\}$, $A_2 = \{1,4,5,7,11\}$, $A_3 = \{5,8\}$, $A_4 = \{8,13\}$, $A_5 = \{2,3,4,11,12\}$ $A_6 = \{5,6,10,13\}$, $A_7 = \{10,13\}$, $A_8 = \{5,8,10,13\}$, $A_9 = \{1,5,8\}$, $A_{10} = \{1,5,8,10,13\}$.

6. (a) Define Euler function $\phi(n)$. Let $n \ge 2$ be an integer whose prime factorization is $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$. Show that $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$.

   (b) How many necklaces can be constructed using three red beads, two white beads and one blue bead?

7. (a) Let G be a group of permutations of a non empty finite set X, and let x be any chosen element of X. Then show that $|Gx| \times |G_x| = |G|$ where $Gx = \{g(x)/g \in G\}$ and $G_x = \{g \in G/g(x) = x\}$.

   (b) Solve the recurrence relation: $a_n = \sum_{k=1}^{n-1} a_k a_{n-k}$, $n \ge 2$ subject to initial value $a_1 = 1$.

8. (a) Define expectation and variance of a discrete random variable. Explain geometric random variable and calculate its expectation and variance.

   (b) What is the probability of the following events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

   a) The first 13 letters of the permutations are in alphabetic order.

   b) a immediately precedes z in the permutations.

16 : 2nd half.12-AM(c) Paper II : Analysis - I 17/10/12 Paper-

Con. 6161-12. **YG-8317**

Scheme A (External)] (3 Hours) **[Total Marks : 100**

Scheme B (Internal / External)] (2 Hours) **[Total Marks : 40**

N.B. : (1) **Scheme A students** should answer any **five** questions.
(2) **Scheme B students** should answer any **three** questions.
(3) **All** questions carry **equal** marks.
(4) Mention clearly the **Scheme** under which you are appearing.

1. (a) For every real $x > 0$ and every integer $n > 0$, prove that there is a unique real $y > 0$ such that $y^n = x$.
   (b) If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$ then prove that there exists a rational number q such that $x < q < y$.

2. (a) If $\{ K_\alpha \}$ is a collection of compact subsets of metric space X such that the intersection of every finite sub-collection of $\{ K_\alpha \}$ is non-empty, then prove that $\cap K_\alpha$ is non-empty.
   (b) If $\overline{E}$ is the closure of a set E in a metric space X, then prove that diam $\overline{E}$ = diam E.

3. (a) If X is a compact metric space and if $\{ p_n \}$ is a Cauchy sequence in X, then prove that $\{ p_n \}$ converges to some point of X.
   (b) Prove that —

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

4. (a) If $\sum x_n$ is a series of real numbers which converges absolutely, then prove that every rearrangement of $\sum x_n$ converges and they all converge to the same sum.
   (b) Discuss the convergence or divergence of the series

$$\frac{1}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{3}{5 \cdot 7} + \frac{4}{7 \cdot 9} + \cdots$$

5. (a) State and prove the Weierstrass M – test.
   (b) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$ converges uniformly for all real x.

6. (a) State and prove the test for locating maxima and minima of real valued function with continuous second-order partial derivatives at a stationary point in $\mathbb{R}^2$.
   (b) Divide 24 into three parts such that the continued product of the first, the square of the second and the cube of the third may be maximum.

7. (a) Suppose f is a continuous mapping of $[a,b] \subseteq \mathbb{R}$ into $\mathbb{R}^n$ and f is differentiable in (a, b). Prove that there exists x in (a, b) such that

$$\| f(b) - f(a) \| \leq (b-a) \| f'(x) \|$$

   (b) Expand $e^x \log(1 + y)$ in a Taylor series in the neighbourhood of the point (1, 0).

8. (a) State and prove the Fubini's theorem.
   (b) By differentiating unde

Paper III : Topology Oct. 2012

P4-RT-Exam.-Oct.-12-223

Con. 6168–12. **YG–8329**

External (Scheme A)] (3 Hours) **[Total Marks : 100**

Internal (Scheme B)] (2 Hours) **[Total Marks : 40**

N.B. : (1) **Scheme A** Students answer any **five** questions.
(2) **Scheme B** Students answer any **three** questions.
(3) **All** questions carry **equal** marks.
(4) Write the **scheme** under which you are appearing on **top** of the answer book.

1. (a) Prove that there is no surjective map from a set X onto $P(X) = \{A \mid A \subseteq X\}$.

   (b) Given two sets X, Y, prove that there is either an injective map from X to Y or there is an injective map from Y to X.

2. (a) Let X be a topological space. For a subset A of X, define the closure $\overline{A}$. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for two subsets A, B of X.

   (b) Let X be a topological space and $A \subseteq X$. Prove that $A = \overline{A} \Leftrightarrow A$ is closed in X.

3. (a) Define a connected topological space. Prove that a continuous image of a connected topological space is connected. 157

   (b) Prove that $\mathbb{R}^2 \setminus \{(0,0)\}$ is a connected topological space. 156

4. (a) Prove that there does not exist a continuous, injective map from $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ into $\mathbb{R}$.

   (b) Let A be a non-empty subset of $\mathbb{R}^n$.

   For any $x \in \mathbb{R}^n$, define $d_A(x) = \inf\{\|x - a\| \mid a \in A\}$.

   Prove that $d_A : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function.

5. (a) Define a second countable space. Define a separable space. Prove that a second countable space is seperable.

   (b) Prove that $\mathbb{R}^n$ is a separable space.

6. (a) If X, Y are compact topological spaces then prove that $X \times Y$ is compact when considered with the product topology.

   (b) Let $A = \{X \in Q \mid 0 \leq |x| < \sqrt{2}\}$. Is A a compact subset of Q ? Justify your answer.

7. (a) Let $A \subseteq X$ and $r : X \longrightarrow A$ be a continuous map such that $r(a) = a$, $\forall a \in A$. Prove that r is a quotient map.

   (b) Define $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x,y) = x$, $\forall (x,y) \in \mathbb{R} \times \mathbb{R}$. Is f a quotient map ? Justify your answer.

8. (a) Define path homotopy. If $f : X \longrightarrow Y$ is a continuous map with $f(x_0) = y_0$, then prove that f induces a group homomorphism from $\pi_1(X, x_0)$ into $\pi_1(Y, y_0)$.

   (b) Prove that $f(z) = z^2$, $\forall z \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a covering map from S^1 onto S^1.