V-Ex-I-09-C-5 M.A. & M.Sc. (Mathematics) (Part - I)

Mathematics : Paper-II Analysis - I

**Con. 1417-09.** **MS-6978**

17/4/09

**Internal (Scheme B)]** (2 Hours) **[Total Marks : 40**

**External (Scheme A)]** (3 Hours) **[Total Marks : 100**

**N.B. :** (1) Write on the top of your answer book the Scheme underwhich you are appearing.

(2) Students of **Scheme B** answer **three** questions ; Students of **Scheme A** answer **five** questions.

(3) **All** questions carry **equal** marks.

1. (a) Show that every bounded monotone sequence of real numbers is convergent.

   (b) Let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that lim inf $a_n$ = lim sup $a_n$. Show that $\{a_n\}$ either converges or diverges.

   (c) Find limit inferior and limit superior of the sequence

$$\left\{\left(1+\frac{1}{n}\right)\cos n\pi : n \in \mathbb{N}\right\}.$$

2. (a) State and prove Leibniz theorem on the convergence of an alternating series.

   (b) Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditionally convergent.

   (c) Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^2}$ is convergent by quoting explicitly the results used.

3. (a) Find the derivative of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1x_2$ at the point (1, 1).

   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by—

$$f(x_1, x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2} \text{ if } (x_1, x_2) \neq (0, 0)$$

$$f(0, 0) = 0.$$

   Verify if f is differentiable at (0, 0) and if the partial derivatives of f at (0, 0) exist or not.

   (c) Let E be an open set in $\mathbb{R}^n$ and f be a real valued function on E such that all partial derivatives $\frac{\partial f}{\partial x_j}$, $1 \leq k \leq n$ are continuous on E. Prove that f is differentiable on E and that its derivative at a point $a \in E$ corresponds to the linear operator L, given by—

$$L(t_1, t_2, \ldots, t_n) = \sum_{k=1}^{n} t_k \frac{\partial f}{\partial x_k}(a),$$

   for $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$.

[TURN OVER

4. (a) Prove that any increasing function $f : \mathbb{R} \to \mathbb{R}$ has atmost countable number of discontinuities and that they are of first kind.
   (b) If $f : [a, b] \mathbb{R}$ is differentiable on (a, b) and has bounded derivative, prove that f is of bounded variation on [a, b].

5. (a) Show that a continuous function f on [a, b] is Riemann-integrable.
   (b) If f is also non-negative in [a, b] and its integral over [a, b] varishes, show that f = 0 in [a, b].

6. (a) If f is continuous on [a, b], prove that $F'(x) = f(x)$ for all $x \in [a, b]$, where $F(x) = \int_a^x f(t)dt$.
   (b) If f and g are two Riemann integrable functions on [a, b] and $h : [a, b] \times [a, b] : \to \mathbb{R}$ is defined by $h(x, y) = f(x) \cdot g(y)$, is h Riemann integrable ? Justify your answer.

7. (a) State and prove Taylor's theorem for n–times continuously differentiable function of two variables.
   (b) Find and classify the extreme values of the following functions :—
   (i) $f(x) = x^2 + y^2 + xy + x + y$ and (ii) $f(x, y) = y^2 - x^3$

8. (a) State only Inverse Function Theorem. Use it to prove that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is one-one, continuously differentiable and has invertible Jacobian matrix at every point then f is an open mapping and $f^{-1} : f(\mathbb{R}^2) \to \mathbb{R}^2$ is differentiable.
   (b) Construct at $C^\infty$ real valued function on $R^2$ such that it is 1 on $\{ (x, y) : x^2 + y^2 \leq 1 \}$ and it is zero on $\mathbb{R}^2 \backslash \{(x, y) : x^2 + y^2 > 2 \}$.

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Analysis

**Con. 1041–08.** **NB–4228**

**Internal Scheme B ]** ( 2 Hours ) **[ Total Marks : 40**

**External Scheme A ]** ( 3 Hours ) **[ Total Marks : 100**

N.B. (1) Write on the top of your answer book the scheme under which you are appearing.
(2) Scheme-B students answer any three questions selecting atleast one from each section.
(3) Scheme-A students answer any five questions selecting atleast two from each section.
(4) All questions carry equal marks.
(5) Answers to both the sections are to be written in the same answer book.

## SECTION I

1(a) For a non-empty bounded subset $E$ of $\mathbb{R}$, define the least upper bound lub($E$) and the greatest lower bound glb($E$). Prove: If $E$ and $F$ are non-empty, bounded subset of $\mathbb{R}$ then (i) $\text{lub}(E+F) = \text{lub}(E) + \text{lub}(F)$ and (ii) $\text{lub}(4E) = 4\text{lub}(E)$.

1(b) Prove: If $x$ and $y$ are any two real numbers with $x > 0$, then there exists a natural number $n$ such that $n \cdot x > y$.

2(a) Let $(x_n : n \in \mathbb{N})$ and $(y_n : n \in \mathbb{N})$ be sequences of real numbers converging to the real numbers $l$ and $m$ respectively. Prove that the sequence $(x_n \cdot y_n : n \in \mathbb{N})$ converges to $l \cdot m$.

2(b) Let the sequence $(x_n : n \in \mathbb{N})$ be given by $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + \sqrt{x_{n-1}}}$ for $n > 1$. Prove that the sequence $(x_n : n \in \mathbb{N})$ converges and find the limit.

3(a) $A$ is a non-empty, closed subset of $\mathbb{R}$. Define $f : \mathbb{R} \to [0, \infty)$ by $f(x) = \text{glb}\{|x-a| \ : \ a \in A\}$. Prove: (i) $f$ is uniformly continuous on $\mathbb{R}$ and (ii) $f(x) = 0$ if and only if $x \in A$.

3(b) Let $A$ be a non-empty, closed subset and $U$ an open subset of $\mathbb{R}$ such that $A \subset U$. Prove that there exists a continuous function $f : \mathbb{R} \to [0,1]$ such that $f \equiv 1$ on $A$ and $f \equiv 0$ on $\mathbb{R} \setminus U$.

4(a) Let $A$ be a non-empty subset of $\mathbb{R}$. Let $(f_n : A \to \mathbb{R} \ : \ n \in \mathbb{N})$ be a sequence of functions converging to a function $f : A \to \mathbb{R}$ the convergence being uniform on $A$. Suppose the limits $\lim_{x \to a} f_n(x)$ $(n \in \mathbb{N})$ and $\lim_{x \to a} f(x)$ exist for all $a$ in $A$. Prove: the double limit $\lim_{n\to\infty} \lim_{x \to a} f_n(x)$ exists and is equal to $\lim_{x\to a} f(x)$ for all $a$ in $A$.

4(b) If in question 4(a) above all functions $f_n$ are continuous then prove that $f$ is also continuous on $A$.

## SECTION II

5(a) Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $p \in \Omega$. Define differentiability of $f : \Omega \to \mathbb{R}^m$ at $p$. Suppose $f : \Omega \to \mathbb{R}^m$ is differentiable at $p$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $f(p)$. Prove that $g \circ f$ is differentiable at $p$ and obtain the expression for the derivative of $g \circ f$ in terms of the derivatives of $f$ and $g$.

5(b) Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear form. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = B(x,x)$ for all $x \in \mathbb{R}^n$. Prove that $f$ is differentiable at every point $p \in \mathbb{R}^n$ and find an expression for $Df(p)$.

6(a) State without proof the inverse function theorem for a continuously differentiable function $f : \Omega \to \mathbb{R}^n$, $\Omega$ being an open subset of $\mathbb{R}^n$ and $p$, a point in $\Omega$ with the property that the derivative $Df(p)$ of $f$ at $p$ is non-singular.

6(b) State the implicit function theorem for a continuously differentiable map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and deduce the same from the inverse function theorem.

7(a) Define- giving all the relevant details- the Riemann integrability and the Riemann integral of a bounded function $f : R \to \mathbb{R}$, $R$ being a closed, bounded rectangle in $\mathbb{R}^n$.

7(b) Explain how a continuous and compactly supported function $g : \mathbb{R}^n \to \mathbb{R}$ can be integrated in the sense of Riemann.

8(a) Prove: If $f, g : \mathbb{R} \to \mathbb{R}$ on a closed,bounded rectangle $R$ in $\mathbb{R}^n$ are Riemann integrable, then $f + g$ is also Riemann integrable.

8(b) Prove: If $f : \mathbb{R} \to \mathbb{R}$ on a closed,bounded rectangle $R$ in $\mathbb{R}^n$ is Riemann integrable then $|f|$ is also Riemann integrable.

Analysis I

Con. 3813–07.

KD–22

Internal (Scheme B) (2 Hours) [Total Marks : 40

External (Scheme A) (3 Hours) [Total Marks : 100

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.

## SECTION I

1(a) Show that for any positive real number $x$ and every natural number $n$ there is a unique positive real number $y$ such that $y^n = x$.

1(b) If $m, n, p, q$ are integers, $n > 0, q > 0$ and $r = \frac{m}{n} = \frac{p}{q}$, prove that if $b > 0$ then $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$. Deduce that it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$

2(a) Define a subsequential limit of a sequence $\{p_n\}$ in $\mathbb{R}$ and show that the set of all subsequential limits of $\{p_n\}$ is a closed subset of $\mathbb{R}$.

2(b) Give an example of a sequence which has $11, 12, 13, 14, 15$ as subsequential limits and no more.

3(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f$ maps an interval onto an interval.

3(b) Let $f : [0,1] \to [0,1]$ be continuous. Show that there is $x \in [0,1]$ such that $f(x) = x$.

4(a) Determine the differentiability of the function if $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

4(b)) $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_{12}f(x,y)$ and $D_{21}f(x,y)$ exist at all points. If the functions $D_{12}f$ and $D_{21}f$ are continous, prove that $D_{12}f(x,y) = D_{21}f(x,y)\ \forall (x,y) \in \mathbb{R}^2$.

## SECTION II

5(a) Let $S$ be an open subset of $\mathbb{R}^n$. Let $f = (f_1, f_2, \cdots, f_n) : S \to \mathbb{R}^n$ be such that $D_j f_i$ are all continuous on $S$ and that $J_f(a) \neq 0$ for some $a \in S$. Prove the following part of the Inverse Function Theorem: $f$ is one-one on some neighborhood of $a$.

5(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^x \cos y, e^x \sin y)$. Is $f$ one-one? Is $f$ onto $\mathbb{R}^2$? Justify.

6(a) State and prove second derivative test for local maxima and minima for function of two variables. State only it's analogue for function of $n-$variables.

6(b) Test the following functions for absolute maxima and minima.

(i) $f(x,y) = x^4 + y^4 - 2x^2 + 8y^2 + 4$ (ii) $f(x,y) = x^2 + xy + 3x + 2y$ (iii) $f(x,y) = 1 - x^2y^2$

7(a) Construct a $C^\infty$-function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = 1$ if $\|x\| < 1$ and $f(x) = 0$ if $\|x\| \geq 2$.

7(b) When is a function $f : [0,1] \to \mathbb{R}$ is said to be of bounded variation. Show by means of an example that a continuous function need not be of bounded variation.

8(a) Prove that every continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ is Riemann integrable.

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|---|---|---|
| Internal(Scheme B) | (Time 2 Hours) | [Marks 40 |
| External(Scheme A) | (Time 3 hours ) | [Marks 100 |

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
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(5) Answers to both the sections are to be written in the same answer book.

## SECTION I

1(a) Define the term 'ordered field' and show that the field $\mathbb{C}$ of complex number is not an ordered field.

1(b) State the archimedean property of the field $\mathbb{R}$ of real numbers and use it to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

2 Let $\{s_n\}$ be a given sequence of real numbers and $s^* = \limsup s_n$.

2(a) Show that there is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \to s^*$.

2(b) Show that if $x > s^*$ then there is an integer $N$ such that $s_n < x$ for all $n > N$.

3(a) Show that a function $f = (f_1, f_2, \cdots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if each of $f_1, f_2, \cdots, f_m$ is continuous.

3(b) Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Show that for any $x \in (a, b)$ we have $f(x-) \le f(x) \le f(x+)$ and use it to prove that $f$ has atmost countably many discontinuities.

4(a) Show by means of an example that a function $f : \mathbb{R}^n \to \mathbb{R}$ may have all its partial derivatives without it being differentiable. Justfy your statements.

4(b) Prove that a real valued function defined on $\mathbb{R}^n$ such that all its partial derivatives exist at all points and define continuous functions is differentiable.

## SECTION II

6(a) State only Taylor's Theorem for real valued function of a real variable with any one form of remainder. Use it to state and prove its anlogue for function of $n-$variables.

6(b) Prove that any triangle of maximal area inscribed in a circle of radius $r$ is an equilateral triangle.

7(a) State only inverse function Theorem. If a $C^1$-function $f : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and $Df(x)$ is non-singular for every $x \in \mathbb{R}^n$ then prove that $G = f(\mathbb{R}^n)$ is open in $\mathbb{R}^n$ and that $f$ is a diffeomorphism from $\mathbb{R}^n$ onto $G$.

7(b) A continuous functon $f : [0, 1] \to \mathbb{R}$ is differentiable on $(0, 1)$ and has bounded derivative. Prove that $f$ is of bounded variation on $[0, 1]$.

8(a) Let $I = [0, 1]$ and $R = I \times I$. When is a bounded function $h : R \to \mathbb{R}$ said to be Riemann integrable on $R$? Suppose $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are two Riemann integrable functions and $h(x, y) = f(x) \cdot g(y)\ \forall\ (x, y) \in R$. Is $h$ Riemann integrable on $R$? Justify your answer.

8(b) Evaluate using spherical polar coordinates $\int_E f(x, y, z) d(x, y, z)$ where

$$E = \{(x, y, z) \in \mathbb{R}^3 / 1 \le x^2 + y^2 + z^2 \le 4\} \quad \text{and} \quad f(x, y, z) = xy(x^2 + y^2 + z^2)$$

mathematics : Paper - I - Algebra - I



**Con. 1644-09.** **MS-81**

15/4/09

**External [Scheme A]** **(3 Hours)** **[Total Marks : 100**

**Internal [Scheme B]** **(2 Hours)** **[Total Marks : 40**

N.B.: (i) **External (Scheme A)** Candidates should attempt any **five (5)** questions.

(ii) **Internal (Scheme B)** candidates should attempt any **three (3)** questions.

(iii) **All** questions carry equal marks.

(iv) **Mark clearly the scheme under which you are appearing for the examination.**

1. (a) Let $G$ be a finite cyclic group of order $n$. Show that the number of generators of $G$ is the number of positive integers less than $n$ and are prime to $n$.
   (b) Show that there is no nontrivial homomorphism from $\mathbb{Z}_{14}$ to $\mathbb{Z}_{15}$.

2. (a) Define the terms Euclidean domain, 'Principal Ideal Domain' (PID) and 'Unique Factorisation Domain' (UFD). Show that every PID is a UFD.
   (b) Show that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean.

3. (a) Show that the center of $S_n$, the permutation group on $\{1, 2, ..., n\}$ is trivial for all $n \geq 3$.
   (b) Show that every group of order 4 is abelian.

4. (a) State and prove Eisenstein's criterion for irreducibility of a polynomial.
   (b) Show that $\mathbf{R}[X]/(X^2 + 1)$ is a field.

5. (a) Let $V$ and $W$ be two vector spaces over a field $k$. Assume that $Dim(V) = n$ and $Dim(W) = m$. Show that there is a 1-1 correspondence between the set of linear transformations from $V$ to $W$ and the set of $(m \times n)$ matrices over $k$.
   (b) In $k^3$, define $T$ by $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2, -2x_2 + 2x_3)$. Show that $T$ is a linear transformation. Calculate kernel of $T$.

6. (a) Define the terms *Bilinear form; Quadratic form.* Assume that the characteristic of the field is zero. Show that given a symmetric bilinear form $f$ on a finite dimensional vector space $V$, there exists an ordered basis for $V$ in which matrix of $f$ is a diagonal matrix.
   (b) Let $V$ be the vector space $\mathbf{R}$ over itself. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Define $f(\alpha, \beta) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$. Find a basis for $V$ in which the matrix of $f$ is diagonal.

7. (a) Define the Determinant function on $M_n(k)$, where $k$ is any field. Show that Determinant function exists.
   (b) Let $A$ be a $5 \times 5$ matrix with characteristic polynomial $f(x) = (x - 2)^3(x + 7)^2$ and minimal polynomial same as characteristic polynomial. What is the Jordan canonical form of $A$?

8. (a) Let $T$ be a linear operator on a finite dimensional vector space $V$ over a field $k$. Show that $T$ is triangulable if and only if the minimal polynomial of $T$ is a product of distinct polynomials over $k$.
   (b) Let $T$ be a linear operator on a finite dimensional vector space $V$. Let $W$ be a subspace

M.A. MSc (Part - I) (Paper - I) Alg...

Mathematics.

Con. 4306–08. SM–4040

External [Scheme A] (3 Hours) [ Total Marks : 100

External [Scheme B] (2 Hours) [ Total Marks : 40

**Instructions to the candidates.**

(i) State clearly on the **Top Left Hand Corner** of the answer sheet, the scheme under which you are appearing for the examination.

(ii) Candidates appearing for the **Internal Scheme (Scheme B)** should attempt **three** questions.

(iii) Candidates appearing for the **External Scheme (Scheme A)** should attempt **five** questions.

(iv) All questions carry **equal** marks.

(v) Through out the paper, $R$ denotes a commutative ring with identity and $K$ denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

(1) (a) Define *conjugacy classes* and *centre* of a group $G$. Let $G$ be a finite group. Let $Z(G)$ denote its centre and $C_g$ denote the conjugacy class of $g \in G$. Prove the class equation:

$$|G| = |Z(G)| + \sum_{g \in G, |C_g| > 1} |C_g|.$$

(b) Prove that $|Z(G)| > 1$, if $G$ is a group of order $p^n$, $p$ a prime and $n \in \mathbb{N}$.

(2) (a) Let $G$ be an abelian group. If $a, b \in G$ are elements of order $m, n$ respectively, prove that there exists an element in $G$ whose order is the least common multiple of $m$ and $n$.

(b) Let $G = (\mathbb{Z}/n\mathbb{Z})^*$ denote the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$. Prove that $(\mathbb{Z}/n\mathbb{Z})^*$ is cyclic if $n$ is a prime. Determine $G$ if $n = 55$.

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(3) (a) If $I, J$ are ideals of $R$ such that $I + J = R$, prove that $I \cap J = IJ$ and that $R/IJ \cong R/I \times R/J$.

(b) Give an example of a ring $R$ and ideals $I, J$ such that $I \cap J \neq IJ$.

(4) (a) Prove that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain.

(b) Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a Euclidean domain.

(5) (a) Let $V$ be a vector space over a field $K$. If $V$ has a finite basis, prove that every basis of $V$ is finite and in that case prove that any two bases of $V$ have the same number of elements.

(b) Let $f(X) \in K[X]$ be a polynomial of degree $n$. Prove that $K[X]/\langle f(X)\rangle$ is a vector space over $K$ of dimension $n$, where $\langle f(X)\rangle$, denotes the ideal of $K[X]$, generated by $f(X)$.

(6) (a) If $A$ is a square matrix over $K$, prove that there exists a monic polynomial $f(X) \in K[X]$ such that $f(A) = 0$.

(b) Find a monic polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(A) = 0$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(7) (a) If $A$ is a real symmetric matrix, prove that there exists an orthogonal matrix $P$ such that $PAP^{-1}$ is diagonal.

(b) Prove that the eigen values of a real symmetric matrix are real.

(8) (a) State and prove Silvester's law of inertia.

(b) Determine the signature of the following real symmetric matrix.

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Con/1209-07.

Algebra

SQ-4



External(Scheme A) (Time 3 hours ) Marks 100

Internal(Scheme B) (Time 2 Hours) Marks 40

N.B. (1) Scheme-B students answer any three questions.

(2) Scheme-A students answer any five questions.

(3) All questions carry equal marks.

(4) Write on the top of your answer book the scheme under which you are appearing.

(5) Answers to both the sections are to be written in the same answer book.

(6) Throughout the paper, $R$ denotes a commutative ring with identity and $F$ denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

## Section I

1. (a) If $p$ is a prime number, determine upto isomorphism all abelian groups of order $p^3$.

   (b) If the order of a finite abelian group $G$ is divisible by 10 then show that $G$ has a cyclic subgroup of order 10.

2. (a) Let $G$ be a group, let $Z(G)$ denote the center of $G$ and let $\text{Inn}(G)$ denote the group of inner automorphisms of $G$. Show that $Z(G)$ is a normal subgroup of $G$ and $G/Z(G)$ is isomorphic to $\text{Inn}(G)$.

   (b) Let $A_4$ denote the group of even permutations on the set $\{1, 2, 3, 4\}$. Show that $A_4$ does not have a subgroup of order 6.

3. (a) Show that $\mathbb{Z}[i]$ , the ring of Gaussian integers is a euclidean domain.

   (b) In $\mathbb{Z}[\sqrt{-5}]$ , show that 21 does not factor uniquely as a product of irreducibles.

4. (a) Let $F$ be a field and let $p(X) \in F[X]$. Show that $< p(X) >$ is a maximal ideal in $F[X]$ if and only if $p(X)$ is irreducible over $F$.

   (b) Find all monic irreducible polynomials of degree 2 over $\mathbb{Z}/5\mathbb{Z}$

## Section II

5. (a) If $V$ is a finite dimensional vector space over a field $F$, and $V^*$ denotes the dual vector space of $V$, prove that $V \cong V^*$.

   (b) Let $V$ be a finite dimensional vector space over a field $F$. For any subspace $W$ of $V$ let $W^0$ denote the annihilator of $W$. Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

6. (a) Let $T$ be a linear operator on $V$, where $V$ is as above. If $M$, $N$ are matrices of $T$ corresponding to two ordered bases of $V$, prove that the determinant of $M$ equals the determinant of $N$.

   (b) Let $V$ be the space of all $n \times 1$ column matrices over a field $F$. Show that every linear operator on $V$ is left multiplication by a unique $n \times n$ matrix over $F$.

7. (a) Let $T$ be a linear operator on a finite dimensional vector space $V$. Define characteristic and minimal polynomials of $T$ and show that they have the same roots except for multiplicities.

   (b) Let $T$ be a diagonalizable linear operator on $\mathbb{R}^5$ with set of characteristic values $\{1, 3, 7\}$. What will be the minimal polynomial of $T$? Justify your answer.

8. (a) For any linear operator $T$ on a finite dimensional inner product space $V$ with inner product $< \ , \ >$, show that there exists a unique linear operator $T^*$ on $V$ such that $< Tv, \overline{w} > = < v, T^*w >$ for all $v, w \in V$.

   (b) Let $V$ be a finite dimensional complex inner product space. Let $T$ be a linear operator on $V$. If $T$ is unitary, prove that there is a basis of $V$ with respect to which the matrix of $T$ is diagonal.

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Algebra – 2

30 : 2ndH107

Con: 3798-07. KD-2205

External Scheme A ] (3 Hours) [ Total Marks : 100
External Scheme B ] (2 Hours) [ Total Marks : 40

N.B.(1) **Scheme B** students answer any **three** questions.
(2) **Scheme A** students answer any **five** questions.
(3) **All** questions carry **equal** marks.
(4) Write on the **top** of your answer-book the **scheme** under wich you are appearing.
(5) Answers to **both** the sections are to be written in the **same** answer-book.
(6) Throughout the paper, $R$ denotes a commutative ring with identity and $F$ denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

## Section I

1. (a) Let $G$ be a finite abelian group of order $p^e m$ where $p$ is a prime that does not divide $m$. Show that $G$ is the internal direct product of two subgroups $H$ and $K$ where $|H| = p^e$ and $|K| = m$.
   (b) Prove or disprove: Every abelian group of order 180 has a cyclic subgroup of order 18.

2. (a) Prove that the center of a group of order $p^n$, where $p$ is a prime number and $n$ is a natural number, is non trivial.
   (b) Let $p$ be a prime number. Prove that a group of order $p^2$ is abelian.

3. (a) Show that $\mathbf{Z}[\omega]$, where $\omega$ is a primitive cube root of unity, is a Euclidean domain.
   (b) Let$R = \mathbf{Z}[\omega]$. Prove or disprove: $R[X]$ is a unique factorization domain.

4. (a) Let $F$ be a field and let $p(X) \in F[X]$. Show that $< p(X) >$ is a maximal ideal in $F[X]$ if and only if $p(X)$ is irreducible over $F$.
   (b) Find all monic irreducible polynomials of degree 2 over $\mathbf{Z}/2\mathbf{Z}$.

[ TURN OVER

## Section II

5. (a) Let $V$ be a vector space over a field $F$. If there exists an infinite subset of $V$ which is linearly independent over $F$, prove that the diemnsion of $V$ must be infinte.

   (b) If $V$ is a finite dimensional vector space over a field $F$, and $V^*$ denotes the dual vector space of $V$, prove that $V \cong V^*$.

6. (a) Let $V$ be a vector space of dimension $n$ over a field $F$. Fixing an ordered basis of $V$, prove that there is a one to one correspondence between linear operators on $V$ and $n \times n$ matrices over $F$.

   (b) Let $T$ be a linear operator on $V$, where $V$ is as above. If $M$, $N$ are matrices of $T$ corresponding to two ordered bases of $V$, prove that determinant of $M$ equals the determinant of $N$.

7. (a) Let $V$ be a finite dimensional vector space over a field $F$ and suppose characteristic of $F$ is not equal to 2. If $B$ is a non singular symmetric bilinear form on $V$, prove that there exists a basis of $V$ with respect to which the matrix of $B$ is diagonal.

   (b) Let $V = \mathbb{R}^2$. Let $B : V \times V \to \mathbb{R}$ be the symmetric bilinear form defined by $B((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_2y_2$. Find the matrix of $B$ with respect to the standard basis of $V$ and determine its signature.

8. (a) Show that every complex $n \times n$ matrix is similar over $\mathbb{C}$ to an upper triangular matrix.

   (b) If $A$ is a complex nilpotent matrix, prove that the eigen values of $I + A$ are all equal to 1.

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M.A & M.Sc (Mathematics) (Part - I)

Mathematics : Paper - III - Topology

Con. 1692–09. MS–81

For Internal (Scheme B)] (2 Hours) [Total Marks : 40

For External (Scheme A)] (3 Hours) [Total Marks : 100

20/4/09

N.B. : (1) Write on the **top** of your answer book the **Scheme** under which you are appearing.

(2) Students of **Scheme B** answer **three** questions with at least **one** question from **each section**; Students of **Scheme A** answer **five** questions with at least **two** questions from **each** section.

(3) **All** questions carry **equal** marks. Answers to **both** the sections are to be written in the **same** answer book.

**Section I**

1. (a) Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of finite sets. If J is finite, then show that the sets

$$\bigcup_{\alpha \in J} A_\alpha \quad \text{and} \quad \prod_{\alpha \in J} A_\alpha$$

are both finite sets.

(b) Show that for any non-empty set A, the cardinality of the power set of A is strictly greater than that of A.

2. (a) Define a basis and a subbasis for a topology on X. Give an example of a subbasis which is not a basis.

(b) Show that the countable collection $\mathbf{B} = \{ (a,b) : a, b \in Q \}$ is a basis for the standard topology on R, while the countable collection $\mathbf{B} = \{ (a,b) : a, b \in Q \}$ generates a topology different from the lower limit topology on R.

(c) Let X be a topological space. Let $\wedge$ denote the subset $\{ x \times x : x \in X \}$ of $X \times X$. Show that X is a Hausdorff space if and only if $\wedge$ is a closed subset of $X \times X$.

3. (a) State and prove the pasting lemma to prove continuity of a given function.

(b) Give an example of continuous bijection from one topological space to the other which is not a homeomorphism.

(c) Let $f : X \longrightarrow Y$ be a function where X is a metric space. Show that the function f is continuous iff for every convergent sequence $x_n \longrightarrow x$ in X the sequence $f(x_n)$ converges to f(x) in Y.

4. (a) Define a connected subspace of a topological space X. Show that if A is a connected subspace of X and if $A \subseteq B \subseteq \bar{A}$, then B is connected.

(b) Let $p \in X$ and let $\{ A_i : i \in I \}$ be a family of connected subsets of X such that $p \in A_i$ for every $i \in I$. Show that $\cup_{i \in I} A_i$ is connected.

(c) Show that product of two connected topological spaces is connected.

## Section II

5. (a) Let Y be a subspace of X. Show that Y is compact iff every covering of Y by sets that are open in X has a finite subcollection covering Y.

   (b) Show that every closed subset of a compact space is compact.

   (c) Show that every locally compact Hausdorff space X which is not compact, has a one-point compactification Y such that Y is compact Hausdorff and $\bar{X}$ equals Y.

6. (a) Let X be the set R in the lower limit topology. Show that X is a Lindeloff space but X × X is not a Lindeloff space.

   (b) Show that a closed subspace of a normal space is normal.

7. (a) Let X be a metric space. Show that if every Cauchy sequence in X has a convergent subsequence, then X is complete.

   (b) Let X be a metric space. Show that X is compact iff X is a complete and totally bounded metric space.

   (c) Let C be the set of all continuous real valued functions on [0, 1] equipped with the sup metric. Let F be a subset of C. Show that if F is an equicontinuous family, then so is $\bar{F}$.

8. (a) Let $p : E \to B$ be a covering map. Let B be connected. Show that, if for some $b_0$, the set $p^{-1}(b_0)$ has k elements, then for every b, the set $p^{-1}(b)$ has k elements.

   (b) Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Show that any path $f : [0, 1] \to B$ beginning at $b_0$ has a unique lifting to a path $\bar{f}$ in E beginning at $e_0$.

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Mathematics.

Con. 4352–08. SM–4067

Topology

**Scheme A]** (3 Hours) [Total Marks : 100

**N.B. :** Answer any **four** questions.

**Scheme B]** (2 Hours) [Total Marks : 40

**N.B. :** Answer any three questions.

Topology

1. (a) Give an example with details to show that a countable product of countable sets need not be countable.

   (b) Let $\mathcal{A} = \{f \mid f : \mathbb{N} \longrightarrow \{0,1\}\}$ be the collection of all maps from $\mathbb{N}$ to $\{0,1\}$. Construct an injective map from $\mathbb{R}$ into $\mathcal{A}$.

2. (a) State and prove the 'Pasting Lemma'.

   (b) Let $f, g : [0,1] \longrightarrow X$ be continuous maps into a topological space $X$ such that $f(1) = g(0)$. Define $h : [0,1] \longrightarrow X$ by $h(s) = f(2s)$ for all $s \in [0, \frac{1}{2}]$ and $h(s) = g(2s-1)$ for all $s \in [\frac{1}{2}, 1]$. Verify that $h$ is a continuous map.

3. (a) Let $X$ be a topological space. Prove that $X$ is a connected space if and only if every continuous map from $X$ to the discrete space $\{0,1\}$ is a constant function.

   (b) Define a path connected space. Prove that any open, connected subset of $\mathbb{R}^n$ is path connected.

4. (a) Prove that any compact subset of a Hausdorff space $X$ is closed in $X$.

   (b) State and prove the 'Tube Lemma'.

5. Let $X, Y, Z$ be topological spaces. $S^1$ denotes the unit circle in $\mathbb{R}^2$ with center at $(0,0)$

   (a) Let $f : X \longrightarrow Y$ be a map. When do you say $f$ is a 'quotient map'. Show that $g : \mathbb{R} \longrightarrow S^1$ defined by $g(x) = (\cos x, \sin x)$ $(x \in \mathbb{R})$ is a quotient map.

   (b) Let $\eta : X \longrightarrow Y$ be a quotient map. Suppose $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ be maps with $f$ being continuous and $g \circ \eta = f$. Then show that $g$ is a continuous map.

6. (a) Define the 'interior of a subset' in a space. Define a 'Baire space'. Prove that a compact, Hausdorff space is a Baire space.

   (b) Prove that $\mathbb{Q}$ can not be written as intersection of countably many open subsets of $\mathbb{R}$.

7. (a) Define the notion of 'path homotopy'. Let $\alpha : [0,1] \longrightarrow X$ be a path in a space $X$ with $\alpha(1) = q$. Prove that $\alpha * c_q$ is path homotopic to $\alpha$ $(c_q(s) = q$ for all $s \in [0,1])$.

M.A. & M.Sc. (PART - I)
Mathematics : paper III - Topology
21st April

P4/RT-Ex-08-424

Con. 1335–08.

NB–5323

**Scheme A]** (3 Hours) **[Total Marks : 100**

**N.B. :** Answer any **four** questions.

**Scheme B]** (2 Hours) **[Total Marks : 40**

**N.B. :** Answer any **three** questions.

1. (a) Prove that a finite product of countable sets is countable.
   (b) Consider $\mathcal{A} = \{S \subset \mathbb{N} \mid S \text{ is an infinite set}\}$. Prove that $\mathcal{A}$ is an uncountable set.

2. (a) State and prove the 'Pasting Lemma'.
   (b) Consider the subsets $A, B, C$ of $\mathbb{R}^2$ defined by
   $A = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}$
   $B = \{(x, y) \in \mathbb{R}^2 \mid (x-1/2)^2 + y^2 = 1/4\}$
   and $C = \{(x, y) \in \mathbb{R}^2 \mid (x+1/2)^2 + y^2 = 1/4\}$.
   Construct a continuous bijection from $A \cup B$ onto $A \cup C$.

3. (a) Give an example of a connected space which is not path connected. Justify.
   (b) Prove that $\mathbb{R}^n \setminus \{0\}$ $(n > 1)$ is connected.

4. (a) State and prove the 'Tube Lemma'.
   (b) Let $X, Y$ be topological spaces. If $Y$ is compact, then prove that the projection $\pi_1 : X \times Y \longrightarrow X$ is a closed map.

5. (a) Define the terms: Second Countable Space, Separable Space. Prove that a second countable space is separable.
   (b) Find a countable dense subset of the irrational numbers. Justify.

6. (a) Prove that there does not exist a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous only at rational numbers.
   (b) Prove that a locally compact, Hausdorff space is a Baire space.

7. (a) Let $p : E \longrightarrow B$ be a covering map. Assume $B$ is path connected. For any $x, y \in B$, prove that $p^{-1}(x), p^{-1}(y)$ have same cardinality.
   (b) Define $c(t) = (\cos 2\pi t, \sin 2\pi t) \in S^1$ $(t \in [0,1])$. Show that $c$ is not path homotopic to a constant path in the space $S^1$.

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M.Sc (Mathematics) - I - ~~Analysis~~ Topology (I)

KD–2214

3823–07.

rnal (Scheme B) (2 Hours) [Total Marks : 40

xternal (Scheme A) (3 Hours) [Total Marks : 100

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.

## SECTION I

1(a) Define a countable set. $A$ and $B$ are countable sets. Prove that $A \times B$ is also countable.

1(b) Lat $S$ be the set of sequences $s = (s_n)$ such that $s_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. Show that $S$ is not countable.

2(a) Let $X$ be a topological space. Let $A \subset X$. Define $\partial A$-the boundry of $A$. Prove that $\partial A$ is empty if and only if $A$ is both open and closed.

2(b) Show that a separable metric space is second countable.

3(a) Let $f : X \to Y$ be a map of topological spaces. Prove that the fllowing two conditions are the equivalent: (i) $f^{-1}(F)$ is a closed set for every closed subset $F$ of $Y$. (ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset $A$ of $X$.

3(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x, |y|)$. Prove that $f$ is a closed map.

4(a) Define a ' quotient map'. Suppose $\eta : X \to Y$ is a quotient map. Suppose $f : X \to Z$ and $g : Y \to Z$ are maps of topological spaces such that $g \circ \eta = f$. Show that $f$ is continuous if and only if $g$ is continuous.

4(b) If $(X_1, \tau_1)$ and $(X_2, \tau_2)$ are topological spaces, define product topology on $X = X_1 \times X_2$. Prove that the produc space $X$ is separable if and only if both $X_1$ and $X_2$ are separable.

## SECTION II

5(a) Prove that $[0, 1]$ is a compact subset of $\mathbb{R}$ provided with ususal topology.

5(b) Prove thayt every continuous map $f : X \to \mathbb{R}$ on a compact metric space $X$ is uniformly continuous, bounded and attains the bounds.

6(a) Prove that every open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

6(b) Find the connected components of $\{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \mid xy = 0\}$

7(a) Let $\alpha, \beta$ and $\gamma$ be loops in the topological space $X$ based at a point $a \in X$. Define $\alpha * \beta$. Prove that $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are path homotopic.

7(b) Let $p : G \to B$ be a covering map. Assume that $B$ is path connected. Show that there is a bijection from $p^{-1}(a)$ and $p^{-1}(b)$ for any two points $a$ and $b$ in $B$.

8(a) Consider the covering map $p : S^1 \to S^1$ defined by $p(z) = z^2$. Define $\gamma : [0, 1] \to S^1$ by $\gamma(s) = \cos(\pi s) + \sqrt{-1}\sin(\pi s)$. Find a path $\mu : [0, 1] \to S^1$ such that $\mu(0) = -1$ and $(p \circ \mu)(s) = \gamma(s)$ for all $s \in [0, 1]$.

M.A./M.Sc (Mathematics) (Part – I

V Ex-I-09-C-19

Mathematics Paper – IV – Complex Analysis

**Con. 1433-09.** **MS-6987**

27/4/09

**Scheme A]** (3 Hours) **[Total Marks : 100**

**Scheme B]** (2 Hours) **[Total Marks : 40**

**N.B. :** (1) **All** questions carry **equal** marks.
(2) Candidates of **Scheme A** should attempt any **five** questions.
(3) Candidates of **Scheme B** should attempt any **three** questions.

1. (a) Prove that all points $z \in \mathbb{C}$ satisfying—

$$\left|\frac{z+1}{z+4}\right| = 2$$

lie on a circle. Find its centre and radius.

(b) let $z_1, z_2, z_3, z_4$ be four distinct points in $\mathbb{C}_\infty$. Prove that their cross ratio is real if and only if the four points lie on a circle.

2. (a) Let w be an n-th root of unity. If $w \neq 1$, show that—

$$1 + w + w^2 + \ldots + w^{n-1} = 0.$$

(b) Find all the fourth roots of $8 + i8\sqrt{3}$.

(c) Define the principal branch of the complex logarithm. Evaluate $i^i$, taking the logarihm in its principal branch.

3. (a) Prove that the real and imaginary parts of a holomorphic function satisfy the Cauchy-Riemann equations in its domain of holomorphy.

(b) For the function f(z) defined by—

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

prove that the Cauchy-Riemann equations are satisfied at the origin, but the function f(z) is not differentiable at the origin.

4. (a) Show that the power series $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=k}^{\infty} j(j-1)\ldots(j-k+1)\, a_j z^{j-k}$ have the same radius of convergence for $k \in \mathbb{N}$.

(b) Find the radius of convergence of—

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} (z/2)^{2n}$ (ii) $\sum_{j=2}^{\infty} z^{2j} / j(j-1)$

**[TURN OVER**

5. (a) State and prove the Cauchy theorem for a triangle.
   (b) Using the Cauchy integral formula evaluate—

$$\int_{\Gamma} \frac{\sin \pi z}{z^2 + 1}$$

   where $\Gamma$ is the circle $| z | = 3$.

6. (a) State and prove the open mapping theorem for a holomorphic function.
   (b) Show that a holomorphic function which only takes real values is a constant.

7. (a) Prove that a complex polynomial of degree n has exactly n zeros.
   (b) Prove that the zeros of a holomorphic function are isolated.

8. (a) State and prove Rouché's theorem.
   (b) Determine the number of zeros, counting multiplicities, of the polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ inside the circle $| z | = 2$.

9. (a) Let f(z) be an analytic function with an essential singularity at $z = a$. For any given complex number w, show that there exists a sequence $\{ z_n \}$ converging to a such that the sequence $\{ f(z_n) \}$ converges to w.
   (b) Use the Cauchy Residue Theorem to evaluate the real integral—

$$\int_0^{\infty} \frac{\cos x}{x^2 + 1}.$$

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Con. 4356–08. M.A & M.Sc (Part-I) Mathematics. P. IV — Complex Analysis SM–4062

Scheme A (External) (3 Hours) [ Total Marks : 100

Scheme B (Internal) (2 Hours) [ Total Marks : 40

20/10/08

**N.B:** Scheme A ( External) students should attempt **five** questions selecting at least **two** from **each section.**

Scheme B ( Internal) students should attempt **three** questions selecting at least **one** from **each section.**

Answers to both the sections are to be written in the same answerbook.

All questions carry equal marks.

**Note:** $\mathbb{C}$ denotes the set of all complex numbers. A holomorphic function on an open set $G \subset \mathbb{C}$ is a complex function differentiable at every point of $G$.

## Section I

Q. 1 a] Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let $r$ be given by

$$\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}} \quad , \ 0 < r \le \infty .$$

Prove : (i) if $|z| < r$, the series converges absolutely;

( ii) if $0 < \rho < r$ then the series converges uniformly on $\{z : |z| \le \rho\}$.

b] Find the radius of convergence of the series

(i) $\sum_{n=0}^{\infty} n^n z^n$ (ii) $\sum_{n=0}^{\infty} \frac{z^n}{n}$

Q. 2 a] Let $G \subset \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be defined by $f(z) = u(z) + iv(z)$ where $u$ and $v$ are real valued functions defined on $G$. If $u$ and $v$ have continuous partial derivatives and if they satisfy the Cauchy-Riemann equations then prove that $f$ is complex differentiable at every point of $G$.

b] Find a holomorphic function $f(z) = u(z) + iv(z)$ whose real part is $2xy + 2x$.

Q. 3 a] Prove that a Mobius transformation takes circles onto circles.

b] Find the image of the circle $x^2 + y^2 + 2x = 0$ in the complex plane under the transformation $w = \frac{1}{z}$.

Q. 4 a] If $G \subset \mathbb{C}$ is open and connected and $f$ is a branch of $\log z$ on $G$, prove that the totality of branches of $\log z$ are the functions $f(z) + 2\pi ni$, $n \in Z$ (set of integers).

b] Prove that the function $f(z)$ is not complex differentiable at any point $z$ in the complex plane $\mathbb{C}$ if $f(z) = \bar{z} = x - iy$.

**[ TURN OVER**

## Section II

Q. 5 a] State and prove Cauchy's theorem for a triangle.

b] Use Cauchy-integral formula to evaluate

$$\int_{\gamma} \frac{\cos(e^z)}{z(z+2)} dz \qquad \text{where } \gamma \text{ is the circle } |z| = 1.$$

Q.6a] Let $D = \{z : |z| < 1\}$ be the unit disk and suppose $f$ is holomrphic on $D$ with $f(0) = 0$ and $|f(z)| < 1$ for $z$ in $D$. Prove that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z$ in $D$.

b] Let $\alpha$ be a complex number in the unit disk $D = \{z : |z| < 1\}$. Find a one-one holomorphic function from $D$ onto itself taking $\alpha$ to 0.

Q. 7 a] State and prove Casorati-Weierstrass theorem.

b] Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for

( i) $1 < |z| < 2$, ( ii) $|z| < 1$.

Q. 8 a] State and prove Residue theorem.

b] Use Residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4}$$

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VI March 08 234

Con. 1045–08. NB–4240

Complex An...

Scheme A (External) ( 3 Hours ) [ Total Marks : 100

Scheme B (Internal ) ( 2 Hours ) [ Total Marks : 40

N.B: Scheme A ( External) students should attempt **five** questions selecting at least **two** from **each section**.
**Scheme B ( Internal)** students should attempt **three** questions selecting at least **one** from **each section**.
Answers to both the sections are to be written in the same answerbook.
All questions carry equal marks.

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**Note:** $\mathbb{C}$ denotes the set of all complex numbers. A holomorphic function on an open set $G \subset \mathbb{C}$ is a complex function differentiable at every point of $G$.

## Section I

Q. 1 a] If the power series $\sum a_n z^n$ converges for a particular value $z_0 (\neq 0)$ of $z$, prove that it converges ( absolutely) for every $z$ for which $|z| < |z_0|$.

b] Prove that the power series $\sum_{n=0}^{\infty} \frac{(z+2)^n}{(n+2)^3 4^{n+1}}$ converges for every $z$ such that $|z+2| < 4$.

Q. 2 a] Let $f(z) = u(z) + iv(z)$ be a holomorphic function on a domain $G$, prove that $f(z)$ satisfies the Cauchy-Riemann equations on $G$.

b] Suppose $f: G \to \mathbb{C}$ is holomorphic and that $G$ is connected. Show that if $f(z)$ is real for all $z$ in $G$ then $f$ is constant.

Q. 3 a] Define a Mobius transformation. If $z_2, z_3, z_4$ are distinct points in $\mathbb{C}_\infty$ and $T$ is any Mobius transformation, prove that
$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4) \quad \text{for any point } z_1.$$

b] Find a Mobius transformation which maps the points $z = 1, i, -1$ onto the points $w = 0, 1, \infty$. Also find the image of the circle $|z| = 1$ in the complex plane under this Mobius transformation.

Q. 4 a] Find the fourth roots of $(-1+i)$ and locate them graphically.

b] Prove that $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

c] Define the function $\cos z$ and prove that it is entire function.

**[ TURN OVER**

## Section II

Q. 5a] Let $\Omega$ be a star shaped domain with respect to $\alpha$ and $f$ be an holomorphic function on $\Omega$. Prove that there exists a holomorphic function $F$ on $\Omega$ such that $F'(z) = f(z)$ in $\Omega$.

b] Use Cauchy-Integral formula to evaluate

$$\int_\gamma \frac{\cos \pi z}{z^2 - 1} dz \qquad \text{where } \gamma \text{ is the circle } |z-1| = 1.$$

Q. 6a] If $f$ is bounded entire function, prove that $f$ is constant.

b] Let $f(z) = u(x, y) + i\, v(x, y)$ where $z = x + iy$ be an entire function. If $u(x, y) = Re\, f(z)$ is bounded for all $z$ in complex plane $\mathbb{C}$, prove that $u(x, y)$ and $v(x, y)$ are constant functions.

Q. 7 a] State and prove Rouche's theorem.

b] Determine the number of zeros, counting multiplicities, of the polynomial $z^4 + 3z^3 + 6$ inside the circle $|z| = 2$.

Q. 8 a] Define: Isolated singularity. Let $z = z_0$ be an isolated singularity of $f$ and let

$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$ be its Laurent series in $ann(z_0; 0, R)$. Prove that $z = z_0$

is a removable singularity of $f$ iff $a_n = 0$ for $n \leq -1$.

b] Use Cauchy-Residue theorem to evaluate

$$\int_0^\infty \frac{dx}{x^2 + 1}$$

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Complex A

P3/D/UPQ/-OLE07-9-12 47

Con. 3842-07.

KD- 2238

Scheme A (External) ( 3 Hours) [Total Marks : 100

Scheme B (Internal) ( 2 Hours). [Total Marks : 40

**N.B:** Scheme A(External) students should attempt **five** questions selecting at least **two** from **each section**.
Scheme B(Internal) students should attempt **three** questions selecting at least **one** from **each section**.
Answers to both the sections are to be written in the same answerbook.
All questions carry equal marks.

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**Note:** A holomorphic function on an open set $G \subset \mathbb{C}$ is a complex function differentiable at every point of $G$.

## Section I

1. (a) Define: The cross ratio $(z_1, z_2, z_3, z_4)$ of $z_1, z_2, z_3$ and $z_4$. If $z_2, z_3, z_4$ are distinct points in $\mathbb{C}_\infty$ and $T$ is any Mobius transformation, prove that for any point $z_1$

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

   (b) Let $H = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ and let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Find a Mobius transformation $g$ such that $g(H) = D$ and $g(i) = 0$. Justify your claims.

2. (a) If $\sum_{n=0}^{\infty} a_n(z-a)^n$ is a given power series with radius of convergence $R$, then prove that

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

   if this limit exists.

   (b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ is 1. Discuss the convergence of this series at points on the boundary of the disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

3. (a) Construct a branch of logarithm $l(z)$ on $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$. If $f$ is any other branch of logarithm on $G$ then show that there exists an integer $k$ such that $f(z) = l(z) + 2\pi ik$ for all $z$ in $G$.

   (b) Compute the following values when $\log z$ is defined by its principal value on the open set $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$.
   (i)$i^i$ (ii)$2^i$.

4. (a) Let $G$ be either the whole complex plane $\mathbb{C}$ or some open disc in $\mathbb{C}$. If $u : G \to \mathbb{R}$ is a harmonic function, prove that $u$ has a harmonic conjugate.

   (b) If $G$ is an open subset of $\mathbb{C}$ and if $u : G \longrightarrow \mathbb{R}$ is a harmonic function, prove that $u$ is infinitely differentiable.

   (c) Let a function $f$ be holomorphic on an open connected subset $G$ of $\mathbb{C}$. If $f(z)$ is real for all $z$ in $G$ then prove that $f$ must be constant on $G$.

[ TURN OVER

## Section II

5. (a) State and prove Cauchy's Theorem for a triangle.

   (b) Use Cauchy's integral formula to evaluate

$$\int_{\gamma} \frac{\sin z}{(z-\pi)(z-(\pi/2))}\, dz \qquad \text{where } \gamma \text{ is the circle } |z| = 2.$$

6. (a) Let $f$ be a holomorphic function on an open connected subset $G$ of $\mathbb{C}$. If there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \cdots$ then prove that $f$ is identically zero on $G$.

   (b) Let $f$ be an entire function. Suppose there is a constant $M$, an $R > 0$ and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that $f$ is a polynomial of degree $\leq n$.

7. (a) Define : (i) Removable singularity (ii) Essential singularity

   (b) Let $A_\delta$ be the annulus $\{z \in \mathbb{C} \mid 0 < |z - a| < \delta\}$ and let $f$ be a function holomorphic on $A_\delta$. If $f$ has an essential singularity at $a$ then show that $f(A_\delta)$ is dense in $\mathbb{C}$.

   (c) Expand

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

   in a Laurent series about $z = 1$ and name the singularity.

8. (a) State Riemann Mapping Theorem.

   (b) State and prove Argument Principle.

   (c) Use Residue Theorem to evaluate

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

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External (Scheme A) ] (3 Hours) [ Total Mar...

Internal/External (Scheme B) ] (2 Hours) [ Total Marks...

4/5/09

N.B. (1) **Scheme-A** students answer any **five** questions.

(2) **Scheme-B** students answer any **three** questions.

(3) **All** questions carry **equal** marks.

(4) **Write** on the **top** of your **answer book** the **scheme** under which you are appearing.

1. (a) Find the number of 3-element subsets {a, b, c} of {1, 2, ... , 2008} such that 3 divides a + b + c.

   (b) 6 boys and 5 girls are to be seated around a table. Find the number of ways this can be done such that no two girls are adjacent.

2. (a) Show that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

   (b) How many minimum numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add to 7 ?

3. (a) For each $n \in \mathbb{N}$, show that the number of partitions of n into parts each of which appears at most twice, is equal to the number of partitions of n into parts the sizes of which are not divisible by 3.

   (b) For each $n \in \mathbb{N}$, let $a_r$ denote the number of r-digit ternary sequence that contain an odd number of 0's and an even number of 1's. Find $a_r$.

4. (a) Define matching in the bipartite graph $G = (X \cup Y, E)$. Show that the bipartite graph $G = (X \cup Y, E)$ has a complete matching if and only if $|J(A)| \geq |A|$ for all $A \subseteq X$, where $J(A) = \{ y \in Y / xy \in E \text{ for some } x \in A \}$.

   (b) Find the largest number of sets in the family $A_1, A_2, \ldots, A_{10}$ which together have a system of distinct representatives, where $A_1 = \{1, 8, 10, 13\}$, $A_2 = \{1, 4, 5, 7, 11\}$, $A_3 = \{5, 8\}$, $A_4 = \{8, 13\}$, $A_5 = \{2, 3, 4, 11, 12\}$, $A_6 = \{5, 6, 10, 13\}$, $A_7 = \{10, 13\}$, $A_8 = \{5, 8, 10, 13\}$, $A_9 = \{1, 5, 8\}$, $A_{10} = \{1, 5, 8, 10, 13\}$.

5. (a) Define $D_n$, derangement of n objects. Show that —

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

   (b) Solve recurrence relation $a_n - 3a_{n-1} = 2 - 2n^2$ with initial condition $a_0 = 3$.

6. (a) Let G be a group of permutations of a non empty finite set X, and let x be any chosen element of X. Then show that $|Gx| \times |G_x| = |G|$ where $Gx = \{g(x)/g \in G\}$ and $G_x = \{g \in G / g(x) = x\}$.

   (b) What is the smallest number of colors required to color faces cubes to get 57 different colored cubes ?

7. (a) Calculate the number of words that can be formed by rearranging the letters ALLAHABAD so that no letters appears at one of its original positions.

   (b) State and prove Mobius inversion formula.

8. (a) You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails 1 – p. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins ?

   (b) For independent random variables X and Y, show that $E(XY) = E(X)E(Y)$, where $E(XY)$, $E(X)$ and $E(Y)$ denote expectations of XY, X and Y respectively.

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Comb

NB



External (Scheme A)] (3 Hours) [Total Marks: 10

[redacted]ernal/External (Scheme B)] (2 Hours) [Total Marks: 40

N.B. 1) Scheme-A students answer any five questions.
2) Scheme-B students answer any three questions.
3) All questions carry equal marks.
4) Write on the top of your answer book the scheme under which you are appearing.

1. (a) How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?
(b) What is the probability that each player has a hand containing an ace when the 52 cards of a standard deck are dealt to four players?

2. (a) During a month with 30 days a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 45 games.
(b) Prove that for integer $m$, $0 \leq m \leq \binom{n}{r} - 1$, can be uniquely expressed as

$$m = \sum_{i=1}^{r} \binom{v_i - 1}{i}, \text{ for some integers } 1 \leq v_1 \leq v_2 \leq \ldots \leq v_r \leq n.$$

3. (a) How many different strings can be made from the letters in MISSISSIPPI, using all letters?
(b) Determine the number of different ways ten identical balloons can be given to four children if each child receives at least two balloons.

4. (a) Define Stirling numbers of second kind $S(n,k)$ and prove the following identity:
$x^n = \sum_{k=1}^{n} S(n,k)[x]_k$, where $[x]_k$ denotes the falling factorial.
(b) Let $R_{n,m}$ be the rook polynomial of $n \times m$ chess board. Derive a recurrence relation connecting $R_{n,m}, R_{n-1,m}, R_{n-1,m-1}$.

5. (a) Solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$ and $a_1 = 5$.
(b) State and prove Mobius inversion formula.

6. (a) Let $G = (X \cup Y, E)$ be a bipartite graph. Show that the largest number of edges in a matching in $G$ equals the smallest number of vertices that cover the edges.
(b) $G$ be a group of permutations of a set $X$. Show that number of orbits of $G$ on $X$ is $\frac{1}{|G|} \sum_{g \in G} |F(g)|$ where $F(g) = \{x \in X / g(x) = x\}$.

7. (a) Find the cycle index of a group of symmetries of a regular pentagon. Hence find the number of ways of colouring exactly two vertices black, one white and two red.
(b) Show that the number of partitions of a positive integer $n$ into at most $k$ parts is equal to the number of partions of $n$ in which every part is less than or equal to $k$.

8. (a) Let $X$ and $Y$ be independent random variables with binomial distribution $B(m,p)$ and $B(n,p)$, respectively. What is the distribution of $X + Y$.
(b) You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random ar

M.A. MSC (part - II) (paper - V) ...

Mathematics.

Con. 4360–08. SM–4055

External (Scheme A)] (3 Hours) [Total Marks: 100

Internal/External (Scheme B)] (2 Hours) [Total Marks: 40

N.B. 1) Scheme-A students answer any five questions.
2) Scheme-B students answer any three questions.
3) All questions carry equal marks.
4) Write on the top of your answer book the scheme under which you are appearing.

1. (a) How many positive integers between 100 and 999 both inclusive are not divisible by either 3 or 4?
(b) Let $m, n$ and $r$ be non-negative integers with $r$ not exceeding either $m$ or $n$. Show that $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$.

2. (a) Show that every sequence of $n^2+1$ distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.
(b) How many solutions are there to the inequality $x_1+x_2+x_3 \leq 11$ where $x_1$, $x_2$ and $x_3$ are non-negative integers ?

3. (a) How many 10 letters words are there in which each of the letter $e$ and $n$ occur at least once and each of the letter $r, s$ occur at least twice.
(b) Define Stirling number $S(n,k)$ of second kind for $1 \leq k \leq n$. Show that $S(n,1) = 1 = S(n,n)$ and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $2 \leq k \leq n-1$.

4. (a) Let $X$ be finite set with $n$ number of elements. Show that number of subset of $X$ with even number of elements is same as number of subsets of $X$ with odd number of elements.
(b) Find number of primes less than 200 using the principal of inclusion exclusion.

5. (a) In how many ways can 25 identical donuts be distributed to four police officers so that each officer gets at least three but no more than seven donuts?
(b) In how many ways can a $2 \times n$ rectangular board be tiled using $1 \times 2$ and $2 \times 2$ pieces?

6. (a) Define matching $M$ in the bipartite graph $G = (X \cup Y, E)$. When $M$ is said to be a maximum matching? Show that if $M$ is not a maximum matching then $G$ contains alternating path for $M$.
(b) Show that $r \times n$ Latin rectangle can always be completed to a Latin square of order $n$.

7. (a) How many round necklaces can be constructed using two red beads, three white beads and two blue beads?
(b) In how many ways can Rs. 100 be exchanged for notes of the value Rs. 50, Rs. 20, Rs. 10 and Rs. 5?

8. (a) A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, he needs to answer all three questions. What is the probability that the student will pass the examination if he knows the answers to 90 questions on the list?
(b) State and prove Bayes' Theorem.

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Con. 3758–07.

KD-

External (Scheme A)] (3 Hours) [Total Marks :

Internal / External (Scheme B)] (2 Hours) [Total Marks : 4

N.B. (1) Scheme-A students answer any **five** questions.

(2) Scheme-B students answer any **three** questions.

(3) All questions carry equal marks.

(4) Write on the top of your answer book the scheme under which you are appearing.

1. (a) Find the number of ways of choosing 3 distinct integers frm the set $X = \{1, 2, \ldots, 100\}$ so that their sum will be divisible by 3.

   (b) Determine the total number of combinations (of any size) of a multiset of $k$ distinct objects with finite repetition numbers $n_1, n_2, \ldots, n_k$ respectively.

2. (a) Consider the set of words of length $n$ generated from the alphabet $\{0, 1, 2\}$.

   i. Show that the number of words in each of which the digit 0 appears an even number of times is $\frac{3^n+1}{2}$.

   ii. Using (i) above or otherwise, prove the identity:

$$\binom{n}{0}2^n + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{q}2^{n-q} = \frac{3^n+1}{2},$$

   where $q = n$ if $n$ is even and $q = n - 1$ if $n$ is odd.

   (b) Use a combinatorial argument to prove that $\frac{(2n)!}{2^n}$ and $\frac{(3n)!}{3^n \times 2^n}$ are integers.

3. (a) Prove that of any 10 points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{3}$.

   (b) Find the number of positive integral solutions of

$$(x_1 + x_2 + x_3)(y_1 + y_2 + y_3 + y_4) = 77.$$

4. (a) Find the number of permutations of the nine positive digits in which the blocks 415, 12, and 23 do not appear.

   (b) Given a sequence of $2n$ elements, find the number of their derangements such that the first $n$ elements of each derangement are (i) the first $n$ elements of the sequence, and (ii) the last $n$ elements of the sequence.

5. Attempt any TWO.

   (a) Using an argument based on Ferrer diagrams or otherwise show that the number of partitions of $n$ which have at most $m$ parts is equal to the number of partitions of $n + \frac{1}{2}m(m+1)$ in which there are $m$ parts, all of them different.

   (b) Explain what is meant by Hypergeometric random variable and calculate its expectation and variance.

   (c) Solve the recurrence relation: $a_n = \sum_{k=1}^{n-1} a_k a_{n-k}$, $n \geq 2$ subject to initial value $a_1 = 1$.

6. (a) For two independent throws of a balanced die, find the expectations of the following variables

   i. Twice the larger score minus the second.

   ii. The total number of fours and sixes.

   (b) In a sample 2% of the population have a certain blood disease in a serious form; 10% have it in a mild form; and 88% don't have it at all. A new blood test is developed; the probability of testing positive is $\frac{9}{10}$ if the subject has the serious form, $\frac{6}{10}$ if the subject has the mild form, and $\frac{1}{10}$ if the subject doesn't have the disease. I have just tested

7. (a) Let $A_1, \cdots, A_n$ be $n$ subsets of a set $S$ such that $|A_i| = m$ for each $i, 1 \leq i \leq n$ and such that each element of $S$ occurs in exactly $m$ of the sets from $A_1, \ldots, A_n$. Use Hall's theorem to show that the sets $A_1, \ldots, A_n$ posses a system of distinct representatives.

   (b) Define $S(n,k)$,the Stirling numbers of second kind. Calculate : (i)$S(n,2)$, (ii)$S(n,n-1)$ and (iii) $S(7,3)$.

8. (a) State and prove Burnside Frobenius theorem about the number of orbits under an action of a finite group acting on a finite set.

   (b) Find the number of distinguishable necklaces made of of 9 spherical stones using three colors. How many of these use 2 blue, 3 green and 4 red stones?