Subject CT8 — Financial Economics
Core Technical

EXAMINERS' REPORT

April 2009

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairman of the Board of Examiners

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1 Put in simple terms, an *arbitrage opportunity* is a situation where we can make a sure profit with no risk. This is sometimes described as a free lunch. Put more precisely an arbitrage opportunity means that:

(a) We can start at time 0 with a portfolio which has a net value of zero (implying that we are long in some assets and short in others).

(b) At some future time T:

- the probability of a loss is 0
- the probability that we make a strictly positive profit is greater than 0

2 (i) Semi-strong form because linked to publically available information.

(ii) It is true that some ratios have predictive power.

This may not violate market-efficiency as the ratios may be acting as a proxy for risk.

(iii) Active managers believe market is not fully efficient hence they attempt to detect mispricings.

Passive managers believe in efficiency and just diversify across the whole market.

3 (i) (a) Direct statistical evidence shows volatility varies over time. Volatility implied from option prices also shows volatility/volatility expectations vary over time.

(b) Good theoretical reasons to expect this to vary over time. Equities should give a risk premium over bonds and bond yields vary over time. Empirically difficult to test.

(c) Empirically unsettled. Some evidence for mean reversion but rests heavily on the aftermath of a few dramatic crashes also conversely some evidence of momentum effects.

(d) Strong empirical evidence that prices are non-normal. Crashes happen more than would be expected. In addition more days with small/no changes than one would expect.

(ii) Some problems with random walk. Can consider the points in (i) again:

(a) Random walk assumes constant volatility — ARCH would be better in this respect. Also processes with non-normal returns can give a similar effect.

(b) Random walk assumes drift is constant.

(c) No allowance for mean reversion in random walk.

(d) Random walk does assume normality. Quite a few alternatives. Levy processes, jump processes like Poisson.

The above answer incorporates what is in the core reading. Quite a few processes which are not mentioned are valid e.g. GARCH, EGARCH, QGARCH.

4 (i) Select on the basis of expected return and variance of return over a single time horizon.

(ii) A portfolio is efficient if an investor cannot find a better one in the sense that it has both a higher expected return and a lower variance.

(iii) The basic idea is that the efficient frontier is a straight line which is the tangent to the efficient frontier (of risky assets) which passes through the point in (s.d., return) space corresponding to the risk-free asset.

Initially need to find the portfolio using A and B that maximises

(expected return – 4%)/standard deviation

Put say proportion x of assets in A and $(1 - x)$ in B.

Expected return of risky portfolio is $0.09x + 0.05(1 - x)$

Standard deviation of risky portfolio is

$[(0.18x)^2 + (0.08(1 - x))^2]^{0.5}$

Thus need to find x to maximise

$[0.09\text{x} + 0.05(1 - x) - 0.04]/ [(0.18x)^2 + (0.08(1 - x))^2]^{0.5}$

Method is to take logs.

Need to maximise

$\ln[0.01 + 0.04x] - 0.5\ln[0.0324x^2 + 0.0064(1 - x)^2]$

$= \ln[0.01 + 0.04x] - 0.5\ln[0.0064 - 0.0128x + 0.0388x^2]$

Differentiate and set to zero.

$0.04/[0.01 + 0.04x] - 0.5[- 0.0128 + 0.0388.2x]/ [0.0064 - 0.0128x + 0.0388x^2] = 0$

x is found to be 0.4969

(Can also calculate x using Lagrangian multipliers)

When x is 0.4969
Expected return of risky portfolio is 0.069876
Standard deviation of risky portfolio is 0.09808

Thus the efficient frontier is the straight line through (0.04.0) and (0.069876, 0.09808).

(iv) Portfolio would be that corresponding to the point where the utility indifference curve of the investor touched the efficient frontier.

5 (i) The market price of risk is $(E_m - r)/\sigma_{m\cdot}$

(ii) $E_m =$ (100 × (0% × 0.1 + 5% × 0.7 + 10% × 0.2) +
50 × (1% × 0.1 + 3% × 0.7 + 7% × 0.2) +
100 × (2% × 0.1 + 3% × 0.7 + 3% × 0.2)) ÷ 250

= 4.08%

$\sigma^2_{m\cdot} =$ [(100 × 0% + 50 × 1% + 100 × 2%) ÷ 250 – 3.36%]^2 × 0.1 +
[(100 × 5% + 50 × 3% + 100 × 3%) ÷ 250 – 3.36%]^2 × 0.7 +
[(100 × 10% + 50 × 7% + 100 × 3%) ÷ 250 – 3.36%]^2 × 0.2

= 0.00022736

Thus the market price of risk is (4.08% – 2.5%)/1.5078%

= 1.0479

(iii) The investor should consider their own human capital which is likely to be large and unpredictable compared to their other assets.

6 **Structural models**

Structural models are explicit models of a corporate entity issuing both equity and debt. They aim to link default events explicitly to the fortunes of the issuing corporate entity. An example of a structural model is the Merton model.

Reduced form models

Reduced form models are statistical models which use observed market statistics rather than specific data relating to the issuing corporate entity. The market statistics most commonly used are the credit ratings issued by credit rating agencies such as Standard and Poor's and Moody's.

Reduced form models use market statistics along with data on the default-free market to model the movement of the credit rating of the bonds issued by a corporate entity over time. The output of such models is a distribution of the time to default.

Intensity-based models

Intensity-based models model the factors influencing the credit events which lead to default and typically (but not always) do not consider what actually triggers the credit event.

7 (i) $S_t = S_0 \exp(\sigma Z_t + (r - ½\sigma^2)t)$, where Z is a standard Brownian motion.
Alternative solution: dS_t/S_t = r dt + σ dZ_t

(ii) Apply B-S formula with $K = 2$, $S = 1.8$, $r = 0.05$, $t = 1$ and sigma = 0.2, gives $c = 0.10$. With sigma = 0.4, $c = 0.24$. Try sigma = 0.2 + 0.2 * 0.1/0.14 = 0.34 and get $c = 0.20$.

(iii) The unique fair price is $V = E_P[e^{-rT}D]$, where P is the EMM. Thus

$V = e^{-r}[P(S_{0.5}$>£2 and $S_1 > 2S_{0.5}) + 0.5\ P(S_{0.5}$ < £2 and $S_1 > 2S_{0.5})]$

= e^{-r}{integral from 2 to infinity Prob[S_1 > 2x] * $f(x)$ dx +
Integral from 0 to 2 0.5 Prob[S_1 > 2x] $f(x)$ dx}

= e^{-r}{integral from 2 to infinity Phi((ln(2x) – ln(1.8) – 0.05)/0.34)
phi((ln(x) – ln(1.8) – 0.025)/0.24) dx +

Integral from – infinity to 2 0.5 Phi((ln(2x) – ln(1.8) – 0.05)/0.34) phi
((ln(x) – ln(1.8) – 0.025)/0.24)dx}

Where Phi is the cumulative standard normal distribution function and phi is the standard normal density function. Standard deviation of ln(S_0.5) is 0.34/sqrt(2) = 0.24.

There is another way to solve the problem and the price can be calculated directly from

$P(S_{0.5} > 2 \text{ and } S_1 > 2\, S_{0.5}) = P(S_{0.5} > 2 \text{ and } S_1/S_{0.5} > 2)$
$= P(S_{0.5} > 2) * P(S_1/S_{0.5} > 2)$ since $S_1/S_{0.5}$ is independent of $S_{0.5}$.

$P(S_{0.5} > 2) = P(\sigma Z_{0.5} + (r - 0.5\sigma^2) *0.5 > \ln(2) - \ln(1.8))$
$= P(Z > x)$

with

$x = [1/(\sigma * \sqrt{0.5})] * [\ln(2) - \ln(1.8)) - (r - 0.5\sigma^2) * 0.5]$

Similarly, defining

$y = [1/(\sigma * \sqrt{0.5})] * [\ln(2)) - (r - 0.5\,\sigma^2) * 0.5\,]$

we get

$V = e^{-r} * [(1 - \varphi_x) * (1 - \varphi_y) + 0.5 * \varphi_x * (1 - \varphi_y)]$
where φ_x is the cumulative normal distribution at x.

8 (i) Suppose that at time t we hold the portfolio (ϕ_t, ψ_t) where ϕ_t represents the number of units of S_t held at time t and ψ_t is the number of units of the cash bond held at time t. We assume that S_t is a tradeable asset as described above. The only significant requirement on (ϕ_t, ψ_t) is that they are previsible: that is, that they are F_{t-}-measurable (so ϕ_t and ψ_t are known based upon information up to but not including time t).

Let $V(t)$ be the value at time t of this portfolio: that is, $V(t) = \phi_t S_t + \psi_t B_t$.

Now consider the instantaneous pure investment gain in the value of this portfolio over the period t up to $t + dt$: that is, assuming that there is no inflow or outflow of cash during the period $[t, t + dt]$. This is equal to

$$\phi_t dS_t + \psi_t dB_t$$

The instantaneous change in the value of the portfolio, allowing for cash inflows and outflows, is given by

$$dV(t) \equiv V(t + dt) - V(t) = \phi_t dS_t + S_t d\phi_t + d\phi_t . dS_t + B_t d\psi_t + \psi_t dB_t .$$

The portfolio strategy is described as self-financing if $dV(t)$ is equal to $\phi_t dS_t + \psi_t dB_t$: that is, at $t + dt$ there is no inflow or outflow of money necessary to make the value of the portfolio back up to $V(t + dt)$.

(ii) Delta is just one of what are called the Greeks. The Greeks are a group of mathematical derivatives which can be used to help us to manage or understand the risks in our portfolio.

Let $f(t, s)$ be the value at time t of a derivative when the price of the underlying asset at t is $S_t = s$.

The delta for an individual derivative is

$$\Delta = \frac{\partial f}{\partial s} \equiv \frac{\partial f}{\partial s}(t, S_t).$$

(iii) In the martingale approach we showed that there exists a portfolio strategy (ϕ_t, ψ_t) which would replicate the derivative payoff. We did not say what ϕ_t actually is or how we work it out. In fact this is quite straightforward.

First we can evaluate directly the price of the derivative

$$V_t = e^{-r(T-t)} E_Q[X \mid F_t]$$

either analytically (as in the Black-Scholes formula) or using numerical techniques.

In general, if S_t, represents the price of a tradeable asset

$$\phi_t = \frac{\partial V}{\partial s}(t, S_t).$$

ϕ_t is usually called the Delta of the derivative.

The martingale approach tells us that provided:

- we start at time 0 with V_0 invested in cash and shares
- we follow a self-financing portfolio strategy
- we continually rebalance the portfolio to hold exactly ϕ_t units of S_t with the rest in cash

then we will precisely replicate the derivative payoff.

9 (i) $B(t,T) = \exp\left[-\int_t^T f(t,u)du\right].$

(ii) Since the bond market is complete, the discounted price of a zero-coupon bond is a martingale with respect to the risk-neutral probability measure. Using Itô, the dynamics for the discounted zero-coupon bond price $\overline{B}(t,T)$ are:

$$\frac{d\overline{B}(t,T)}{\overline{B}(t,T)} = \left(m(t,T) - r(t)\right)dt + S(t,T)dW_t$$

As to be a martingale, the drift term should be equal to 0:

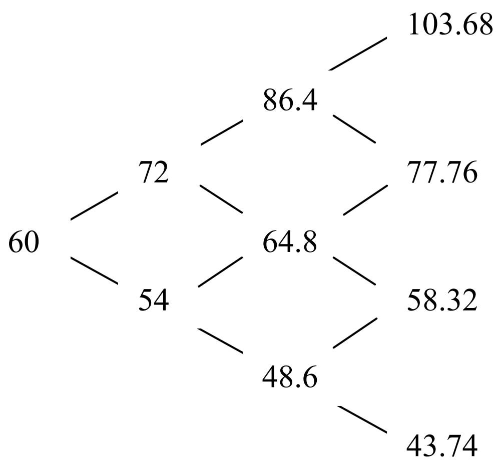
$$\left(m(t,T) - r(t)\right) = 0$$

In other words

$$\int_t^T a(t,s)ds = \left(\int_t^T \sigma(t,s)ds\right)^2.$$

10 (i) (a) There is no arbitrage in the market since $d = 0.9 < 1.11 < 1.2$.

(b)



(ii) To price the call option, we use the risk-neutral pricing formula. We use the following simplifying notation:

$$C_{uuu} = \left(u^3 S_0 - K\right)^+ ;$$

$$C_{uud} = \left(u^2 d S_0 - K\right)^+ ;$$

$$C_{udd} = \left(u d^2 S_0 - K\right)^+ ;$$

$$C_{ddd} = \left(d^3 S_0 - K\right)^+ .$$

At time 2, we get in the upper state,

$$C_2(uu) = \frac{1}{1+r}\left[qC_{uuu} + (1-q)C_{uud}\right],$$

in the medium state,

$$C_2(ud) = \frac{1}{1+r}\left[qC_{uud} + (1-q)C_{udd}\right]$$

and in the lowest state,

$$C_2(dd) = \frac{1}{1+r}\left[qC_{udd} + (1-q)C_{ddd}\right],$$

where the risk-neutral probability of an upward move is

$$q = \frac{(1+r)-d}{u-d}.$$

At time 1, we get in the upper state,

$$C_1(u) = \frac{1}{1+r}\left[qC_2(uu) + (1-q)C_2(ud)\right],$$

and in the lower state,

$$C_1(d) = \frac{1}{1+r}\left[qC_2(ud) + (1-q)C_2(dd)\right].$$

At time 0,

$$C_0 = \frac{1}{1+r}\left[qC_1(u) + (1-q)C_1(d)\right].$$

Hence

$$C_0 = 16.68.$$

(iii) Two paths are relevant for this knock-in option: "up-up-up" and "up-up-down". The associated payoff are respectively C_{uuu} with probability q^3 and C_{uud} with probability $q^2(1-q)$. The price at time 0 of the option is therefore:

$$Knock-in_0 = \frac{1}{(1+r)^3}\left[q^3C_{uuu} + q^2(1-q)C_{uud}\right] = 12.82$$

11 The proof of this result is an adaptation of that of the standard spot-forward parity. Two (self-financing) portfolios are considered:

- Portfolio A: buying the forward contract at time *t*. Its value at time t is 0 and at time *T*, it is $S_T - F_t^T$.

- Portfolio B: buying the underlying asset and borrowing $\left(F_t^T - c\right)\exp\left(-r\left(T-t\right)\right)$ at time *t*. Its value at time t is then $\left(F_t^T - c\right)\exp\left(-r\left(T-t\right)\right) - S_t$. Its value at maturity is $S_T - F_t^T$ by taking into account the storage costs.

Using the absence of arbitrage opportunity, both portfolios should have the same value at any intermediate time, in particular at time *t*. Hence:

$$F_t^T = S_t \exp\left(r\left(T-t\right)\right) + c .$$

END OF EXAMINERS' REPORT