PART – I

TYPICAL QUESTIONS & ANSWERS

OBJECTIVE TYPE QUESTIONS

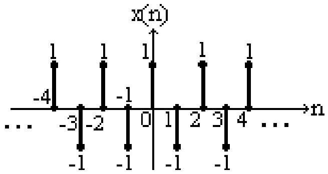
Each Question carries 2 marks.

Choose the correct or best alternative in the following:

Q.1 The discrete-time signal x (n) = $(-1)^n$ is periodic with fundamental period

(A) 6 **(B)** 4
(C) 2 **(D)** 0

Ans: C Period = 2

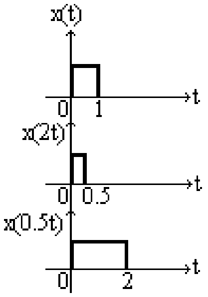


Q.2 The frequency of a continuous time signal x (t) changes on transformation from x (t) to x (αt), $\alpha > 0$ by a factor

(A) α. **(B)** $\frac{1}{\alpha}$.
(C) α^2. **(D)** $\sqrt{\alpha}$.

Ans: A $x(t)\xrightarrow{\text{Transform}}x(\alpha t),\ \alpha > 0$

$\alpha > 1 \Longrightarrow$ compression in t, expansion in f by α.
$\alpha < 1 \Longrightarrow$ expansion in t, compression in f by α.



Q.3 A useful property of the unit impulse $\delta(t)$ is that

(A) $\delta(at) = a\,\delta(t)$. **(B)** $\delta(at) = \delta(t)$.
(C) $\delta(at) = \frac{1}{a}\delta(t)$. **(D)** $\delta(at) = [\delta(t)]^a$.

Ans: C Time-scaling property of δ(t):

$$\delta(at) = \frac{1}{a}\delta(t),\ a > 0$$

Q.4 The continuous time version of the unit impulse $\delta(t)$ is defined by the pa relations

(A) $\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$. **(B)** $\delta(t) = 1,\ t = 0$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$.

(C) $\delta(t) = 0, t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$. **(D)** $\delta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$.

Ans: C $\delta(t) = 0, t \neq 0 \rightarrow \delta(t) \neq 0$ at origin

$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1 \rightarrow$ Total area under the curve is unity.

[$\delta(t)$ is also called Dirac-delta function]

Q.5 Two sequences $x_1(n)$ and $x_2(n)$ are related by $x_2(n) = x_1(-n)$. In the z- domain, their ROC's are

(A) the same. **(B)** reciprocal of each other.
(C) negative of each other. **(D)** complements of each other.

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Ans: B
$$\left.\begin{aligned} &x_1(n) \overset{z}{\longleftrightarrow} X_1(z), \text{RoC } R_x \\ &x_2(n) = x_1(-n) \overset{z}{\longleftrightarrow} X_1(1/z), \text{RoC } 1/R_x \end{aligned}\right\} \text{Reciprocals}$$

Q.6 The Fourier transform of the exponential signal $e^{j\omega_0 t}$ is

(A) a constant. **(B)** a rectangular gate.
(C) an impulse. **(D)** a series of impulses.

Ans: C Since the signal contains only a high frequency ω_o its FT must be an impulse at $\omega = \omega_o$

Q.7 If the Laplace transform of $f(t)$ is $\frac{\omega}{(s^2 + \omega^2)}$, then the value of $\lim_{t\rightarrow\infty} f(t)$

(A) cannot be determined. **(B)** is zero.
(C) is unity. **(D)** is infinity.

Ans: B $f(t) \overset{L}{\longleftrightarrow} \frac{\omega}{s^2 + \omega^2}$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s\,F(s) \quad \text{[Final value theorem]}$$

$$= \lim_{s \rightarrow 0} \left(\frac{s\omega}{s^2 + \omega^2} \right) = 0$$

Q.8 The unit impulse response of a linear time invariant system is the unit step function $u(t)$. For t > 0, the response of the system to an excitation

$e^{-at}u(t),\ a > 0,$ will be

(A) ae^{-at}. **(B)** $\frac{1-e^{-at}}{a}$.

(C) $a(1-e^{-at})$. **(D)** $1-e^{-at}$.

Ans: B

$h(t) = u(t);\quad x(t) = e^{-at}\,u(t),\ a > 0$

$$\text{System response } y(t) = L^{-1}\left[\frac{1}{s}.\frac{1}{s+a}\right]$$

$$= L^{-1}\frac{1}{a}\left[\frac{1}{s}-\frac{1}{s+a}\right]$$

$$= \frac{1}{a}(1 - e^{-at})$$

Q.9 The z-transform of the function $\sum_{k=-\infty}^{0}\delta(n-k)$ has the following region of convergence

(A) $|z|>1$ **(B)** $|z|=1$

(C) $|z|<1$ **(D)** $0<|z|<1$

Ans: C $x(n) = \sum_{k=-\infty}^{0}\delta(n-k)$

$$x(z) = \sum_{k=-\infty}^{0} z^{-k} = \ldots..+ z^3 + z^2 + z + 1 \quad \text{(Sum of infinite geometric series)}$$

$$= \frac{1}{1-z}, \qquad |z| < 1$$

Q.10 The auto-correlation function of a rectangular pulse of duration T is

(A) a rectangular pulse of duration T.
(B) a rectangular pulse of duration 2T.
(C) a triangular pulse of duration T.
(D) a triangular pulse of duration 2T.

Ans: D

$R_{XX}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, x(t+\tau)\, d\tau$ ⇨ triangular function of duration 2T.

Q.11 The Fourier transform (FT) of a function x (t) is X (f). The FT of $dx(t)/dt$ will be

(A) $dX(f)/df$. **(B)** $j2\pi f\ X(f)$.
(C) $jf\ X(f)$. **(D)** $X(f)/(jf)$.

Ans: B $(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)\ e^{j\omega t}\ d\omega$

$$\frac{dx}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\ X(f)\ e^{j\omega t}\ d\omega$$

$$\therefore\ \frac{dx}{dt} \leftrightarrow j\ 2\pi\ f\ X(f)$$

Q.12 The FT of a rectangular pulse existing between $t = -T/2$ to $t = T/2$ is a

(A) sinc squared function. **(B)** sinc function.
(C) sine squared function. **(D)** sine function.

Ans: B $x(t) = \begin{cases} 1, & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$

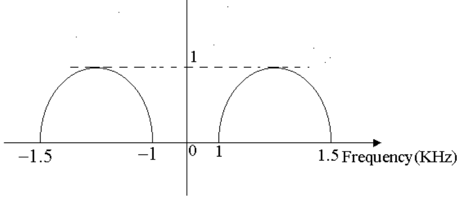
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\ e^{-j\omega t}\ dt = \int_{-T/2}^{+T/2} e^{-j\omega t}\ dt = \left.\frac{e^{-j\omega t}}{j\omega}\right|_{-T/2}^{+T/2}$$

$$\left\{\begin{array}{l} = -\frac{1}{j\omega}(e^{-j\omega T/2} - e^{j\omega T/2}) = \frac{2}{\omega}\left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right) \\ = \frac{2}{\omega}\sin\frac{\omega T}{2} = \frac{\sin(\omega T/2)}{\omega T/2}.T \end{array}\right.$$

Hence X(jω) is expressed in terms of a sinc function.

Q.13 An analog signal has the spectrum shown in Fig. The minimum sampling rate needed to completely represent this signal is

(A) 3 KHz.
(B) 2 KHz.
(C) 1 KHz.
(D) 0.5 KHz.



Ans: C For a band pass signal, the minimum sampling rate is twice the bandwidth, which is 0.5kHz here.

Q.14 A given system is characterized by the differential equation:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t).$$

The system is :

(A) linear and unstable. **(B)** linear and stable.
(C) nonlinear and unstable. **(D)** nonlinear and stable.

Ans:A $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$, $x(t) \longrightarrow \underset{\text{system}}{\boxed{h(t)}} \longrightarrow y(t)$

The system is linear . Taking LT with zero initial conditions, we get
$s^2Y(s) - sY(s) - 2Y(s) = X(s)$

or, $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$

Because of the pole at s = +2, the system is unstable.

Q.15 The system characterized by the equation $y(t) = ax(t) + b$ is

(A) linear for any value of b. **(B)** linear if b > 0.
(C) linear if b < 0. **(D)** non-linear.

Ans: D The system is non-linear because x(t) = 0 does not lead to y (t) = 0, which is a violation of the principle of homogeneity.

Q.16 Inverse Fourier transform of $u(\omega)$ is

(A) $\frac{1}{2}\delta(t) + \frac{1}{\pi t}$. **(B)** $\frac{1}{2}\delta(t)$.
(C) $2\delta(t) + \frac{1}{\pi t}$. **(D)** $\delta(t) + \text{sgn}(t)$.

Ans: A $x(t) = u(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \pi \frac{\delta(\omega)}{J\omega} + 1$

Duality property: $X(jt) \longleftrightarrow 2\pi x(-\omega)$

$u(\omega) \longleftrightarrow \frac{1}{2}\delta(t) + \frac{1}{\pi t}$

Q.17 The impulse response of a system is $h(n) = a^n\ u(n)$. The condition for the system be BIBO stable is

(A) a is real and positive. **(B)** a is real and negative.
(C) $|a| > 1$. **(D)** $|a| < 1$.

Ans: D Sum $S = \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} | a^n u(n) |$

$$\leq \sum_{n=0}^{+\infty} |a|^n \qquad (\because u(n) = 1 \text{ for } n \geq 0)$$

$$\leq \frac{1}{1-|a|} \qquad \text{if } |a| < 1.$$

Q.18 If R_1 is the region of convergence of x (n) and R_2 is the region of convergence of y(n), then the region of convergence of x (n) convoluted y (n) is

(A) R_1+R_2. **(B)** R_1-R_2.
(C) $R_1 \cap R_2$. **(D)** $R_1 \cup R_2$.

Ans:C $x(n) \overset{z}{\longleftrightarrow} X(z)$, RoC R_1

$y(n) \overset{z}{\longleftrightarrow} Y(z)$, RoC R_2

$x(n) * y(n) \overset{z}{\longleftrightarrow} X(z).Y(z)$, RoC at least $R_1 \cap R_2$

Q.19 The continuous time system described by $y(t) = x(t^2)$ is

(A) causal, linear and time varying.
(B) causal, non-linear and time varying.
(C) non causal, non-linear and time-invariant.
(D) non causal, linear and time-invariant.

Ans: D

$y(t) = x(t^2)$

y(t) depends on $x(t^2)$ i.e., future values of input if $t > 1$.

$\therefore$ System is anticipative or <u>non-causal</u>

$\alpha\, x_1(t) \rightarrow y_1(t) = \alpha\, x_1(t^2)$

$\beta\, x_2(t) \rightarrow y_2(t) = \beta\, x_2(t^2)$

$\therefore \alpha\, x_1(t) + \beta\, x_2(t) \rightarrow y(t) = \alpha x_1(t^2) + \beta\, x_2(t^2) = y_1(t) + y_2(t)$

$\therefore$ System is <u>Linear</u>

System is time varying. Check with $x(t) = u(t) - u(t-z) \rightarrow y(t)$ and

$x_1(t) = x(t-1) \rightarrow y_1(t)$ and find that $y_1(t) \neq y(t-1)$.

Q.20 If G(f) represents the Fourier Transform of a signal g (t) which is real and odd symmetric in time, then G (f) is

(A) complex. **(B)** imaginary.
(C) real. **(D)** real and non-negative.

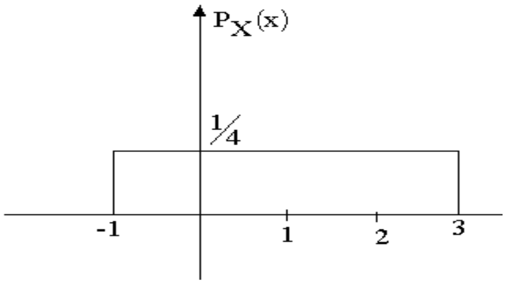
Ans:B $g(t) \xleftrightarrow{FT} G(f)$

g(t) real, odd symmetric in time

$G^*(j\omega) = -G(j\omega)$; $G(j\omega)$ purely imaginary.

Q.21 For a random variable x having the PDF shown in the Fig., the mean and the variance are, respectively,

(A) $\frac{1}{2}$ and $\frac{2}{3}$.
(B) 1 and $\frac{4}{3}$.
(C) 1 and $\frac{2}{3}$.
(D) 2 and $\frac{4}{3}$.



Ans:B Mean $= \mu_x(t) = \int_{-\infty}^{+\infty} x\, f_{x(t)}(x)\, dx$

$$= \int_{-1}^{3} x \frac{1}{4} dx = \frac{1}{4} \frac{x^2}{2} \Big|_{-1}^{3} = \left(\frac{9}{2} - \frac{1}{2}\right) \frac{1}{4} = 1$$

$$\text{Variance} = \int_{-\infty}^{+\infty} (x - \mu_x)^2 f_x(x)\, dx$$

$$= \int_{-1}^{3} (x-1)^2 \frac{1}{4} d(x-1)$$

$$= \frac{1}{4} \frac{(x-1)^3}{3} \Big|_{-1}^{3} = \frac{1}{12}[8 + 8] = \frac{4}{3}$$

Q.22 If white noise is input to an RC integrator the ACF at the output is proportional to

(A) $\exp\left(\frac{-\lvert\tau\rvert}{RC}\right)$.	**(B)** $\exp\left(\frac{-\tau}{RC}\right)$.
(C) $\exp(\lvert\tau\rvert RC)$.	**(D)** $\exp(-\tau RC)$.

Ans: A

$$R_N(\tau) = \frac{N_0}{4RC}\left(\exp - \frac{\mid\tau\mid}{RC}\right)$$

Q.23 $x(n) = a^{|n|}, |a| < 1$ is

(A) an energy signal.
(B) a power signal.
(C) neither an energy nor a power signal.
(D) an energy as well as a power signal.

Ans: A

$$\text{Energy} = \sum_{n=-\infty}^{+\infty} x^2(n) = \sum_{n=-\infty}^{\infty} a^{2|n|} = \sum_{n=-\infty}^{\infty} (a^2)^{|n|} = 1 + 2\sum_{n=1}^{\infty} a^2$$

$= \text{finite since } |a| < 1$

$\therefore$ This is an energy signal.

Q.24 The spectrum of x (n) extends from $-\omega_o$ to $+\omega_o$, while that of h(n) extends from $-2\omega_o$ to $+2\omega_o$. The spectrum of $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$ extends from

(A) $-4\omega_o$ to $+4\omega_o$.	**(B)** $-3\omega_o$ to $+3\omega_o$.
(C) $-2\omega_o$ to $+2\omega_o$.	**(D)** $-\omega_o$ to $+\omega_o$

Ans: D Spectrum depends on $H(e^{j\omega}) \longrightarrow X(e^{j\omega})$ Smaller of the two ranges.

Q.25 The signals $x_1(t)$ and $x_2(t)$ are both bandlimited to $(-\omega_1, +\omega_1)$ and $(-\omega_2, +\omega_2)$ respectively. The Nyquist sampling rate for the signal $x_1(t)x_2(t)$ will be

(A) $2\omega_1$ if $\omega_1 > \omega_2$.	**(B)** $2\omega_2$ if $\omega_1 < \omega_2$.
(C) $2(\omega_1+\omega_2)$.	**(D)** $(\omega_1+\omega_2)/2$.

Ans: C Nyquist sampling rate = 2(Bandwidth) = $2(\omega_1 - (-\omega_2)) = 2(\omega_1 + \omega_2)$

Q.26 If a periodic function f(t) of period T satisfies $f(t) = -f\left(t + T/2\right)$, then in its Fourie

series expansion,

(A)the constant term will be zero.
(B)there will be no cosine terms.
(C)there will be no sine terms.
(D)there will be no even harmonics.

Ans:

$$\frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left(\int_0^{T/2} f(t)\,dt + \int_{T/2}^{T} f(t)\,dt\right) = \frac{1}{T}\left(\int_0^{T/2} f(t)\,dt + \int_0^{T/2} f(\tau + T/2)d\tau\right) = 0$$

Q.27 A band pass signal extends from 1 KHz to 2 KHz. The minimum sampling frequency needed to retain all information in the sampled signal is

(A)1 KHz.
(B) 2 KHz.
(C) 3 KHz.
(D) 4 KHz.

Ans: B

Minimum sampling frequency = 2(Bandwidth) = 2(1) = 2 kHz

Q.28 The region of convergence of the z-transform of the signal

$$2^n\,u(n) - 3^n\,u(-n-1)$$

(A) is $|z| > 1$.
(B) is $|z| < 1$.
(C) is $2 < |z| < 3$.
(D) does not exist.

Ans:

$$2^n u(n) \longleftrightarrow \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$$3^n u(-n-1) \longleftrightarrow \frac{1}{1-3z^{-1}}, \quad |z| < 3$$

$\therefore$ ROC is $2 < |z| < 3$.

Q.29 The number of possible regions of convergence of the function $\dfrac{\left(e^{-2}-2\right)z}{\left(z-e^{-2}\right)(z-2)}$

is

(A) 1.
(B) 2.
(C) 3.
(D) 4.

Ans: C

Possible ROC's are $|z| > e^{-2}$, $|z| < 2$ and $e^{-2} < |z| < 2$

Q.30 The Laplace transform of u(t) is A(s) and the Fourier transform of u(t) is $B(j\omega)$. Then

(A) $B(j\omega) = A(s)\big|_{s=j\omega}$. **(B)** $A(s) = \frac{1}{s}$ but $B(j\omega) \neq \frac{1}{j\omega}$.

(C) $A(s) \neq \frac{1}{s}$ but $B(j\omega) = \frac{1}{j\omega}$. **(D)** $A(s) \neq \frac{1}{s}$ but $B(j\omega) \neq \frac{1}{j\omega}$.

Ans: B $u(t) \overset{L}{\Longleftrightarrow} A(s) = \frac{1}{s}$

$$u(t) \overset{F.T}{\Longleftrightarrow} B(j\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)$$

$$\therefore\ A(s) = \frac{1}{s} \text{ but } B(j\omega) \neq \frac{1}{j\omega}$$

PART – II

NUMERICALS & DERIVATIONS

Q.1. Determine whether the system having input x (n) and output y (n) and described by relationship : $y(n) = \sum_{k=-\infty}^{n} x(k+2)$
is (i) memoryless, (ii) stable, (iii)causal (iv) linear and (v) time invariant. **(5)**

Ans:

$$y(n) = \sum_{k=-\infty} x(k+2)$$

(i) Not memoryless - as y(n) depends on past values of input from $x(-\infty)$ to x(n-1) (assuming)$n > 0$)

(ii) Unstable- since if $|x(n)| \le M$, then $|y(n)|$ goes to ∞ for any n.

(iii) Non-causal - as y(n) depends on x(n+1) as well as x(n+2).

(iv) Linear - ∵ the principle of superposition applies (due to ∑ operation)

(v) Time – invariant - ∵ a time-shift in input results in corresponding time-shift in output.

Q.2. Determine whether the signal x (t) described by
x (t) = exp [- at] u (t), a > 0 is a power signal or energy signal or neither. **(5)**

Ans:

$x(t) = e^{-at} u(t), a > 0$

x(t) is a non-periodic signal.

$$\text{Energy } E = \int_{-\infty}^{+\infty} x^2(t)\, dt = \int_{0}^{\infty} e^{-2at} dt = \frac{e^{-2at}}{-2a}\Bigg|_{0}^{\infty} = \frac{1}{2a} \text{ (finite, positive)}$$

The energy is finite and deterministic.

∴ x(t) is an energy signal.

Q.3. Determine the even and odd parts of the signal x (t) given by

$$x(t) = \begin{cases} A e^{-\alpha t} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$x(t) = \begin{cases} Ae^{-\alpha t} & \overline{t > 0} \\ 0 & t < 0 \end{cases}$$

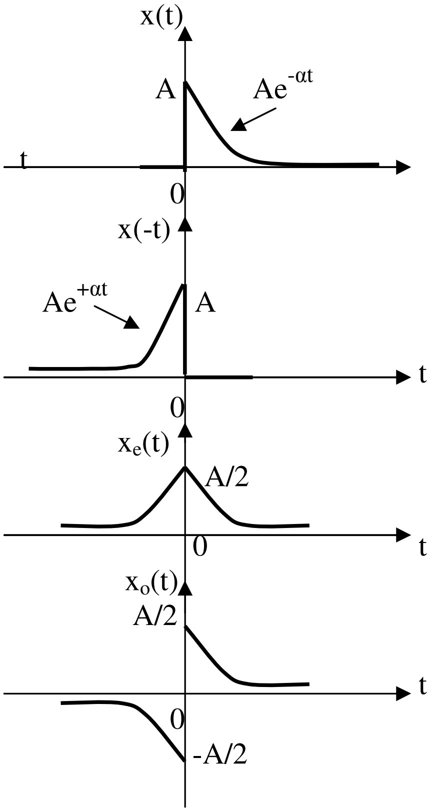
(5)

Ans:

Assumption : $\alpha > 0, A > 0, -\infty < t < \infty$

Even part $x_e(t) = \frac{x(t) + x(-t)}{2}$

Odd part $x_o(t) = \frac{x(t) - x(-t)}{2}$



Q.4. Use one sided Laplace transform to determine the output y (t) of a system described by

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y\,(t)=0 \text{ where } y(0-)=3 \text{ and } \left.\frac{dy}{dt}\right|_{t=0-}=1 \qquad \textbf{(7)}$$

Ans:

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2\,y(t)=0,\quad y(0-)=3,\quad \left.\frac{dy}{dt}\right|_{t=0^-}=1$$

$$\left(s^2\,Y(s)-s\,y(0)-\left.\frac{dy}{dt}\right|_{t=0}\right)+3\,[s\,Y(s)-y(0)]+2\,Y(s)=0$$

$$(s^2+3s+2)\,Y(s) = sy(0)+\left.\frac{dy}{dt}\right|_{t=0}+3\,y(0)$$

$$(s^2+3s+2)\,Y(s) = 3s+1+9=3s+10$$

$$Y(s) = \frac{3s+10}{s^2+3s+2} = \frac{3s+10}{(s+1)(s+2)}$$

$$=\frac{A}{s+1}+\frac{B}{s+2}$$

$$A = \frac{3s + 10}{s + 2}\Bigg|_{s = -1} = 7 \,; \qquad B = \frac{3s + 10}{s + 1}\Bigg|_{s = -2} = -4$$

$$\therefore Y(s) = \frac{7}{s + 1} - \frac{4}{s + 2}$$

$$\therefore y(t) = L^{-1}[Y(s)] = 7e^{-t} - 4e^{-2t} = e^{-t}(7 - 4e^{-t})$$

$\therefore$ The output of the system is $y(t) = e^{-t}(7 - 4e^{-t})\,u(t)$

Q. 5. Obtain two different realizations of the system given by $y(n) - (a+b)\,y(n-1) + ab\,y(n-2) = x(n)$. Also obtain its transfer function. **(7)**

Ans:

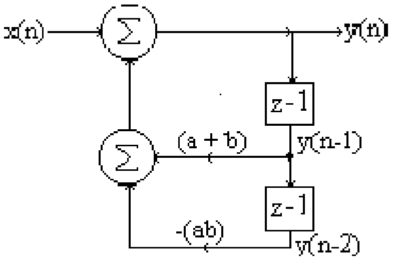
$$y(n) - (a + b)\,y(n-1) + ab\,y(n-2) = x(n)$$

$$\therefore Y(z) - (a+b)\,z^{-1}\,Y(z) + ab\,z^{-2}\,Y(z) = X(z)$$

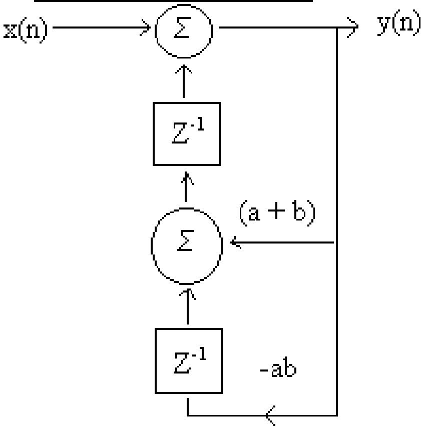
$$\text{Transfer function } H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - (a+b)\,z^{-1} + ab\,z^{-2}}$$

$$y(n) = x(n) + (a + b)\,y(n-1) - ab\,y(n-2)$$

Direct Form I/II realization



Alternative Realisation



Q. 6. An LTI system has an impulse response $h(t) = \exp[-at]\,u(t)$; when it is excited by an input signal $x(t)$, its output is $y(t) = [\exp(-bt) - \exp(-ct)]\,u(t)$ Determine its input $x(t)$. **(7)**

Ans:

$h(t) = e^{-at}\,u(t)$ for input $x(t)$

Output $y(t) = (e^{-bt} - e^{-ct})\,u(t)$

$$h(t) \overset{L}{\longleftrightarrow} H(s),\quad y(t) \overset{L}{\longleftrightarrow} Y(s),\quad x(t) \overset{L}{\longleftrightarrow} X(s)$$

$$H(s) = \frac{1}{s + a}\,; \quad Y(s) = \frac{1}{s + b} - \frac{1}{s + c} = \frac{s + c - s - b}{(s + b)(s + c)} = \frac{c - b}{(s + b)(s + c)}$$

As $H(s) = \frac{Y(s)}{X(s)}$, $X(s) = \frac{Y(s)}{H(s)}$

$$\therefore X(s) = \frac{(c - b)(s + a)}{(s + b)(s + c)} = \frac{A}{s + b} + \frac{B}{s + c}$$

$$A = \left.\frac{(c - b)(s + a)}{(s + c)}\right|_{s = -b} = \frac{(c - b)(-b + a)}{(-b + c)} = a - b$$

$$B = \left.\frac{(c - b)(s + a)}{(s + b)}\right|_{s = -c} = \frac{(c - b)(-c + a)}{(-c + b)} = c - a$$

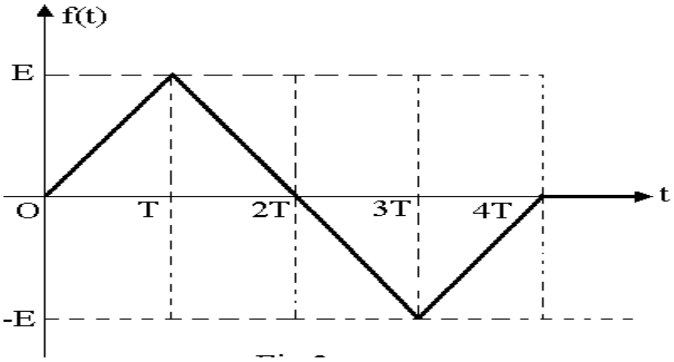
$$\therefore X(s) = \frac{a - b}{s + b} + \frac{c - a}{s + c}$$

$$x(t) = (a - b)\, e^{-bt} + (c - a)\, e^{-ct}$$

$\therefore$ The input $x(t) = [(a - b)\, e^{-bt} + (c - a)\, e^{-ct}]\, u(t)$

Q.7. Write an expression for the waveform $f(t)$ shown in Fig. using only unit step function and powers of t. **(3)**

Ans:

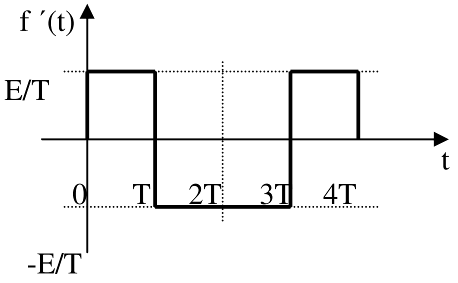


$$\therefore f(t) = \frac{E}{T}\, [\, t\, u(t) - 2(t - T)\, u(t - T) + 2(t - 3T)\, u(t - 3T) - (t - 4T)\, u(t - 4T)]$$

Q.8. For f(t) of Q7, find and sketch $f'(t)$ (prime denotes differentiation with respect to t).

Ans:

$$f(t) = \frac{E}{T}\,[\,t\,u(t) - 2(t - T)\,u(t - T) + 2(t - 3T)\,u(t - 3T) - (t - 4T)\,u(t - 4T)]$$



$$\therefore\ f'(t) = \frac{E}{T}\,[u(t) - 2\,u(t - T) + 2\,u(t - 3T) - u(t - 4T)]$$

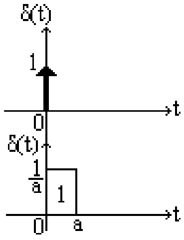
Q.9. Define a unit impulse function $\delta(t)$. **(2)**

Ans:

. Unit impulse function δ(t) is defined as:

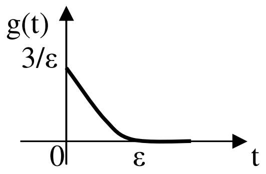
$$\begin{cases} \delta(t) = 0,\ t \neq 0 \\ \int_{-\infty}^{+\infty} \delta(t)\,dt = 1 \end{cases}$$

It can be viewed as the limit of a rectangular pulse of duration a and height 1/a when a ⟶ 0, as shown below.



Q.10. Sketch the function $g(t) = \frac{3}{\in^3}(t-\in)^2[u(t)-u(t-\in)]$ and show that $g(t) \to \delta(t)\,\text{as}\in \to 0$. **(6)**

Ans:



As ε ⟶ 0, duration ⟶ 0, amplitude ⟶ ∞

$$\int_0^{\varepsilon} g(t)\,dt = 1$$

Q.11. Show that if the FT of x (t) is $X(j\omega)$, then the FT of $x\left(\frac{t}{a}\right)$ is $|a|\,X(ja\omega)$.

Ans:

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

Let $x\left[\frac{t}{a}\right] \xleftrightarrow{FT} X_1(j\omega)$, then

$$X_1(j\omega) = \int_{-\infty}^{+\infty} x\left[\frac{t}{a}\right] e^{-j\omega t}\,dt \qquad \text{Let } \frac{t}{a} = \alpha \quad \therefore\ dt = a\,d\alpha$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j\omega a\alpha}\, a\,d\alpha \text{ if } a > 0$$

$$- \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j\omega a\alpha}\, a\,d\alpha \text{ if } a < 0$$

Hence $X_1(j\omega) = |a| \int_{-\infty}^{+\infty} x(\alpha)\, e^{-j\omega a\alpha}\, d\alpha = |a|\, x(j\omega a)$

Q.12. Solve, by using Laplace transforms, the following set of simultaneous differential equations for x (t). **(14)**

Ans:

$$2x'(t) + 4x(t) + y'(t) + 7y(t) = 5u(t)$$
$$x'(t) + x(t) + y'(t) + 3y(t) = 5\delta(t)$$

The initial conditions are : $x(0-) = y(0-) = 0$.

$$2\,x'(t) + 4\,x(t) + y'(t) + 7\,y(t) = 5\,u(t)$$

$$x'(t) + x(t) + y'(t) + 3\,y(t) = 5\,\delta(t)$$

$$x(t) \xleftrightarrow{L} X(s),\ x'(t) \xleftrightarrow{L} s\,X(s),\ \delta(t) \xleftrightarrow{L} 1,\ u(t) \xleftrightarrow{L} \frac{1}{s}$$

(Given zero initial conditions)

$$\therefore 2\,sX(s) + 4\,X(s) + sY(s) + 7\,Y(s) = \frac{5}{s}$$

$$sX(s) + X(s) + sY(s) + 3\,Y(s) = 5$$

$$(2s + 4)\,X(s) + (s+7)\,Y(s) = \frac{5}{s}$$

$$(s + 1)\,X(s) + (s+3)\,Y(s) = 5$$

$$X(s) = \frac{\begin{vmatrix} \frac{5}{S} & s+7 \\ 5 & s+3 \end{vmatrix}}{\begin{vmatrix} 2s+4 & s+7 \\ s+1 & s+3 \end{vmatrix}}$$

Or, $X(s) = -\dfrac{5s + 35 - 5 - 15/s}{2s^2 + 6s + 4s + 12 - s^2 - 8s - 7}$

$$= -\frac{5s^2 + 30s - 15}{s(s^2 + 2s + 5)} = -\frac{5}{s}\left(\frac{s^2 + 6s - 3}{s^2 + 2s + 5}\right) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

Then $A(s^2 + 2s + 5) + B s^2 + Cs = -5(s^2 + 6s - 3)$

$\therefore$ $A + B = -5$
$2A + C = -30$
$5A = 15$

Thus $A = 3$, $B = -8$, $C = -36$ and we can write

$$X(s) = \frac{3}{s} - 8\frac{s+1}{(s+1)^2 + 2^2} - 14\frac{2}{(s+1)^2 + 2^2}$$

$\therefore x(t) = (3 - 8e^{-t}\cos 2t - 14e^{-t}\sin 2t)\,u(t)$

Q.13. Find the Laplace transform of $t \sin \omega_0 t\, u(t)$. **(6)**

Ans:

$$\sin(\omega_0 t) \overset{L}{\longleftrightarrow} \frac{\omega_0}{s^2 + \omega_0^2}$$

Using $t f(t) \overset{L}{\longleftrightarrow} -\dfrac{d}{ds}[F(s)]$,

$$L[t \sin(\omega_0 t) u(t)] = -\frac{d}{ds}\left[\frac{\omega_0}{s^2 + \omega_0^2}\right]$$

$$= \left[\frac{0 - \omega_0(2s)}{(s^2 + \omega_0^2)^2}\right] = \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}$$

Q.14. Find the inverse Laplace transform of $\dfrac{s-2}{s(s+1)^3}$. **(8)**

Ans:

$$F(s) = \frac{s-2}{s(s+1)^3} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$A = \left.\dfrac{s-2}{(s+1)^3}\right|_{s=0} = -2$

$A(s+1)^3 + Bs(s+1)^2 + Cs(s+1) + Ds = s-2$

$s^3 : A + B = 0$ $\boxed{B = 2}$

$D = \left.\dfrac{s-2}{s}\right|_{s=-1} = 3$

$s^2 : 3A + 2B + C = 0$ $\boxed{C = 2}$

$\boxed{A = -2}$ $\boxed{D = 3}$

$$F(s) = \frac{-2}{s} + \frac{2}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{(s+1)^3}$$

$$\therefore\ f(t) = -2 + 2\,e^{-t} + 2\,t\,e^{-t} + \frac{3}{2}\,t^2\,e^{-t}$$

$$\therefore f(t) = [-2 + e^{-t}(\frac{3}{2}t^2 + 2t + 2)]\,u(t)$$

Q.15. Show that the difference equation $y(n)-\alpha y(n-1)=-\alpha x(n)+x(n-1)$ represents an all-pass transfer function. What is (are) the condition(s) on α for the system to be stable? **(8)**

Ans:

$$y(n) - \alpha\, y(n-1) = -\,\alpha\, x(n) + x(n-1)$$

$$Y(z) - \alpha\, z^{-1}\, Y(z) = -\,\alpha\, X(z) + z^{-1}\, X(z)$$

$$(1-\alpha\, z^{-1})\, Y(z) = (-\alpha + z^{-1})\, X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\alpha + z^{-1}}{1-\alpha\, z^{-1}} = \frac{1-\alpha\, z}{z-\alpha}$$

Zero : $z = \frac{1}{\alpha}$

Pole : $z = \alpha$

As poles and zeros have reciprocal values, the transfer function represents an all pass filter system.

Condition for stability of the system :

For stability, the pole at $z = \alpha$ must be inside the unit circle, i.e. $|\alpha| < 1$.

Q.16. Give a recursive realization of the transfer function $H(z)=1+z^{-1}+z^{-2}+z^{-3}$ **(6)**

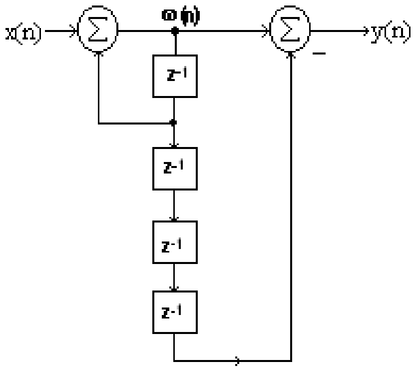
Ans:

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} = \frac{1-z^{-4}}{1-z^{-1}} \left(\begin{array}{c}\text{Geometric series of 4 terms}\\ \text{First term} = 1\text{, Common ratio} = z^{-1}\end{array}\right)$$

As $H(z) = \frac{Y(z)}{X(z)}$, we can write

$$\therefore\ (1-z^{-1})\, Y(z) = (1-z^{-4})\, X(z) \text{ or } Y(z) = \frac{X(z)}{(1-z^{-1})}(1-z^{-4}) = W(z)(1-z^{-4})$$

The realization of the system is shown below.



Q.17 Determine the z-transform of $x_1(n) = \alpha^n u(n)$ and $x_2(n) = -\alpha^n u(-n-1)$ and indicate their regions of convergence. **(6)**

Ans:

$x_1(n) = \alpha^n u(n)$ and $x_2(n) = -\alpha^n u(-n-1)$

$X_1(z) = \dfrac{1}{1-\alpha z^{-1}}$ RoC $|\alpha z^{-1}| < 1$ i.e., $|z| > \alpha$

$$X_2(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} \alpha^{-n} z^n = -(\alpha^{-1}z + \alpha^{-2}z^2 + \alpha^{-3}z^3 + \ldots\ldots\ldots)$$

$$= -\alpha^{-1}z\,(1 + \alpha^{-1}z + \alpha^{-2}z^2 + \ldots\ldots..)$$

$$= \frac{-\alpha^{-1}z}{1-\alpha^{-1}z} = \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}\ ; \quad \text{RoC} \quad |\alpha^{-1} z| < 1 \text{ i.e., } |z| < |\alpha|$$

Q.18. Determine the sequence $h(n)$ whose z-transform is

$$H(z) = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}, \quad |r| < 1.$$ **(6)**

Ans:

.

$$H(z) = \frac{1}{1-2r\cos\theta\, z^{-1} + r^2 z^{-2}}, \quad |r| < 1$$

$$= \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}, \quad |r| < 1$$

$$= \frac{A}{(1-re^{j\theta}z^{-1})} + \frac{B}{(1-re^{-j\theta}z^{-1})} = |r| < 1$$

where $A = \left.\dfrac{1}{(1-r\,e^{j\theta}\,z^{-1})}\right|_{r\,e^{j\theta}\,z^{-1}=1} = \dfrac{1}{1-e^{-j2\theta}}$

$B = \left.\dfrac{1}{(1-r\,e^{j\theta}\,z^{-1})}\right|_{r\,e^{-j\theta}\,z^{-1}=1} = \dfrac{1}{1-e^{j2\theta}}$

$$\therefore h(n) = \frac{1}{1-e^{-2j\theta}}\,(r\,e^{j\theta})^n + \frac{1}{1-e^{2j\theta}}\,(r\,e^{-j\theta})^n$$

$$\therefore h(n) = r^n\left[\frac{e^{jn\theta}}{1-e^{-j2\theta}} + \frac{e^{-jn\theta}}{1-e^{j2\theta}}\right]u(n)$$

$$= r^n\,\frac{e^{j(n+1)\theta} - e^{-j(n+1)\theta}}{e^{j\theta} - e^{-j\theta}}\quad u(n)$$

$$= \frac{r^n\ \sin(n+1)\,\theta}{\sin\theta}\quad u(n)$$

Q.19. Let the Z- transform of x(n) be X(z).Show that the z-transform of x (-n) is $X\left(\frac{1}{z}\right)$. **(2)**

Ans:

$$x(n) \overset{z}{\longleftrightarrow} X(z) \qquad \text{Let } y(n) = x(-n)$$

$$\text{Then } Y(z) = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{r=-\infty}^{\infty} x(r)\,z^{+r} = \sum_{r=-\infty}^{\infty} x(r)\,(z^{-1})^{-1} = X(z^{-1})$$

Q.20. Find the energy content in the signal $x(n) = e^{-n/10}\sin\left(\frac{2\pi n}{4}\right)$. **(7)**

Ans:

$$x(n) = e^{-0.1n}\sin\left(\frac{2\pi n}{4}\right)$$

$$\text{Energy content } E = \sum_{n=-\infty}^{+\infty} |x^2(n)| = \sum_{n=-\infty}^{+\infty} e^{-0.2\,n}\left(\sin\left(\frac{2\pi n}{4}\right)\right)^2$$

$$E = \sum_{n=-\infty}^{+\infty} e^{-2n}\quad \sin^2\frac{n\pi}{2}$$

$$E = \sum_{n=-\infty}^{+\infty} e^{-2n}\quad \frac{1-\cos n\pi}{2}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{+\infty} e^{-2n}\quad [1-(-1)^n]$$

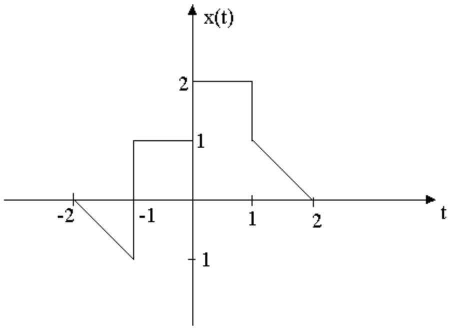
Now $1 - (-1)^{n} = \begin{cases} 2 \text{ for n odd} \\ 0 \text{ for n even} \end{cases}$

Also Let n = 2r +1 ; then $E = \sum_{r=-\infty}^{\infty} e^{-.2(2r+1)} = \sum_{r=-\infty}^{\infty} e^{-.4r} \; e^{-.2}$

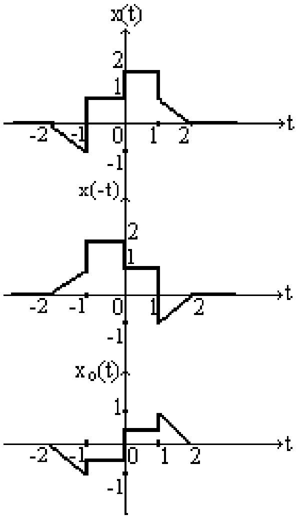
$= e^{-..2}\left[\sum_{r=0}^{\infty} e^{-.4r} + \sum_{r=1}^{\infty} e^{.4r}\right]$ The second term in brackets goes to infinity . Hence E is infinite.

Q.21. Sketch the odd part of the signal shown in Fig. **(3)**

Ans:

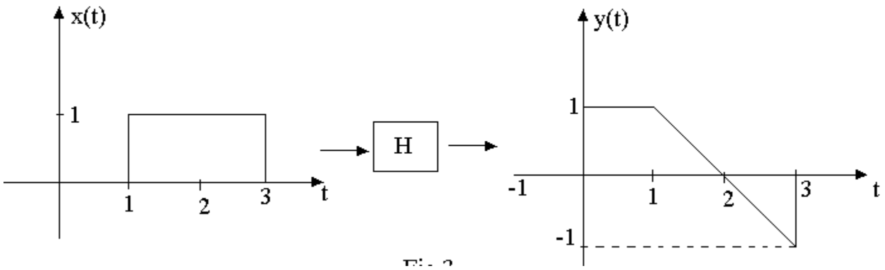


Odd part $x_o(t) = \dfrac{x(t) - x(-t)}{2}$



Q.22. A linear system H has an input-output pair as shown in Fig. Determine whether the system is causal and time-invariant. **(4)**

Ans



System is <u>non-causal</u> $\because$ the output y(t) exists at t = 0 when input x(t) starts only at t = +1.

System is <u>time-varying</u> $\because$ the expression for y(t) = [u (t) – u (t-1)(t –1) + u (t –3) (t – 3) – u (t-3)] shows that the system H has time varying parameters.

Q.23. Determine whether the system characterized by the differential equation $\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} + 2y(t) = x(t)$ is stable or not. **(4)**

Ans:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} + 2y(t) = x(t)$$

$y(t) \overset{L}{\longleftrightarrow} Y(s)$; $x(t) \overset{L}{\longleftrightarrow} X(s)$; Zero initial conditions

$s^2\,Y(s) - sY(s) + 2Y(s) = X(s)$

System transfer function $\frac{Y(s)}{X(s)} = \frac{1}{s^2 - s + 2}$ whose poles are in the right half plane.

Hence the system is not stable.

Q.24 Determine whether the system $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ is invertible. **(5)**

Ans:

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

<u>Condition for invertibility</u>: $H^{-1}H = I$ (Identity operator)

$$\left\{ \begin{array}{l} H \longrightarrow \text{Integration} \\ H^{-1} \longrightarrow \text{Differentiation} \end{array} \right.$$

$x(t) \longrightarrow y(t) = H\{x(t)\}$

$H^{-1}\{y(t)\} = H^{-1}\,H\{x(t)\} = x(t)$

$\therefore$ The system is invertible.

Q.25 Find the impulse response of a system characterized by the differential equation $y'(t) + a\,y(t) = x(t)$. **(5)**

Ans:

$y'(t) + a\,y(t) = x(t)$

$x(t) \overset{L}{\longleftrightarrow} X(s)$, $y(t) \overset{L}{\longleftrightarrow} Y(s)$, $h(t) \overset{L}{\longleftrightarrow} H(s)$

sY(s) + aY(s) = X(s), assuming zero initial conditions

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s + a}$$

∴ The impulse response of the system is $h(t) = e^{-at} u(t)$

Q.26. Compute the Laplace transform of the signal $y(t) = (1 + 0.5\sin t)\sin 1000t$. **(4)**

Ans:

$$\begin{aligned} y(t) &= (1 + 0.5 \text{ sint}) \text{ sin}1000t \\ &= \sin 1000t + 0.5 \text{ sint} \sin 1000t \\ &= \sin 1000t + 0.5\left(\frac{\cos 999t - \cos 1001t}{2}\right) \\ &= \sin 1000t + 0.25 \cos 999t - 0.25 \cos 1001t \end{aligned}$$

$$\therefore Y(s) = \frac{1000}{s^2 + 1000^2} + 0.25\frac{s}{s^2 + 999^2} - 0.25\frac{s}{s^2 + 1001^2}$$

Q.27. Determine Fourier Transform $F(\omega)$ of the signal $f(t) = e^{-\alpha t}\cos(\omega t + \theta)$ and determine the value of $|F(\omega)|$. **(7)**

Ans:

We assume $f(t) = e^{-\alpha t} \cos(\omega t + \theta)\ u(t)$ because otherwise FT does not exist

$$f(t) \overset{FT}{\longleftrightarrow} F(\omega) = \int^{+\infty} e^{-\alpha t}\, \frac{e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}}{2}\, e^{-j\omega t}\, dt$$

$$\therefore F(\omega) = \frac{1}{2}\int^{+\infty} [e^{-\alpha t} e^{-j\omega t} e^{j\omega t + j\theta} + e^{-\alpha t} e^{-j\omega t} e^{-j\omega t - j\theta}]\, dt$$

$$= \frac{1}{2}\int^{+\infty} [e^{-\alpha t + j\theta} + e^{-j\theta} e^{-(\alpha + 2j\omega)t}]\, dt$$

$$|F(\omega)| = \frac{1}{2}\left| e^{j\theta}\frac{e^{-\alpha t}}{-\alpha}\Bigg|_0^{+\infty} + e^{-j\theta}\frac{e^{-(\alpha + 2j\omega)t}}{-(\alpha + 2j\omega)}\right|_0^{\omega}$$

$$= \frac{1}{2}\left|\frac{1}{\alpha}e^{j\theta} + \frac{1}{\alpha + 2j\omega}e^{-j\theta}\right|$$

$$\therefore |F(\omega)| = \frac{1}{2}\left|\frac{(\alpha + 2j\omega)\, e^{j\theta} + \alpha\, e^{-j\theta}}{\alpha\,(\alpha + 2j\omega)}\right|$$

$$= \frac{1}{2}\left|\frac{2\alpha \cos\theta + 2j\omega e^{j\theta}}{\alpha\,(\alpha + 2j\omega)}\right|$$

$$= \left|\frac{\alpha \cos\theta + j\,\omega \cos\theta - j\,\omega \sin\theta}{\alpha\,(\alpha + 2j\omega)}\right|$$

$$\left|F(\omega)\right|^2 = \frac{\alpha^2\cos^2\theta + \omega^2 - 2\alpha\omega\sin\theta + \cos\theta}{\alpha^2(\alpha^2+4\omega^2)}$$

$$= \frac{\omega^2 + \alpha^2\cos^2\theta - \alpha\omega\sin 2\theta}{\alpha^2(\alpha^2+4\omega^2)}$$

Q.28. Determine the impulse response h(t) and sketch the magnitude and phase response of the system described by the transfer function **(14)**

$$H(s) = \frac{s^2+\omega_o^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}.$$

Ans:

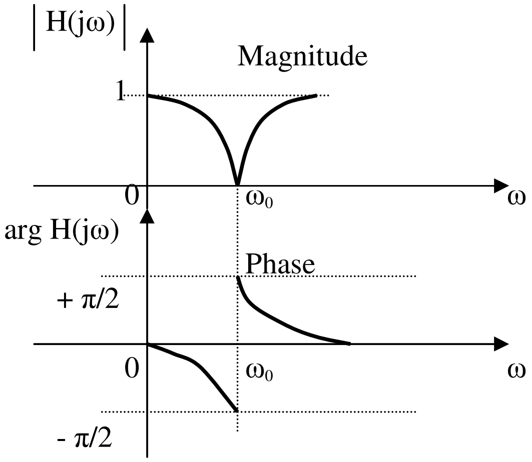
$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$$H(j\omega) = \frac{(j\omega)^2 + \omega_0^2}{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + \omega_0^2} = \frac{\omega_0^2 - \omega^2}{\omega_0^2 - \omega^2 + j\,\omega\,\frac{\omega_0}{Q}}$$

$$\therefore\ \left|H(j\omega)\right| = \frac{\left|\omega_0^2 - \omega^2\right|}{\left[(\omega_0^2-\omega^2)^2 + \omega^2\left(\frac{\omega_0}{Q}\right)^2\right]^{1/2}}$$

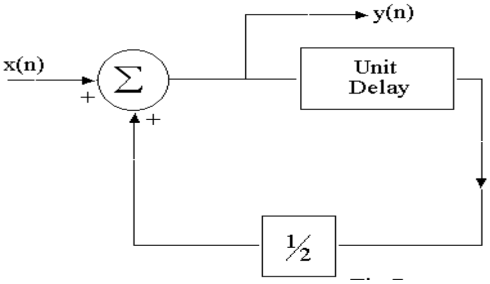
$$\text{Arg } H(j\omega) = -\tan^{-1}\left(\frac{\omega\left(\frac{\omega_0}{Q}\right)}{\omega_0^2-\omega^2}\right)$$

ω	|H(jω)|	Arg H(jω)
0	1	0
∞	1	0
ω_{0-}	0	- π/2
ω_{0+}	0	+ π/2



Q.29. Using the convolution sum, determine the output of the digital system shown in Fig. below.

Assume that the input sequence is $\{x(n)\} = \{\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3\}$ and that the system is initially at rest. **(5)**



Ans:

$x(n) = \{\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3\}$, system at rest initially (zero initial conditions)

$x(n) = 3\delta(n) - \delta(n-1) + 3\delta(n-2)$

$X(z) = 3 - z^{-1} + 3z^{-2}$

Digital system: $y(n) = x(n) + \frac{1}{2}y(n-1)$

$$\therefore Y(z) = \frac{X(z)}{1-\frac{1}{2}z^{-1}} = \frac{3 - z^{-1} + 3z^{-2}}{1-\frac{1}{2}z^{-1}} = -10 - 6\,z^{-1} + \frac{13}{1-\frac{1}{2}z^{-1}}$$

by partial fraction expansion.

Hence $y(n) = -10\,\delta(n) - 6\,\delta(n-1) + 13\left(\frac{1}{2}\right)^n u(n)$

Q.30. Find the z-transform of the digital signal obtained by sampling the analog signal $e^{-4t}\sin 4t\, u(t)$ at intervals of 0.1 sec. **(6)**

Ans:

$x(t) = e^{-4t}\sin 4t\, u(t), \qquad T = 0.1\text{ s}$

$x(n) = x(t \longrightarrow nT) = x(0.1n) = (e^{-0.4})^n \sin(0.4n)$

$x(n) \overset{z}{\longleftrightarrow} X(z)$

$x(n) = \sin \Omega n\, u(n) \overset{z}{\longleftrightarrow} \dfrac{z \sin \Omega}{z^2 - 2z\cos\Omega + 1}$

$\alpha = e^{-0.4} = 0.6703, \ \frac{1}{\alpha} = 1.4918$

$\Omega = 0.4\text{ rad} = 22.92^\circ$

$\sin\Omega = 0.3894; \cos\Omega = 0.9211$

$$\alpha^n x(n) \overset{z}{\longleftrightarrow} X(z/\alpha)$$

$$\therefore\ X(z) = \frac{1.4918z\,(0.3894)}{(1.4918)^2 z^2 - 2(1.4918)z(0.9211) + 1}$$

$$X(z) = \frac{0.5809z}{2.2255\, z^2 - 2.7482z + 1}$$

Q.31. An LTI system is given by the difference equation $y(n) + 2y(n-1) + y(n-2) = x(n)$.

i. Determine the unit impulse response.

ii. Determine the response of the system to the input $(\underset{\substack{\uparrow \\ n = 0}}{3}, -1, 3)$. **(4)**

Ans:

$$y(n) + 2y(n-1) + y(n-2) = x(n)$$

$$Y(z) + 2z^{-1}\,Y(z) + z^{-2}\,Y(z) = X(z)$$

$$(1 + 2z^{-1} + z^{-2})Y(z) = X(z)$$

(i). $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1 + 2z^{-1} + z^{-2}} = \dfrac{1}{(1 + z^{-1})^2}$ (Binomial expansion)

$$= 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + 5z^{-4} - 6z^{-5} + 7z^{-6} - \ldots\ldots\text{(Binomial expansion)}$$

$$\therefore\ h(n) = \delta(n) - 2\delta(n-1) + 3\delta(n-2) - \ldots..$$

$$= \{\underset{\substack{\uparrow \\ n=0}}{1}, -2, 3, -4, 5, -6, 7, \ldots.\}\quad \text{is the impulse response.}$$

(ii). $x(n) = \{\underset{\substack{\uparrow \\ n=0}}{3}, -1, 3\}$

$$= 3\delta(n) - \delta(n-1) + 3\delta(n-2)$$

$$X(z) = 3 - z^{-1} + 3z^{-2}$$

$$\therefore\ Y(z) = X(z).H(z) = \frac{3 - z^{-1} + 3z^{-2}}{1 + 2z^{-1} + z^{-2}} = \frac{3(1 + 2z^{-1} + z^{-2}) - 7z^{-1}}{1 + 2z^{-1} + z^{-2}}$$

$$= 3 - 7\frac{z^{-1}}{(1 + z^{-1})^2}$$

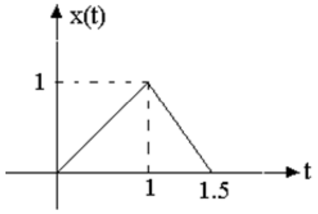
$\therefore\ y(n) = 3\delta(n) + 7nu(n)$ is the required response of the system.

Q.32. The signal x(t) shown below in Fig. is applied to the input of an

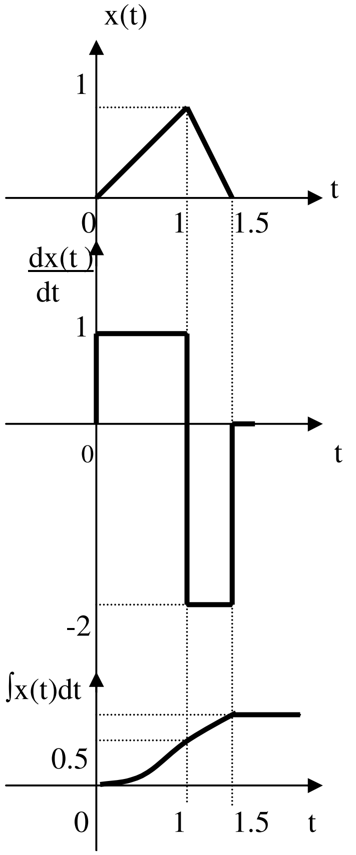
(i) ideal differentiator. (ii) ideal integrator.

Sketch the responses. **(1+4=**

$x(t) = t\,u(t) – 3t\,u(t-1) + 2t\,u(t-1.5)$



Ans:



(i) $0 < t < 1$

$$y(t) = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = 0.5 \text{ (Nonlinear)}$$

(ii) $1 < t < 1.5$

$$y(t) = y(1) + \int_1^t (3-2t)dt$$

$$= 0.5 + (3t – t^2)\Big|_1^t = 0.5 + 3t – t^2 – 3 + 1$$

$$= 3t – t^2 – 1.5 \quad \text{(Nonlinear)}$$

For t =1: $y(1) = 3 – 1 – 1.5 = 0.5$

(iii) $t \geq 1.5 : y(1.5) = 4.5 – 2.25 –1.5 = 0.75$

Q.33. Sketch the even and odd parts of **(1+2+3=6)**

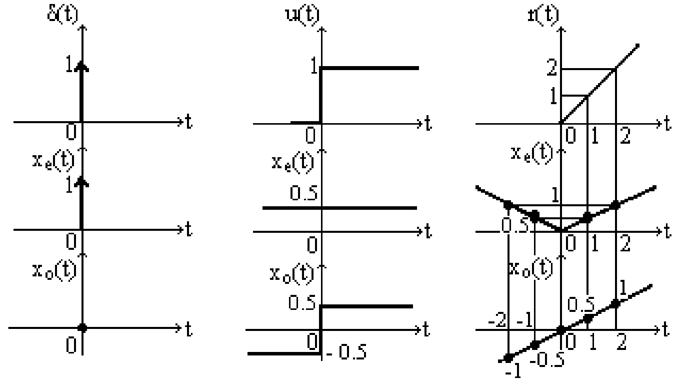
(i) a unit impulse function (ii) a unit step function

(iii) a unit ramp function.

Ans:

Even part $x_e(t) = \frac{x(t) + x(-t)}{2}$

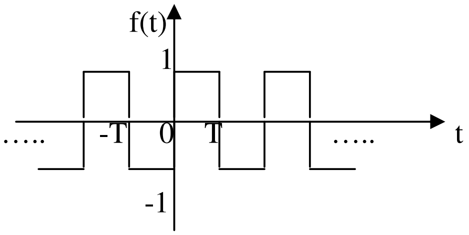
Odd part $x_o(t) = \frac{x(t) - x(-t)}{2}$



(i) unit impulse function (ii) unit step function (iii) unit ramp function

Q.34. Sketch the function $f(t) = u\left(\sin\frac{\pi t}{T}\right) - u\left(-\sin\frac{\pi t}{T}\right)$. (3)

Ans:



f(t) = { 1 0 < t T, 2T < t 3 T1
-1 T< t 2T, …
3 T < t <4T, ……

Q.35. Under what conditions, will the system characterized by $y(n)=\sum_{k=n_o}^{\infty} e^{-ak}\,x(n-k)$ be linear, time-invariant, causal, stable and memory less? (5)

Ans:

y(n) is : linear and time invariant for all k
causal if n_0 not less than 0.
stable if a > 0
memoryless if k = 0 only

Q.36. Let E denote the energy of the signal x (t). What is the energy of the signal x (2t)? (2)

Ans:

Given that

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$\text{To find } E^1 = \int_{-\infty}^{\infty} |x(2t)|^2 \, dt$$

$$\text{Let } 2t = r \text{ then } E^1 = \int_{-\infty}^{\infty} |x(r)|^2 \frac{dr}{2} = \frac{1}{2}\int_{-\infty}^{\infty} |x(r)|^2 \, dr = \frac{E}{2}$$

Q.37. x(n), h(n) and y(n) are, respectively, the input signal, unit impulse response and output signal of a linear, time-invariant, causal system and it is given that $y(n-2)=x(n-n_1)*h(n-n_2)$, where * denotes convolution. Find the possible sets of values of n_1 and n_2. **(3)**

Ans:

$$y(n-2) = x(n-n_1) * h(n-n_2)$$

$$\therefore z^{-2} Y(z) = z^{-n1} X(z) . z^{-n2} H(z)$$

$$z^{-2} H(z) X(z) = z^{-(n_1+n_2)} X(z)H(z)$$

$$\therefore n_1+n_2 = 2$$

Also, $n_1, n_2 \geq 0$, as the system is causal. So, the possible sets of values for n_1 and n_2 are:

$\{n_1, n_2\} = \{(0,2),(1,1),(2,0)\}$

Q.38. Let h(n) be the impulse response of the LTI causal system described by the difference equation $y(n)=a\,y(n-1)+x(n)$ and let $h(n)*h_1(n)=\delta(n)$. Find $h_1(n)$. **(4)**

Ans:

$y(n) = a\,y(n-1) + x(n)$ and $h(n) * h_1(n) = \delta(n)$

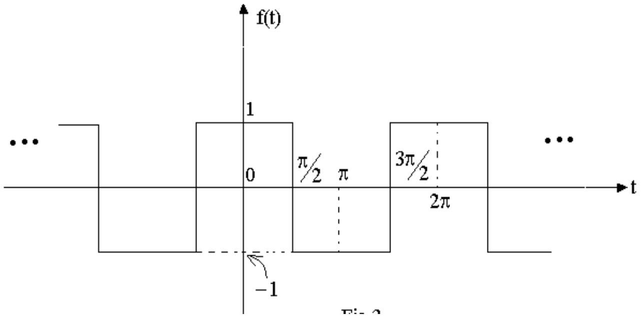
$Y(z) = az^{-1} Y(z) + X(z)$ and $H(z) H_1(z) = 1$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}}$ and $H_1(z) = \frac{1}{H(z)}$

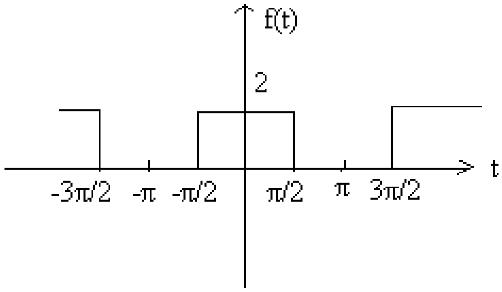
$\therefore H_1(z) = 1-az^{-1}$ or $h_1(n) = \delta(n) - a\,\delta(n-1)$

Q.39. Determine the Fourier series expansion of the waveform f (t) shown below in terms of sines and cosines. Sketch the magnitude and phase spectra. **(10+2+2=14)**

Ans:



Define g(t) = f(t) +1. Then the plot of g(t) is as shown , below and,



ω = 2π/2π = 1
because T =2π

$$g(t) = \begin{cases} 0 & -\pi < t < -\pi/2 \\ 2 & -\pi/2 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases}$$

Let $g(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

Then $a_0 =$ average value of f(t) =1

$$a_n = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} 2\cos nt\,dt \;=\; \frac{2}{\pi}\frac{\sin nt}{n}\Bigg|_{-\pi/2}^{\pi/2} \;=\; 2/n\pi \,.\, 2\sin n\pi/2$$

$$= 4/n\pi \,.\, \sin n\pi/2$$

$$= \begin{cases} 0 & \text{if } n = 2,4,6 \ldots\ldots \\ 4/n\pi & \text{if } n = 1,5,9 \ldots\ldots \\ -4/n\pi & \text{if } n = 3,7,11\ldots\ldots \end{cases}$$

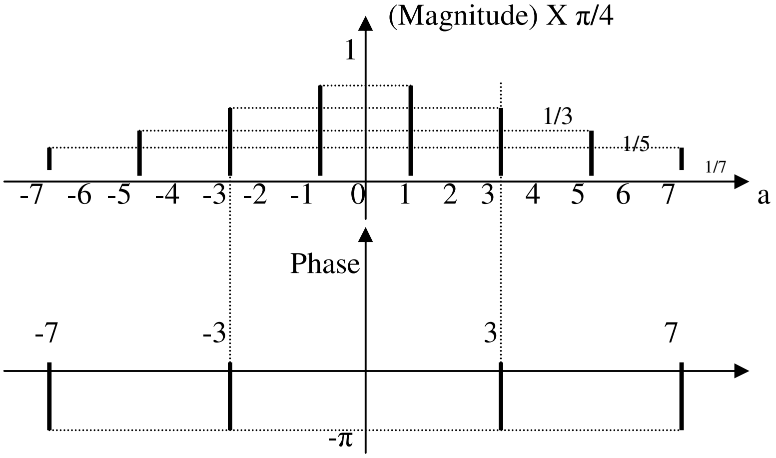
$$\text{Also, } b_n = \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} 2\sin nt\,dt \;=\; \frac{4}{\pi}\frac{\cos nt}{n}\Bigg|_{\pi/2}^{\pi/2} \;=\; 4/n\pi[\cos n\pi/2 - \cos n\pi/2] = 0$$

Thus, we have f(t) = -1 + g(t)

$$= \frac{4\text{cost}}{\pi} - \frac{4\cos 3t}{3\pi} + \frac{4\cos 5t}{5\pi} - \ldots\ldots.$$

$$= 4/\pi \;\; \{\text{cost} - \cos 3t/3 + \cos 5t/5 \;\; \ldots..\}$$

spectra :



Q.40. Show that if the Fourier Transform (FT) of x (t) is $X(\omega)$, then

$$FT\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega).$$

Ans:

$$x(t) \xleftrightarrow{FT} X(j\omega) \text{ or } X(\omega)$$

$$\text{i.e., } x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \frac{d}{dt}[x(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

$$\therefore \frac{d}{dt}[x(t)] \xleftrightarrow{FT} j\omega X(j\omega)$$

Q.41. Show, by any method, that $FT\left[\frac{1}{2}\right] = \pi\,\delta(\omega)$. **(2)**

Ans:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \pi\,\delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2} \qquad \because X(j\omega) = \pi\,\delta(\omega)$$

$$\therefore \frac{1}{2} \xleftrightarrow{FT} \pi\,\delta(\omega)$$

Q.42 Find the unit impulse response, h(t), of the system characterized by the relationship :

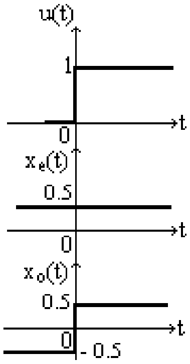
$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$ **(3)**

Ans:

$$y(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, t \geq 0 = u(t) \\ 0, \text{otherwise} \end{cases}$$

Q.43. Using the results of parts (a) and (b), or otherwise, determine the frequency respons
of the system of part (c). (

Ans:



As shown in the figure, u(t) = 1/2 + x(t)

$$\text{where } x(t) = \begin{cases} 0.5, & t > 0 \\ -0.5, & t < 0 \end{cases}$$

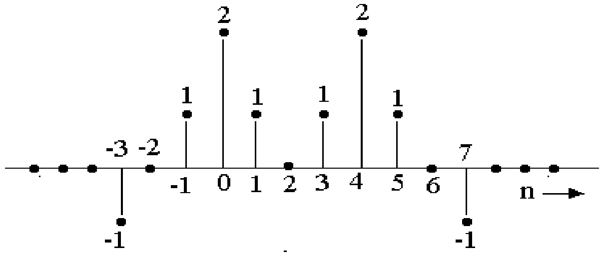
$\therefore$ $dx/dt = \delta(t)$ By (a) $FT[\delta(t)] = j\omega X(\omega)$

$\therefore$ $X(\omega) = 1/j\omega$. Also $FT[1/2] = \pi\delta(\omega)$

Therefore $FT[u(t)] = H(j\omega) = \pi\surd(\omega) + 1/j\omega$.

Q.44. Let $X\left(e^{j\omega}\right)$ denote the Fourier Transform of the signal x (n) shown below .**(2+2+3+5+2=14)**

Ans:



Without explicitly finding out $X\left(e^{j\omega}\right)$, find the following :-

(i) X (1)

(ii) $\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega$

(iii) X(-1)

(iv) the sequence y(n) whose Fourier Transform is the real part of $X\left(e^{j\omega}\right)$.

(v) $\int_{-\pi}^{\pi} \left| X\left(e^{j\omega}\right) \right|^2 d\omega$.

Ans:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$+\infty$

(i) $X(1) = X(e^{j0}) = \sum_{-\infty} x(n) = -1 + 1 + 2 + 1 + 1 + 2 + 1 - 1 = 6$

(ii) $x(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\omega}) e^{j\omega n} d\omega \;;\; \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi\, x(0) = 4\pi$

(iii) $X(-1) = X(e^{j\pi}) = \sum_{n=-\infty}^{+\infty} x(n)\,(-1)^n = 1+ 0-1+2-1+0-1+2-1+0+1 = 2$

(iv) Real part $X(e^{j\omega}) \longleftrightarrow x_e(n) = \frac{x(n) + x(-n)}{2}$

$y(n) = x_e(n) = 0, \quad n < -7, n > 7$

$y(7) = \frac{1}{2} x(7) = -\frac{1}{2} = y(-7)$

$y(6) = \frac{1}{2} x(6) = 0 = y(-6)$

$y(5) = \frac{1}{2} x(5) = \frac{1}{2} = y(-5)$

$y(4) = \frac{1}{2} x(4) = 2 = y(-4)$

$y(3) = \frac{1}{2}[x(3) + x(-3)] = 0 = y(-3)$

$y(2) = \frac{1}{2}[x(2) + x(-2)] = 0 = y(-2)$

$y(1) = \frac{1}{2}[y(1) + y(-1)] = 1 = y(-1)$

$y(0) = \frac{1}{2}[\, y(0) + y(0)] = 2$

(v) Parseval's theorem:

$$\int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} \left| x(n) \right|^2 = 2\pi(1 + 1 + 4 + 1 + 1 + 4 + 1 + 1) = 28\pi$$

Q.45 If the z-transform of x (n) is X(z) with ROC denoted by R_x, find the z-transform of $y(n) = \sum_{k=-\infty}^{n} x(k)$ and its ROC. **(4)**

Ans:

$x(n) \overset{z}{\longleftrightarrow} X(z), \quad \text{RoC } R_x$

$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=\infty}^{0} x(n-k) = \sum_{k=0}^{\infty} x(n-k)$

$\therefore Y(z) = X(z) \underbrace{\sum_{k=0}^{\infty} z^{-k}}_{\text{Geometric series}} = \frac{X(z)}{1 - z^{-1}}$, RoC at least $R_x \cap (|z| > 1)$

Q.46 (i) x (n) is a real right-sided sequence having a z-transform X(z). X(z) has two poles, one of which is at $a\,e^{j\phi}$ and two zeros, one of which is at $r\,e^{-j\theta}$. It is also known that $\sum x(n) = 1$. Determine X(z) as a ratio of polynomials in z^{-1}. **(6)**

(ii) If $a = \frac{1}{2}, r = 2, \theta = \phi = \pi/4$ in part (b) (i), determine the magnitude of X(z) on the unit circle. **(4)**

Ans:

. (i) x(n) : real, right-sided sequence $\xleftrightarrow{z}$ X(z)

$$X(z) = K \frac{(z- re^{-j\theta})(z- re^{j\theta})}{(z- ae^{j\Phi})(z- ae^{-j\Phi})} \quad ; \sum x(n) = X(1) = 1$$

$$= K \frac{z^2 - zr\,(e^{j\theta}+e^{-j\theta}) + r^2}{z^2 - za\,(e^{j\Phi}+ e^{j\Phi}) + a^2}$$

$$= K \frac{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}}{1 - 2a\cos\Phi z^{-1} + a^2 z^{-2}} = K. \frac{N(z^{-1})}{D(z^{-1})}$$

where $K. \frac{1 - 2r\cos\theta + r^2}{1 - 2a\cos\Phi + a^2} = X(1) = 1$

i.e., $K = \frac{1 - 2a\cos\Phi + a^2}{1 - 2r\cos\theta + r^2}$

(ii) $a = ½, r = 2, \theta = \Phi = \pi/4$; $K = \frac{1 - 2(½).(1/\sqrt{2}) + ¼}{1 - 2(2)\,(1/\sqrt{2}) + 4} = 0.25$

$$X(z) = (0.25)\ .\ \frac{1 - 2(2)\,(1/\sqrt{2})\, z^{-1} + 4z^{-2}}{1 - 2(½).(1/\sqrt{2})\, z^{-1} + ¼\, z^{-2}}$$

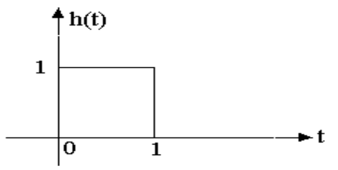
$$= (0.25) \frac{1 - 2\sqrt{2}\, z^{-1} + 4z^{-2}}{1 - (1/\sqrt{2})\, z^{-1} + ¼\, z^{-2}} \Longrightarrow X(e^{j\omega}) = (0.25) \frac{1 - 2\sqrt{2}\, e^{-j\omega} + 4\, e^{-2j\omega}}{1 - (1/\sqrt{2})\, e^{-j\omega} + ¼\, e^{-2j\omega}}$$

$$= \frac{- 2\sqrt{2} + e^{j\omega} + 4\, e^{-j\omega}}{-2\sqrt{2}+ 4e^{j\omega} + e^{-j\omega}}$$

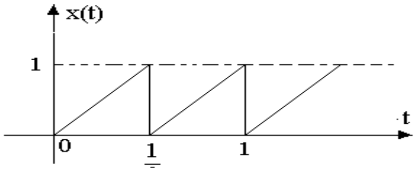
$$\therefore \left|X(e^{j\omega})\right| = 1$$

Q.47 Determine, by any method, the output y(t) of an LTI system whose impulse response h(t) is of the form shown in fig(a). to the periodic excitation x(t) as shown in fig(b). **(14)**

Ans:



Fig(a)



Fig(b)

$h(t) = u(t) - u(t-1) \Rightarrow H(s) = \frac{1-e^{-s}}{s}$

First period of x(t) , $x_T(t) = 2t\,[u(t) - u(t-½)\,]$

$= 2[\,t\,u(t) - (t-1/2)\,u(t-1/2) - 1/2\,u(t-1/2)]$

$\therefore X_T(s) = 2[1/s^2 - e^{-s/2}/s^2 - 1/2\,e^{-s/2}/s\,]$

$X(s) = X_T(s) / 1 - e^{-s/2}$

$$Y(s) = \frac{1-^{e-s}}{s}.\frac{1}{1-e^{-s/2}}\;2\left(\frac{1-e^{-s/2}-0.5se^{-s/2}}{s^2}\right)$$

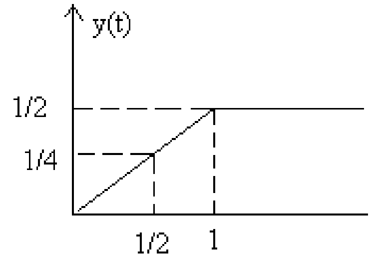
$$=\frac{2}{s^3}\left(1+e^{-s/2}\right)\left[1-e^{-s/2}-0.5\,s\,e^{-s/2}\right]$$

$$=\frac{2}{s^3}\left(1-e^{-s}-0.5s(e^{-s/2}+e^{-s})\right)$$

$$=2\frac{1-e^{-s}}{s^3}-\frac{e^{-s/2}+e^{-s}}{s^2}$$

Therefore $y(t) = t^2\,u(t) - (t-1)^2\,u(t-1) - \left(t-\frac{1}{2}\right)u\left(t+\frac{1}{2}\right)-(t-1)u(t-1)$

This gives $y(t) = \begin{cases} t2 & 0< t < 1/2 \\ t^2 - t + 1/2 & 1/2 < t < 1 \\ 1/2 & t > 1 \end{cases}$



(not to scale)

Q.48 Obtain the time function f(t) whose Laplace Transform is $F(s) = \frac{s^2 + 3s + 1}{(s+1)^3 (s+2)^2}$.

Ans:

$$F(s) = \frac{s^2+3s+1}{(s+1)^3(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2}$$

$$A(s+2)^2(s+1)^2 + B(s+2)^2(s+1) + C(s+2)^2 + D(s+1)^3(s+2) + E(s+1)^3 = s^2+3s+1$$

$$C = \frac{s^2+3s+1}{(s+2)^2}\Bigg|_{s=-1} = \frac{1-3+1}{1} = -1 \qquad \boxed{C = -1}$$

$$E = \frac{s^2+3s+1}{(s+1)^3}\Bigg|_{s=-2} = \frac{4-6+1}{-1} = 1 \qquad \boxed{E = 1}$$

$$A(s^2+3s+2)^2 + B(s^2+4s+4)(s+1) + C(s^2+4s+4) + D(s^3+3s^2+3s+1)(s+2) + E(s^3+3s^2+3s+1)$$
$$= s^2+3s+1$$

$$A(s^4+6s^3+13s^2+12s+4) + B(s^3+5s^2+8s+4) + C(s^2+4s+4) + D(s^4+5s^3+9s^2+7s+2) +$$
$$E(s^3+3s^2+3s+1) = s^2+3s+1$$

s^4 : A+D = 0

s^3 : 6A+ B+ 5D +E = 0 ; A+B+1 = 0 as 5(A+D) = 0, E = 1

s^2 : 13A+5B+C+9D+3E = 1 ; 4A+5B+1 = 0 as 9(A+D) = 0, C = -1, E = 1

s^1 : 12A+8B+4C+7D+3E = 3 ; 5A+8B-4 = 0 as 7(A+D) = 0, C = -1, E = 1

s^0 : 4A+4B+4C+2D+E = 1

A+B = -1 ; 4(A+B)+B+1 = 0 or –4+B+1 = 0 or $\boxed{B = 3}$

$\boxed{A = -4}$

$\therefore$ A = -1-3 = - 4

A+D = 0 or D = -A = 4 $\boxed{D = 4}$

$$\therefore F(s) = \frac{-4}{(s+1)} + \frac{3}{(s+1)^2} + \frac{-1}{(s+1)^3} + \frac{4}{(s+2)} + \frac{1}{(s+2)^2}$$

$$\therefore f(t) = L^{-1}[F(s)] = -4e^{-t} + 3t\,e^{-t} - t^2 e^{-t} + 4e^{-2t} + t\,e^{-2t} = [e^{-t}(-4 + 3t - t^2) + e^{-2t}(4 + t)]\,u(t)$$

$$\therefore f(t) = [e^{-t}(-4 + 3t - t^2) + e^{-2t}(4 + t)]\,u(t)$$

Q.49 Define the terms variance, co-variance and correlation coefficient as applied to random variables.

Ans:

Variance of a random variable X is defined as the second central moment $E[(X-\mu_X)]^n$, n=2, where central moment is the moment of the difference between a random variable X and its mean μ_X i.e.,

$$\sigma_X^2 = \text{var}\,[X] = \int_{-\infty}^{+\infty} (x-\mu_X)^2 f_x(x)\,dx$$

Co-variance of random variables X and Y is defined as the joint moment:

$$\sigma_{XY} = \text{cov}\,[XY] = E[\{X-E[X]\}\{Y-E[Y]\}] = E[XY]-\mu_X\mu_Y$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

Correlation coefficient ρ_{XY} of X and Y is defined as the co-variance of X and Y normalized w.r.t $\sigma_X\sigma_Y$:

$$\rho_{XY} = \frac{\text{cov}\,[XY]}{\sigma_X\sigma_Y} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$$