Q.2 a. Prove that $\cos\alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right) = 0$

Answer:

L.H.S.

$= \cos\alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right)$

$= \cos\alpha + \cos\left(\alpha + 120^\circ\right) + \cos\left(\alpha + 240^\circ\right)$

$= \cos\alpha + \cos\left(90^\circ + \alpha + 30^\circ\right) + \cos(180^\circ + \alpha + 60^\circ)$

$= \cos\alpha - \sin\left(\alpha + 120^\circ\right) - \cos\left(\alpha + 60^\circ\right)$

$= \cos\alpha - (\sin\alpha\cos 30^\circ + \cos\alpha\ \sin 30^\circ)$

$= \cos\alpha - (\sin\alpha\cos 30^\circ + \cos\alpha\sin 30^\circ) - (\cos\alpha\cos 60^\circ - \sin\alpha\sin 60^\circ)$

$= \cos\alpha - (\frac{\sqrt{3}}{2}\sin\alpha + \frac{1}{2}\cos\alpha) - (\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha)$

$= \cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha + \frac{1}{2}\cos\alpha - \frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha$

$= \cos\alpha - \cos\alpha = 0$

b. Prove that $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$

Answer:

L.H.S

$$= \frac{\sec 8A - 1}{\sec 4A - 1}$$

$$= \frac{\frac{1}{\cos 8A} - 1}{\frac{1}{\cos 4A} - 1}$$

$$= \frac{\cos 4A(1 - \cos 8A)}{\cos 8A(1 - \cos 4A)}$$

$$= \frac{\cos 4A}{\cos 8A}.\frac{2\sin^2 4A}{2\sin^2 2A}$$

$$= \frac{2\sin 4A\cos 4A.\sin 4A}{\cos 8A.2\sin^2 2A}$$

$$= \frac{\tan 8A}{\tan 2A} = \text{R.H.S.}$$

Q.3 a. Find the coefficient of x^{18} in the expansion of $\left(x^2 + \frac{3a}{x}\right)^{15}$

Answer:

Given expression is $\left(x^2 + \frac{3a}{x}\right)^{15}$

Let x^{18} occurs in the given expansion in T_{r+1}

$$T_{r+1} = 15_{C_r} (x^2)^{15-r} \left(\frac{3a}{x}\right)^r \qquad \left(\because T_{r+1} = n_{C_r} x^{n-r} a^r\right)$$

$$= \frac{15!}{r!(15-r)!} x^{30-2r} \frac{(3a)^r}{x^r}$$

$$= \frac{15!}{r!(15-r)!} (3a)^r x^{30-2r} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (i)}$$

Since x^{18} occurs in T_{r+1}

$\therefore$ $30 - 3r = 18$

or $-3r = 18 - 30$

or $-3r = -12$

or $r = 4$

Putting r = 4 in equation (i), in get

$$T_5 = \frac{15!}{4!11!}(3a)^4.x^{30-12}$$

$$= \frac{15\times14\times13\times12\times11!}{4\times3\times2\times1\times11!} 81a^4 x^{18}$$

$$= (1365)(81a^4)\ x^{18}$$

$\therefore$ Required coefficient of x^{18} in the given expansion is 110565 a^4

b. If the first term of an A.P is 2 and the sum of first five terms is equal to one fourth of the sum of the next five terms, then (i) show that $T_{20} = -112$ (ii) find the sum of first 30 terms.

Answer:

Let the A.P. be a, a+d, a+2d ------------

$$T_1 + T_2 + T_3 + T_4 + T_5 = \frac{1}{4}\left[T_6 + T_7 + T_8 + T_9 + T_{10}\right]$$

$$\therefore \frac{5}{2}(T_1 + T_5) = \frac{1}{4} \times \frac{5}{2}(T_6 + T_{10})$$

(sum = $\frac{n}{2}$(first term + last term)

$\Rightarrow \frac{5}{2}[a+(a+4d)] = \frac{5}{8}[(a+5d)+(a+9d)]$

$\Rightarrow 2a+4d = \frac{1}{4}(2a+14d)$

$\Rightarrow$ 4a +8a = a + 7d

$\Rightarrow$ d = -3a

$\therefore$ d = -3(2) = -6 ($\therefore$ a = 2 is geven)

$\therefore$ a = 2 & d = -6

(i) T_{20} = a + (20-1)d = 2+19(-6) = 2-114 = -142

(ii) $S_{30} = \frac{30}{2}$[(2(2) + (30-1)(-6)]

= 15[4 - 174]

= 15 (-170) = -2550

Q.4 a. Show that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$$

Answer:

$$\text{L.H.S} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

Operating $R_1 \to R_1 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

Operating $C_1 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ we have

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ b+c & a-b & a-c \end{vmatrix}$$

$$= (a+b+c)(a-b)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & -(a+b) & c+a \\ b+c & 1 & -1 \end{vmatrix}$$

$$= (a+b+c)(a-b)(c-a)\begin{vmatrix} -(a+b) & c+a \\ 1 & -1 \end{vmatrix}$$

$$= (a+b+c)(a-b)(c-a)\begin{vmatrix} a+b & -c-a \end{vmatrix}$$

$$= (a+b+c)(a-b)(b-c)(c-a)$$

b. Using determinants solve the following system of equations:

2y – 3z = 0
x + 3y = – 4
3x + 4y = 3

Answer:

The given equation are

0.x + 2y – 3y = 0
x + 3y +0.3 = -4
3x + 4y + 0.3 = 3

$$\text{Here } \Delta = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

(expanding by C_3)

$$= -3\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -3(4-9)$$

$$= 15 \neq 0$$

Because $\neq$ therefore, system has unique solution

$$\Delta_1 = \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

$$= 0 - 2(0-0) - 3(-16-19)$$

$= 75$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix}$$
$$= -3(3+12) = -45$$

$$\Delta_3 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix}$$
$$= -2(3+12) = -30$$
$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{75}{15} = 5$$
$$y = \frac{\Delta_2}{\Delta} = \frac{-45}{15} = -3$$
$$z = \frac{\Delta_3}{\Delta} = \frac{-30}{15} = -2$$
$$\therefore x = 5, y = -3, z = -2$$

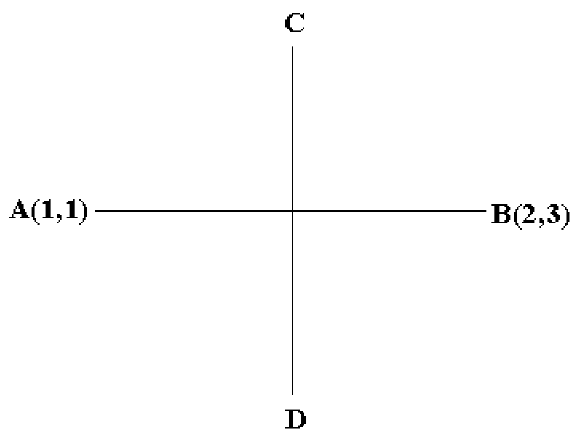
Q.5 a. Find the equation of the right bisector of the segment joining A(1, 1) and B(2, 3)

Answer:

Let AB be a segment joining the points A(1,1) ,B(2,3) & CD be its right bisector. Then CD passes through the midpoint of AB is

$$\left(\frac{1+2}{2}, \frac{1+3}{2}\right)$$
$$\left(\frac{3}{2}, 2\right)$$

This CD passes through $\left(\frac{3}{2}, 2\right)$



Also slope of AB = $\dfrac{3-1}{2-1} = 2$

Now CD $\perp$ AB

$\therefore$ equation of CD which passes through $\left(\dfrac{3}{2}, 2\right)$

and has slop -1/2 is

$y - y_1 = m(x - x_1)$

$y-2 = -\dfrac{1}{2}\left(x - \dfrac{3}{2}\right)$

$4y-8 = -2x + 3$

$2x+4y-11=0$

Which is the required equation.

b. Find the equation of the lines through the origin and making an angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$

Answer:

Let m be the slope of any one of the required lines through (0,0)

Then the equation is $y-0 = m(x-0)$

$y = mx$ _________ (i)

also slop of $x + \sqrt{3}y + 3\sqrt{3} = 0$ is $-\dfrac{1}{\sqrt{3}}$

Angle between the two line is given to be 60°

$$\text{Tan } 60^\circ = \left| \frac{m + \frac{1}{\sqrt{3}}}{1 + m\left(-\frac{1}{\sqrt{3}}\right)} \right| \qquad \left(\text{Using } \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$$

$$\Rightarrow \pm\sqrt{3} = \frac{\sqrt{3}m+1}{\sqrt{3}-m}$$

Taking + ve sign, we here 3- $\sqrt{3}m = \sqrt{3}m+1$

$-2\sqrt{3}m = -2$

$$m = \frac{1}{\sqrt{3}}$$

$$\therefore \text{from(i)}, y = \frac{1}{\sqrt{3}}x$$

Taking –ve sign, we here

$-3+\sqrt{3}m = \sqrt{3}m+1$

Here m does not here a finite value, then the line is x = 0.

($\therefore$ The line passes through the orgin).

Q.6 a. Find the equation of the circle which passes through the points (5, –8), (2, –9) and (2, 1). Find also the co-ordinates of its centre and radius.

Answer:

Let the required equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ _________ (i)

(i) Passes through (5, -8)

$(5)^2 + (-8)^2 + 2g(5) + 2f(-8) + c = 0$

$25 + 64 + 10g – 16f + c = 0$

$10g - 16f + c = -89$ ____________ (ii)

(i) Passes through (2, -9)

$(2)^2 + (-9)^2 + 2g(2) + 2f(9) + c = 0$

$4g – 18f + c -85$ ________ (iii)

(i) Passes through (2, 1)

$(2)^2 + (1)^2 + 2g(2) + 2f(1) + c = 0$

$4g + 2f + c = -5$ _________ (iv)

Solving (ii) & (iii)

$10g - 16f + c – 89$

$4g - 18f + c = -85$

$6g + 2f = -4$ _________ (v)

Solving (iii) & (iv)

$4g – 18f + c = - 85$

$4g + 2f + c = -5$

-20f = - 80
f = 4
Substituting the value of 'f' in (v)
6g + 2 (4) = -4
6g = -4 -8 = -12
g = -2
Substituting the value of 'g' and 'f' in equation (iv)
4 (-2) + 2(4) + c = 14
c = -5
substituting the value of g, f, c in equation (i)
4(-2) + 2(4) + c = -5
c = -5
Substituting the values of g, f, c in equation (i)

$x^2 + y^2 - 4x + 8y - 5 = 0$

c (-g, -f) = c (2, -4) & r = $\sqrt{g^2 + f^2 - c}$

$= \sqrt{4+16+5} = \sqrt{25} = 5$

b. Find the length of major and minor axis, eccentricity, the co-ordinates of vertices and foci, directrices and the length latus rectum of the ellipse $3x^2+2y^2=6$.

Answer:

The given ellipse is $3x^2 + 2y^2 = 6$

Comparing it with $\frac{x^2}{a^2} + \frac{y}{b^2} = 1$

$a^2 = 2$, $b^2 = 3$, $a^2 < b^2$ u $a < b$

Length of the major axis = 2b = $2\sqrt{3}$

Length of the minor axis = 2a = $2\sqrt{2}$

$a^2 = b^2 (1-e^2)$
$2 = 3 (1 – e^2)$

$\frac{2}{3} = 1 - e^2$

$e^2 = 1 - 1 - \frac{2}{3} = \frac{1}{3}$

$e = \frac{1}{\sqrt{3}}$

co- ordinates of vertices are $(0, \pm b)$ ie, $(0, \pm \sqrt{3})$

co-ordinates of foci are $(0, \pm be)$ ie $(0, \pm \frac{1}{\sqrt{3}}\sqrt{3})$

$= (0, \pm 1)$

Equation of the directices are $y = \pm \frac{b}{e}$

$$y = \pm\frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} \pm 3$$

$$\text{length of latus x here} = \frac{2b^2}{a} = \frac{2(2)}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Q.7 a. If $y = \ell og(x + \sqrt{1+x^2})$, Prove that $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$

Answer:

siny = x sin(a+y)

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating both sides w.r.t 'y'

$$\frac{dx}{dy} = \frac{\sin(a+y)\frac{d}{dy}\sin y - \sin y\frac{d}{dy}\sin(a+y)}{\sin^2(a+y)}$$

$$\frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{\sin a}{\sin^2(a+y)}} = \frac{\sin^2(a+y)}{\sin a}$$

b. Find the equation of the tangent to the curve $y = \dfrac{x-7}{(x-2)(x-3)}$ at the point where it cuts x-axis.

Answer:

The given curve is $y = \dfrac{x-7}{(x-2)(x-3)}$ ______________ (i)

This curve cut x – axis; $\therefore$ y = 0

We get $0 = \dfrac{x-7}{(x-2)(x-3)}$

$\therefore$ x = 7

This (i) cuts at x – axis at (7, 0) Diff. (i) w.r.t 'x' both sides

$$\frac{dy}{dx}=\frac{(x^2-5x+6)\frac{d}{dx}(x-7)-(X-7)\frac{d}{dx}(x^2-5x+6)}{(x^2-5x+6)^2}$$

$$\frac{dy}{dx}=\frac{(x^2-5x+6)(1)-(x-7)(2x-5)}{(x^2-5x+6)^2}$$

$$\frac{dy}{dx}=\frac{(x^2-5x+6)-(2x^2-5x-14x+35)}{(x^2-5x+6)^2}$$

$$\frac{dy}{dx}=\frac{x^2-5x+6-2x^2+19x-35}{(x^2-5x+6)^2}$$

$$\frac{dy}{dx}=\frac{-x^2+14x-29}{(x^2-5x+6)^2}$$

$$\left(\frac{dy}{dx}\right)(7,0)=\frac{-(7)^2+14(7)-29}{((7)^2-5(7)+6)^2}=\frac{-49+98-29}{(49-35+6)^2}$$

$$\frac{-78+98}{(20)^2}=\frac{20}{(20)^2}=\frac{1}{20}$$

New equation of tangent at the point (7, 0) sharing slop $\frac{1}{20}$ is

y-0 = $\frac{1}{20}$ (x-7) (Using $y-y_1 = m(x-x_1)$)

20y = x – 7

x - 20y - 7 = 0

Q.8 a. Evaluate $\int \frac{4x+1}{x^2+3x+2}dx$

Answer:

Let $I=\int \frac{4x+1}{x^2+3x+2}dx$

$$4x+1=\lambda\frac{d}{dx}(x^2+3x+2)+\mu$$

$4x + 1 = 2\lambda x + 3\lambda + \mu$

Comparing the coefficients.

$4 = 2\lambda,\ \ 1 = 3\lambda + \mu$
$\lambda = 2,\ \ 1 = 3(2) + \mu$

$\mu = 1\text{-}6 = -5$
Substituting the values of 'λ' & 'μ' in equation (i)

$4x + 1 = 2(2x + 3) - 5$

$$I = \int \frac{2(2x+3)-5}{x^2+3x+2}dx$$

$$= 2\int \frac{2x+3}{x^2+3x+2}dx - 5\int \frac{1}{x^2+3x+2}dx$$

$$= \quad 2\int \frac{2x+3}{x^2+3x+2}dx - 5\int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}+2}dx$$

$$= 2\int \frac{2x+3}{x^2+3x+2}dx - 5\int \frac{1}{\left(x^2+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$= 2\ell og\left|x^3+3x+2\right| - 5\frac{1}{2\left(\frac{1}{2}\right)^2}\ell og\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + c$$

$$= \quad 2\ell og\left|x^3+3x+2\right| - 5\ell og\left|\frac{2x+2}{2x+4}\right| + c$$

$$= 2\ell og\left|x^3+3x+2\right| - 5\ell og\left|\frac{x+2}{x+4}\right| + c$$

b. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

Answer:

Let $I = \int_0^{\pi/4} \ell og(1+\tan x)dx$ ______________ (i)

(Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi/4} \ell og(1+\tan(\frac{\pi}{4}-x)dx$

$= \int_0^{\pi/4} \ell og\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \ell og\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \ell og\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} (\ell og2 - \ell og(1+\tan x))dx$ ________________ (ii)

= Adding (i) & (ii)

$2I = \int_0^{\pi/4} \ell og2dx = \ell og2\int_0^{\pi/4} 1dx$

$= \frac{\pi}{8}\ell og2$

Q.9 a. Solve the initial value problem

$x\frac{dy}{dx} + \cot y = 0$, when $y(\sqrt{2}) = \pi/4$

Answer:

$x\ \frac{dy}{dx} + \cot y = 0$

$\frac{1}{\cot y}dy + \frac{1}{x}dx = 0$ (Using variable separated)

Integrating both sides

$\int \tan dy + \int \frac{1}{x}dx = \int 0dx$

$\ell og|\sec y| + \ell og|x| = \ell og|c|$

Taking antilog both sides

X sec y = c ________ (i)

Also $y\left(\sqrt{2}\right) = \pi/4$

u. when $x = \sqrt{2}$, $y = \frac{\pi}{4}$

put in (i)

$\sqrt{2}\sec\frac{\pi}{4} = c$

$\sqrt{2}\,.\sqrt{2} = c$

Put this value in (i)

x sec y = 2

b. Solve $(x^2 + xy)dy = (x^2 + y^2)dx$

Answer:

$(x^2 + xy)dy = (x^2 + y^2)dx$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \quad \_\_\_\_\_\_\_\_\_\_\_\_ \text{(i)}$$

Clearly it is a homogeneous equation

$\therefore$ Put y = vx ____________ (ii)

Differentiating both sides w.r.t. (x)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \_\_\_\_\_\_\_\_\_\_\_\_ \text{(iii)}$$

Put the values of (ii) and (iii) in (i) weget

$$v + x\,\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + x(vx)} = \frac{x^2 + v^2x^2}{x^2 + vx^2}$$

$$v + x\,\frac{dv}{dx} = \frac{1+v^2}{1+v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{1+v^2 - v - v^2}{1+v} = \frac{1-v}{1+v}$$

$$\frac{1-v}{1+v}\,dv = \frac{1}{x}dx$$

Integrating both sides

$$\int\frac{1+v}{1-v}dv = \int\frac{1}{x}dx$$

$$\int -1 + \frac{2}{1-v}dv = \int\frac{1}{x}dx$$

$$-v + 2\,\frac{\ell og|1-v|}{-1} = \ell og|x| + c$$

$$-\frac{y}{x} - \ell og\left|1 - \frac{y}{x}\right| - \ell og|x| = c$$

$$-\frac{y}{x} - \ell og\left|\left(\frac{x-y}{x}\right)^2 .x\right| = c$$

$$-\frac{y}{x}-\ell og\left|\frac{(x-y)^2}{x}\right|=c$$

$$\frac{y}{x}+\ell og\left|\frac{(x-y)^2}{x}\right|=-c$$

<u>Text Books</u>

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II. A Text book of Comprehensive Mathematics Class XI, Parmanand Gupta, Laxmi Publications (P) Ltd, New Delhi.

III. Engineering Mathematics, H. K. Dass, S, Chand and Company Ltd, 13th Edition, New Delhi.