

Q.2 a. If Z is a homogeneous function of degree n in x and y , show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

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b. Use method of differentiation under integral sign to show that

$$\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2} \log(1+a), a \geq 0$$

Answer:

(b) Let $I(a) = \int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)} dx$

Differentiate under sign of integration w.r.t. a .

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{x}{1+a^2x^2} dx \quad \text{--- (1)}$$

$$= \frac{1}{(1-a^2)} \int_0^{\infty} \left(\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2} \right) dx \quad \text{--- (2)}$$

$$= \frac{1}{1-a^2} \cdot \left[\tan^{-1} x - a \tan^{-1} ax \right]_0^{\infty} \quad \text{--- (3)}$$

$$= \frac{1}{1-a^2} \left(\frac{\pi}{2} - a \frac{\pi}{2} \right) = \frac{\pi}{2} \frac{1}{1-a} \quad \text{--- (4)}$$

Integrate w.r.t. a .

$$I(a) = \frac{\pi}{2} \log(1+a) + C$$

$I(0) = 0 = 0 + C \therefore C = 0$. Hence $I(a) = \frac{\pi}{2} \log(1+a)$

Q.3 a. Change the order of integration and then evaluate $\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx dy$

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b. Find the volume bounded by the xy -plane, the cylinder $x^2 + y^2 = 1$ and the plane $x+y+z = 3$.

Answer:

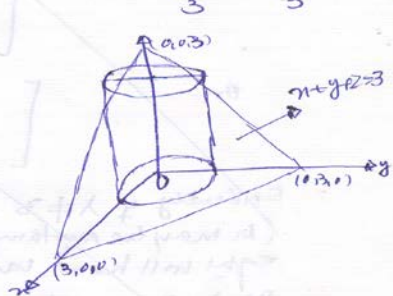
(b) Req'd. Volume = $\int_{-1}^{+1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (3-x-y) dx dy$ --- (1)

$$= \int_{-1}^{+1} \left[3y - xy - \frac{y^2}{2} \right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{+1} \left[6\sqrt{1-y^2} - 2y\sqrt{1-y^2} \right] dy$$

$$= \int_{-\pi/2}^{\pi/2} \left[6 \cos \theta - 2 \sin \theta \cos \theta \right] \cos \theta d\theta$$

putting $y = \sin \theta \therefore dy = \cos \theta d\theta$

$$= 12 \cdot \frac{1}{2} \frac{\pi}{2} + \left| 2 \frac{\cos^3 \theta}{3} \right|_{-\pi/2}^{\pi/2} = 3\pi$$


Q.4

a. Find the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Answer:

(b) Characteristic eqn of the given matrix is

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \text{i.e. } \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

eigen values of the given matrix are roots of characteristic eqn.

These are 2, 2, 8

Eigen vector $[x_1, x_2, x_3]$ corresponding to eigen value $\lambda = 2$ is given by

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{i.e. } \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. Sol. of $2x_1 - x_2 + x_3 = 0$ (remaining eqns are dependent on this)

Choosing $x_3 = 0$, $x_1 = 1$, $x_2 = 2$ i.e. eigen vector is $(1, 2, 0)$

Choosing $x_3 = 0$, $x_1 = 1$, $x_2 = -2$ i.e. eigen vector is $(1, 0, -2)$

These are two independent eigen vectors corresponding to same eigen value $\lambda = 2$.

Similarly eigen vector corresponding to eigen value $\lambda = 8$ is given by

$$[A - 8I]X = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \text{or} \quad \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & 3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 + x_2 - x_3 = 0$
 $x_2 + x_3 = 0$
 $x_2 = 1$
 $x_3 = -1$
 $x_1 = 2$

(b) The equations can be rewritten in matrix form

b. Determine the rank of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

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- Q.5 a. Use Regula-Falsi method to compute real root of $xe^x = 2$ correct to three decimal places.

Answer:

(5) Rewrite the eqn as $f(x) = xe^x - 2 = 0$
 $\therefore f(0) = -2$
 $f(1) = e - 1 = 0.718281828$
 $f(0)$ and $f(1)$ differ by sign. Hence root lies between 0 and 1.
 $\therefore$ By Regula-Falsi method, 1st approx. root is given by

$$x_1 = a - \frac{b-a}{f(b)-f(a)} \cdot f(a) = 0 - \frac{1-0}{0.718281828-(-2)} \cdot (-2) = \frac{-1}{2.718281828} = -0.367879441$$

 $f(x_1) = f(-0.367879441) = -0.367879441 \cdot e^{-0.367879441} - 2 = -1.464423228$
 $\therefore$ Root lies between -0.367879441 and 1.0
 By Regula-Falsi method second approx root is given by

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = \frac{-0.367879441 \times 0.718281828 - 1 \times (-1.464423228)}{0.718281828 - (-1.464423228)} = \frac{-0.264168421 + 1.464423228}{2.182505056} = \frac{1.200254807}{2.182505056} = 0.549802137$$

 $f(x_2) = -0.0562935$
 $\therefore$ Root lies between 0.549802137 and 1.0. By Regula-Falsi method 3rd approx root is

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} = \frac{0.549802137 \times 0.718281828 - 1 \times (-0.0562935)}{0.718281828 - (-0.0562935)} = \frac{0.394781111 + 0.0562935}{0.774575323} = \frac{0.451074611}{0.774575323} = 0.582451567$$

 $f(x_3) = -0.00617137$
 $\therefore$ Root lies between 0.582451567 and 1.0. By Regula-Falsi method

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = \frac{0.582451567 \times 0.718281828 - 1 \times (-0.00617137)}{0.718281828 - (-0.00617137)} = \frac{0.418481111 + 0.00617137}{0.724453201} = \frac{0.424652481}{0.724453201} = 0.586276424$$

 $f(x_4) = -0.00066888$
 Hence correct answer upto 3 decimal places is 0.586

- b. Find by Runge-Kutta method of order four, an approximate value of y at

$x = 0.2$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$.

Answer:

$\therefore \frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}, \quad x_0 = 0, y_0 = 1, h = 0.2$
 $\therefore K_1 = h \cdot f(x_0, y_0) = 0.2 \cdot \frac{1-0}{1+0} = 0.2$
 $K_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.2 \times \frac{(1+\frac{0.2}{2}) - (0+\frac{0.2}{2})}{(1+\frac{0.2}{2}) + (1+\frac{0.2}{2})} = 0.2 \times \frac{1}{1.2} = 0.166666$
 $K_3 = h \cdot f(x_0 + h, y_0 + K_2) = 0.2 \times \frac{(1+\frac{0.4}{2}) - (0+\frac{0.4}{2})}{(1+\frac{0.4}{2}) + (1+\frac{0.4}{2})} = 0.2 \times \frac{0.98333}{1.18333} = 0.166197$
 $K_4 = h \cdot f(x_0 + h, y_0 + K_3) = 0.2 \times \frac{1.166197 - 0.2}{1.166197 + 0.2} = 0.2 \times \frac{0.966197}{1.366197} = 0.141443$
 $K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6} = \frac{0.2 + 3(0.166197) + 0.141443}{6} = \frac{1.007769}{6} = 0.1678615$
 $\therefore y(0.2) = y_0 + K = 1.1678615$

Q.6a. Solve the differential equation $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$

Answer:

Eqn. can be rewritten as
 Separating variables $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$
 $\frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1}$ or $\frac{e^y dy}{2 - e^y} = \frac{dx}{x+1}$
 Integrating, $-\log(2 - e^y) - \log(x+1) = \text{const}$
 or $(2 - e^y)(x+1) = \text{const}$ Ans.

b. Find the orthogonal trajectories of family of curves $ay^2 = x^3$

Answer:

Differentiating $ay^2 = x^3$
 $2ay \frac{dy}{dx} = 3x^2$
 $\therefore$ ~~Eliminate parameter 'a'~~ Eliminate parameter 'a'
 $\frac{x^3}{y^2} \cdot 2y \frac{dy}{dx} = 3x^2$ — (1)
 $\therefore$ Diff. Eqn. of orthogonal trajectory is (replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$)
 $\frac{x^3}{y} \left(-\frac{dx}{dy}\right) = 3x^2$ — (2)
 or $2x dx + 3y dy = 0$
 Integrating $2x^2 + 3y^2 = \text{const}$ is reqd orth. traj.

Q.7 a. Solve the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + x + \sin x$

Q.7 (a) A.E. of the given Diff. Eqn. is
 $m^2 - 6m + 25 = 0 \quad \therefore m = 3 \pm 4i$
 $\therefore$ C.F. = $e^{3x} (C_1 \cos 4x + C_2 \sin 4x)$
 Particular Integral is given by

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 6D + 25} (e^{2x} + x + \sin x) \\ &= \frac{1}{D^2 - 6D + 25} \cdot e^{2x} + \frac{1}{25} \left(1 - \frac{6D}{25} + \frac{D^2}{25}\right)^{-1} x + \frac{1}{D^2 - 6D + 25} \sin x \\ &= \frac{1}{17} e^{2x} + \frac{1}{25} \left(1 + \frac{6D}{25} + \dots\right) x + \frac{1}{4 - 6D + 25} \sin x \\ &= \frac{1}{17} e^{2x} + \frac{1}{25} \left(x + \frac{6}{25} \cdot 1\right) - \frac{1}{6} \frac{1}{D - 4} \sin x \\ &= \frac{1}{17} e^{2x} + \frac{1}{25} \left(x + \frac{6}{25}\right) - \frac{1}{6} \frac{1}{D - 4} \sin x \\ &= \frac{1}{17} e^{2x} + \frac{1}{25} \left(x + \frac{6}{25}\right) + \frac{1}{102} (\cos x + 4 \sin x) \end{aligned}$$

Hence Complete Sol. is

$$y = e^{3x} (C_1 \cos 4x + C_2 \sin 4x) + \frac{1}{17} e^{2x} + \frac{1}{25} \left(x + \frac{6}{25}\right) + \frac{1}{102} (\cos x + 4 \sin x)$$

b. Solve the equation $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} = 12 \frac{\log x}{x^2}$

Answer:

(b) Given eqn can be rewritten as

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 12 \log x \quad \text{--- (1)}$$

Put $x = e^t$ i.e. $t = \log x$, so that $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ (1) becomes

$$D(D-1)y + Dy = 12t \quad \text{--- (2)}$$

or $Dy = 12t$ --- (2)
Its complementary function = $C_1 + C_2 t$

and P.I. = $\frac{1}{D^2} 12t = 12 \cdot \frac{t^3}{2 \times 3} = 2t^3$

Hence complete soln

$$y = C_1 + C_2 t + 2t^3 = C_1 + C_2 \log x + 2(\log x)^3$$

Q.8 a. Obtain the series solution of $(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$

Answer:

Q.8 (b) Here $x=0$ is an ordinary point because coefft of $y'' \neq 0$ at $x=0$

$\therefore$ Let

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad \text{--- (1)}$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \dots$$

Putting these values in the given equation,

$$(1-x^2)[2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \dots]$$

$$- 2x(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots) + 2(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots) = 0$$

equating coeffts of different powers of x on both sides, we get

$$2a_2 + 2a_0 = 0 \quad \text{ie } a_2 = -a_0 \quad (\text{const terms})$$

$$3 \cdot 2a_3 - 2a_1 + 2a_1 = 0 \quad \text{ie } a_3 = 0 \quad (\text{coeffts of } x)$$

$$4 \cdot 3a_4 - 2a_2 - 4a_2 + 2a_2 = 0 \quad \text{ie } a_4 = \frac{a_2}{3} = -\frac{a_0}{3} \quad (\text{coeffts of } x^2)$$

$$5 \cdot 4a_5 - 3 \cdot 2a_3 - 2 \cdot 3a_3 + 2a_3 = 0 \quad \text{ie } a_5 = \frac{10}{20} a_3 = 0$$

Putting values in (1), we get

$$y = a_0 + a_1 x - a_0 x^2 + 0 \cdot x^3 - \frac{a_0}{3} x^4 + 0 \cdot x^5 + \dots$$

$$= a_0(1 - x^2 - \frac{1}{3}x^4 - \dots) + a_1 x \quad \text{as reqd. sol.}$$

b. Show that $\beta(m, n) = \frac{\overline{m} \overline{n}}{\overline{m+n}}$

Answer: Page Number 854 of Text Book 1

Q.9 a. State and prove Rodrigue's formula.

Answer:

Q.9) Legendre's polynomial $P_n(x)$ is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \quad \text{--- (1)}$$

To prove (1), let

$$y = (x^2-1)^n$$

$$\therefore \frac{dy}{dx} = y_1 = n(x^2-1)^{n-1} \cdot 2x = \frac{2nx(x^2-1)^{n-1}}{(x^2-1)} = \frac{2nxy}{(x^2-1)}$$

$$\therefore (x^2-1)y_1 - 2nxy = 0 \quad \text{--- (2)}$$

Differentiate (2) ~~n~~ times using Leibnitz's theorem

$$(x^2-1)y_{n+2} + (n+1)y_1 \cdot 2x + \frac{(n+1)n}{2} y_n \cdot 2 - 2n[x y_{n+1} + (n+1)y_{n+1}] = 0$$

$$(x^2-1)y_{n+2} + (2n+2-2n)xy_{n+1} - n(n+1)y_n = 0$$

$$\text{or } (1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0 \quad \text{--- (3)}$$

If we put $V = y_n = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} (x^2-1)^n$, (3) becomes,

$$(1-x^2)\frac{d^2 V}{dx^2} - 2x\frac{dV}{dx} + n(n+1)V = 0$$

which is Legendre's equation having $V = y_n = \frac{d^n y}{dx^n}$ as its solution or its constant multiple as solution. But its finite series solution is $P_n(x)$, a Legendre's polynomial.

$$\therefore P_n(x) = CV = Cy_n = C \frac{d^n}{dx^n} (x^2-1)^n \quad \text{--- (4)}$$

$$= C \frac{d^n}{dx^n} [(x-1)^n (x+1)^n]$$

To evaluate C , put $x=1$ and remember $P_n(1)=1$,

$$1 = C \left[\frac{1}{2^n n!} (n+1)^2 + \text{terms containing powers of } (n-1) \text{ which become zero at } x=1 \right]$$

$$\therefore C = \frac{1}{2^n n!}$$

$$\text{Hence } P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

b. Prove that $J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$

Answer:

(b) We know that

$$\frac{d}{dx} (x^n J_n(x)) = x^n J_{n-1}(x) \quad \text{--- (1)}$$

$$\text{and } \frac{d}{dx} (x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x) \quad \text{--- (2)}$$

Denoting $\frac{d}{dx} J_n(x) = J'_n(x)$, (1) + (2) can be rewritten as

$$x^n J'_n(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x) \quad \text{--- (3)}$$

$$x^{-n} J'_n(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x) \quad \text{--- (4)}$$

or

$$J'_n(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad \text{--- (5)}$$

$$J'_n(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x) \quad \text{--- (6)}$$

Subtracting (6) from (5),

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

Ans.

TEXT BOOKS

1. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi.
2. Text book of Engineering Mathematics, N.P. Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd.