Stage 2 Higher Revision Sheet

This document attempts to sum up the contents of the Higher Tier Stage 2 Module.

There are two exams, each 25 minutes long. One allows use of a calculator and the other doesn't. Together they represent 15% of the GCSE.

Before you go into the exam make sure you are fully equipped with two pens, two pencils, a calculator, a ruler, a protractor and a pair of compasses. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted... always an area prone to error.

I am always available on `jonathan.m.stone@gmail.com` to answer any questions you may have. Please do not hesitate.

JMS

Index Notation & Standard Form

- Know the basic index rules...

  1. $x^0 = 1$ (provided $x \neq 0$),
  2. $x^{-n} = \dfrac{1}{x^n}$ (moving something from bottom to top (or vice versa) of a fraction $\Rightarrow$ change sign of power),
  3. $x^{1/n} = \sqrt[n]{x}$,
  4. $x^{m/n} = (x^m)^{1/n} = (x^{1/n})^m$ (which is the same as $x^{m/n} = \sqrt[n]{x^m} = [\sqrt[n]{x}]^m$),
  5. $x^m \times x^n = x^{m+n}$ (note that this rule is for multiplying only, *not* adding!),
  6. $x^m \div x^n = \dfrac{x^m}{x^n} = x^{m-n}$,
  7. $(x^m)^n = x^{mn}$,
  8. $(ab)^n = a^n b^n$ (*not* in book).

- ...and be able to apply them. For example simplify the following:

  1. $8^{-2/3} = \dfrac{1}{8^{2/3}} = \dfrac{1}{2^2} = \dfrac{1}{4}$.
  2. $\dfrac{2^{-4} \times 2^7}{2^2} = \dfrac{2^3}{2^2} = 2^1 = 2$.
  3. $\dfrac{(2xy^2)^3 \times 3x^3y}{6x^2y^{10}} = \dfrac{8x^3y^6 \times 3x^3y}{6x^2y^{10}} = \dfrac{24x^6y^7}{6x^2y^{10}} = \dfrac{4x^4}{y^3} = 4x^4y^{-3}$.

- Standard form is a compact way of writing large and small numbers. It is always

  $$[\text{number between 1 and 10}] \times 10^{\text{some power}}.$$

  For example $34500000 = 3.45 \times 10^7$ and $0.0000406 = 4.06 \times 10^{-5}$. On your calculator you use the $\boxed{\text{EXP}}$ button; to use 3.45×10^6 you would enter $3.45\boxed{\text{EXP}}6$.

- For example $\frac{(8\times10^7)\times(4\times10^4)}{(2\times10^2)} = \frac{32\times10^{11}}{(2\times10^2)} = 16 \times 10^9 = 1.6 \times 10^{10}$. Note the last step to make sure the answer is in standard form.

- To estimate a calculation round all the components to one significant figure and carry out the calculation. This is true for calculations in standard form. For example estimate $(3.45 \times 10^2) \times (4.67 \times 10^6) \approx (3 \times 10^2) \times (5 \times 10^6) = 15 \times 10^8 = 1.5 \times 10^9$.

Proportion, Estimating & Accuracy

- Two quantities are said to be in *direct proportion* (or just proportion) if, when one quantity is multiplied or divided by a quantity, the other one is too. We can use this to solve problems in a table. For example if s and t are in direct proportion and $s = 5$ when $t = 7$, what is s when $t = 9$?

s	5	
t	7	9

  We can see that to get from 7 to 9 we multiply by $\frac{9}{7}$. Therefore s would be $s = 5 \times (\frac{9}{7})$.

- If you graph two proportional quantities you obtain a straight line graph through the origin. The equation connecting quantities will therefore be of the form $y = kx$.

  So, in the above example, $s = kt$. Therefore $5 = 7k$, so $k = \frac{5}{7}$. The relationship is therefore completely written as $s = \frac{5}{7}t$. This can then be use to solve the second part of the original s/t problem $\Rightarrow \; s = \frac{5}{7} \times 9$.

- Two quantities are said to be in *inverse proportion* if, when one quantity is multiplied by a quantity, the other one is divided by it (or, equivalently, multiplied by the reciprocal). A table can be used again; u and v are in inverse proportion; $u = 3$ when $v = 7$; what is v when $u = 5$?

u	3	5
v	7	

  So u has been multiplied by $\frac{5}{3}$, so v will be divided by $\frac{5}{3} \Rightarrow \; v = 7 \div \frac{5}{3} = 7 \times \frac{3}{5}$.

- The curve $y = 1.1^x$ is said to be an exponentially growing function. The curve $y = 0.9^x$ is exponentially decaying. It all depends whether the number to which the x-power is applied is greater or less than one. Covered more fully later on.

- Surds are quantities like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{\text{anything not perfect square}}$. It is often more elegant to leave answers in 'surd form' or in terms of π. This is because of the ugly decimal expansions these numbers tend to have. For example the equilateral triangle of side length 2 has area $\sqrt{5}$; (work this out for your self).

- Two theorems you need to be able to use well are

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \text{and} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

  So $\sqrt{20}$ can be written $\sqrt{4}\sqrt{5} = 2\sqrt{5}$. You treat π just like a surd and we can use it as follows; what is the surface area of a cylinder radius 2 and length 10? SA $= 2\pi r^2 + 2\pi rh = 48\pi$.

- In any measurement there is bound to be some uncertainty. When a number is given to a stated degree of accuracy there is an implied upper bound and an implied lower bound. This usually takes the form of half a unit above and below the given value. For example if something is 128cm to the nearest cm then the upper bound is 128.5cm and the lower bound is 127.5cm. You sometimes need to think quite hard about what the error bounds are. If a length is 12300 to 3sf, then the upper is 12350 and the lower is 12250.

- When using values in calculations you need to have your wits about you! You need to *think* what will maximise my variable and what will minimise it. The classic example if the rectangular lawn with length 12m (nearest m) and area 72m^2 (nearest m^2). The upper bound for the area will be $\frac{72.5}{11.5}$ and the lower bound will be $\frac{71.5}{12.5}$. Think hard about this as they sometimes ask hard questions on it.

- Absolute error is difference between value measured and nominal value. The percentage error is the absolute error as a percentage of the nominal value.

Algebra: Graphs

- Understand fully the concept of $y = mx + c$ for straight lines[1].

  The value of c is just where the line 'cuts' the y-axis.

  The value of m is the gradient. For every *one* you step across to the *right* you go *up* the gradient if positive (or *down* if negative).

  Lines with the same gradient are parallel.

- For example; draw the line $4y - x = 2x + 2y + 8$. Firstly this line is clearly *not* in the form $y = mx + c$. We therefore rearrange to find $y = \frac{3}{2}x + 4$. Draw the point $(0, 4)$. From this point we could step one to the right and $\frac{3}{2}$ up, but it is better to step 2 right and 3 up.

- To solve quadratic equations graphically it invariably requires you to draw a quadratic curve and a horizontal line and find (the x coordinates) where they cross. For example to solve '(quadratic expression) $= -2$' you draw '$y =$ (quadratic expression)' and '$y = -2$' and find the x coordinates where they cross.[2]

- For example to solve $x^2+2x-1 = 2x+3$ you would draw $y = 2x+3$ and $y = x^2+2x-1$ and find the x coordinates where they cross. These values would be the solution to $x^2 - 4 = 0$ which is the equation found from rearranging the original.

- To plot any curve you merely need to draw a table of values, and this table would usually be given to you in the exam to complete.

- You must know the general shapes of various types of curves given on P26/27. In harder questions they may give you the general form of the graph and tell you some points that it passes through. You then need to work out the constants.

  For example the curve has equation $y = \frac{a}{x} + b$. It passes through $(1, 5)$ and $(-1, 1)$. What are a and b? Putting in the points we are given we discover $5 = a + b$ and $1 = b - a$. Solving these simultaneous equations we discover $a = 2$ and $b = 3$ so $y = \frac{2}{x} + 3$.

- You must know the graphs of $y = \sin x$ and $y = \cos x$. It may also be useful to know the graphs of things like $y = 2\sin x$, $y = \sin(2x)$, $y = \sin x + 2$. Can I suggest you sit down at a school computer and plot the above graphs and play around until you have a feel for how the different constants distort the graph.

- To solve something like $\sin x = -\frac{1}{2}$ in the range $0°$ to $360°$ you firstly do $\sin^{-1}(-\frac{1}{2})$ on you calculator and you obtain $-30°$. Note this is not in the required range. Next you sketch the sin curve and $y = -\frac{1}{2}$ and notice it cuts twice in the range. Using symmetry on the curve you discover that $x = 330°$ and $x = 210°$.

[1]Don't forget '$x =$ constant' is a vertical line, and '$y =$ constant' is a horizontal line.

[2]This is true in general; to graphically solve $f(x) = g(x)$ you plot $y = f(x)$ and $y = g(x)$ and find the x-coordinates where they cross. The notation $f(x)$ and $g(x)$ just means that $f(x)$ and (x) can be any 'function' of x; for example $y = \sin x$, $y = x^2 + 2x$, $y = 7x + 2$ are all $f(x)$'s.

Working with Algebra

- To multiply bracketed expressions you must ensure that *everything* in the first bracket multiplies *everything* in the second bracket. So the rule becomes $(a+b)(c+d) = ac + ad + bc + bd$. Some examples:

$$(2x+3)(3x-2) = 6x^2 - 4x + 9x - 6 = \underline{6x^2 + 5x - 6}.$$
$$(3x-2)^2 = (3x-2)(3x-2) = 9x^2 - 6x - 6x + 4 = \underline{9x^2 - 12x + 4}.$$

- One thing that is very important is the difference of two squares formula: $(x-y)(x+y) = x^2 - y^2$. You must *always* be on the look out for it in some form or other. So if you see $4x^2 - 9$ you should read that as $[2x]^2 - [3]^2$, which you can then factorise instantly to $(2x+3)(2x-3)$.

- Factorising general quadratics is covered in a separate handout and is *very important*. However I will cover the simple types here.

  1. No constant: Easy! Just pull out that x and anything else that comes. $3x^2 - 9x = 3x(x-3)$. Done.
  2. No x term: Easy! It will be a difference of two squares. $x^2 - 49 = (x+7)(x-7)$. Another; $x^2 - 3 = (x+\sqrt{3})(x-\sqrt{3})$.

- From the index rules we have in the first section we know that we can cancel terms on the top and bottom of fractions. For example $x^7/x^3 = x^4$ and $x^2/x^9 = 1/x^7$. We can use this idea to help cancel more complex terms; for example

$$\frac{3(x+4)^5}{6(x+4)^3} = \frac{(x+4)^2}{2}.$$

- Tying this all together we can tackle tough problems like:

$$\begin{aligned}\frac{x^2+10x+25}{5x-10} \div \frac{x^2+4x-5}{2x^2-4x} &= \frac{x^2+10x+25}{5x-10} \times \frac{2x^2-4x}{x^2+4x-5} \\ &= \frac{(x+5)(x+5)}{5(x-2)} \times \frac{2x(x-2)}{(x+5)(x-1)} \\ &= \frac{2x(x+5)}{5(x-1)}.\end{aligned}$$

- Given two formulae you can take one and (once correctly rearranged) place it into the second to create a new valid formula. For example in a circle $A = \pi r^2$ and $C = 2\pi r$. To find a formula giving A in terms of C we re-write the second to give $r = C/(2\pi)$. We then use this to replace the r in the first; $A = \pi r^2 = \pi(C/(2\pi))^2 = C^2/(4\pi)$.

- Changing the subject of a formula involves getting the letter you want on its own. If it appears only once in the equation then you just need to get rid of everything else using the standard rules of equations. For example make m the subject of $R = \sqrt{\frac{3M+m}{M}}$;

$$R = \sqrt{\frac{3M+m}{M}} \qquad \Rightarrow \qquad R^2M = 3M + m \qquad \Rightarrow \qquad m = R^2M - 3M.$$

- If the letter you're trying to isolate appears twice then you will probably need to use factorisation after getting all the terms with that letter to one side. For example make u the subject of $uf + vf = uv$;

$$uf + vf = uv \qquad \Rightarrow \qquad vf = uv - uf = u(v-f) \qquad \Rightarrow \qquad u = \frac{vf}{v-f}.$$

Equations & Inequalities

- There exists another handout on simultaneous equations that you might like to see on my site. However they usually appear in the form

$$\text{“Solve } \begin{array}{c} 2x+y=-1 \\ 4x-3y=8 \end{array}\text{”.}$$

  You would then make either the number of x's or y's the same and then add or subtract to *eliminate* one of the variables. In this case you should find $x = 0.5$ and $y = -2$: Do it!

- You must know that the solution represents the coordinate where the lines cross in the xy-grid.

- The questions are also sometimes asked where the lines are given in the form $y = mx + c$. These are easy; you just put the two y expressions equal and solve. For example solve $y = 2x + 3$ with $y = 4 - x$.

  Putting equal we find $2x + 3 = 4 - x$, so $x = \frac{1}{3}$. To find y just place the x value back into either of the original equations; $y = 4 - \frac{1}{3} = 3\frac{2}{3}$. The lines cross at $(\frac{1}{3}, 3\frac{2}{3})$.

- Inequalities should be treated just like equations, except when you multiply or divide by a negative number, in which case you flip the inequality. For example

$$\begin{aligned} 5(x+2) &< 7x-4 \\ 5x+10 &< 7x-4 \\ -2x &< -14 \\ x &> 7. \qquad \text{Notice the sign flip because we have divided by } -2. \end{aligned}$$

- With 'triple' inequalities you just do the same thing to all three terms to isolate the x in the middle. For example

$$\begin{aligned} -3 < 2x - 5 &\leqslant 7 \\ 2 < 2x &\leqslant 12 \\ 1 < x &\leqslant 6. \end{aligned}$$

  If the above question was in integers only the solutions would be $\{2, 3, 4, 5, 6\}$.

- Regions can be defined by inequalities. For example the region defined by $x > 1$, $y > 0$ and $y < -2x + 4$ is the triangular region enclosed within the three lines $x = 1$, $y = 0$ and $y = -2x + 4$. If it is a '$>$' or '$<$' inequality then you draw a *dashed* line because the line itself down not satisfy the inequality. If it is '$\geqslant$' or '$\leqslant$' you draw a *full* line.

  To find which side of a line is satisfied by the inequality you draw the line given by the *equality* and then you pick a coordinate away from the line (the origin $(0, 0)$ is best if it doesn't lie on the line) and see if it satisfies the inequality. If it does then that side is allowed and if not then that is the forbidden region.

- **All the important stuff on quadratics is covered in a separate handout.**

Area & Volume

- Area of a triangle can be expressed as $A = \frac{1}{2}ab\sin\theta$ where the angle θ is between the two lengths[3].

[3] This can be derived by considering half of a parallelogram with sides a and b and angle θ between them.

Sometimes you will get an ambiguous angle problem. These problems will reduce to something like $\sin\theta = 0.4$, so $\theta = \sin^{-1}(0.4) = 23.58°$, but $\theta = 180° - 23.58° = 156.42°$ is *also* a valid solution[4]. Example given bottom P66.

- Know the long list of boring formulae on P75. The only hard bit with 3D shapes is whether you are using the right length for the right letter. For example with cones, SA $= \pi rl + \pi r^2$; the l refers to the slanted length, *not* the perpendicular height. You must be prepared to do some work to work out these lengths, for example by Pythagoras.

- The one thing I would add is make sure you know what the different terms of the formulae refer to. For example for cones $SA = \pi rl + \pi r^2$. Know that the πr^2 refers to the circular bottom and the πrl refers to the curved face.

Congruence, Constructions & Loci

- Two shapes are said to be congruent if they are the same size and shape. A image under reflection of one shape is congruent to the original, despite the fact that it is 'flipped'.

- Usually three things are needed to show that two triangles are congruent. (The only times this is not the case is when only the three angles are known, or in the ambiguous case of $\triangle ABC$ with, say, $AB = 10$, $BC = 8$, $B\hat{A}C = 30°$.)

- The times when the triangles are congruent are categorised[5] into four cases:

  1. SSS = All three sides the same.
  2. SAS = Two sides and the included angle the same.
  3. ASA = Two angles and the included side the same.
  4. RHS = Two right angled triangles, hypotenuse the same and one pair of corresponding sides the same.

  In the exam you must make sure you have *three explicit statements* and then link them up with one of the above reasons.

- You must know the constructions for (i) bisecting an angle, (ii) creating a perpendicular bisector between two points, and (iii) creating a perpendicular to a line, through a given point. You *must* leave your construction lines in place since they look for these when marking.

- A locus is a set of points that obeys a given rule. Loci are closely linked to constructions (above). You need to know the loci highlighted on P81 of the Mid-sized book and be able to complete harder problems like the example on the bottom of that page.

Transformations

- The four basic transformations (translation, rotation, reflection & enlargement) should be thoroughly understood from Stage 1. Refer to the Stage 1 Revision Sheet or P85 if you are at all unsure. You must be able to take an *object* and its *image* and describe (fully[6]) the transformation. Also, given an object you must be able to perform any single transformation.

[4] It is worth dwelling on this subtle point… it is poorly understood by most students.

[5] This, incidentally, is the most stupid piece of the entire GCSE course (of any subject).

[6] It is quite staggering just how may students fail to *name* the transformation that has occurred; translation, reflection, etc. You must avoid sloppy language like 'sliding' or 'movement'.

- Combined transformations are when you carry out one transformation, and then carry out another on the image of the first. These combined transformations can be expressed as a single transformation; this is the format of most exam questions.

  For example take $\triangle ABC$ given by $A(1,1)$, $B(2,1)$, $C(1,3)$. If you reflect it in the line $x=-1$ and then reflect it in the line $x=1$ is actually just the translation $\binom{4}{0}$. Question; is that true of any shape you start with?

Vectors

- This section is very geometric so I will be referring to the textbook a lot to see the pretty pictures. Vectors are usually expressed as a column vector. For example the column vector $\binom{-4}{3}$ represents the vector going 4 *left* and 3 *up*.
- Addition and subtraction of vectors is as you would expect. The same goes for multiplication by a vector. For example
$$\begin{pmatrix}-4\\3\end{pmatrix}-2\begin{pmatrix}3\\0\end{pmatrix}+3\begin{pmatrix}1\\-6\end{pmatrix}=\begin{pmatrix}-4\\3\end{pmatrix}-\begin{pmatrix}6\\0\end{pmatrix}+\begin{pmatrix}3\\-18\end{pmatrix}=\begin{pmatrix}-7\\-15\end{pmatrix}=-\begin{pmatrix}7\\15\end{pmatrix}.$$
- You must be comfortable with the geometric interpretation of addition and subtraction given on P91. The parallelogram rule for addition. Subtraction joins the heads of the two vectors in a given direction.
- A *linear combination* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $p\mathbf{a}+q\mathbf{b}$ for any numbers p and q.

  This can be used in questions as follows: Find the scalars p and q such that
$$p\begin{pmatrix}2\\1\end{pmatrix}+q\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}4\\7\end{pmatrix}.$$
  These are just *simultaneous equations in disguise*. Considering the top line we find $2p-q=4$ and the bottom line yields $p+2q=7$. You should then find that $p=3$ and $q=2$.

  A slightly harder example that you should view is Example 4 on P94.
- Whereas the vectors we have considered above are not tied to any point (they can exist anywhere in the xy-plane), *position vectors* are written relative to the origin. So the position vector $\binom{3}{4}$ is the vector joining $(0,0)$ to $(3,4)$.
- Two vectors are parallel if one is a scalar multiple of the other. For example $\binom{4}{1}$ is parallel to $\binom{6}{1.5}$ because the second if 1.5 times the first.
- Given two position vectors $\mathbf{a}$ and $\mathbf{b}$ to points A and B (like the top of P97) we write $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$. We can then see that $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ (and $\overrightarrow{BA}=\mathbf{a}-\mathbf{b}$).
- It is very difficult to explain this with just writing(!) However I will try to do question 9 on P99. Draw a sketch for yourself!
$$\begin{aligned}\overrightarrow{AM}&=\overrightarrow{AB}+\tfrac{1}{2}\overrightarrow{BC}=\underline{\mathbf{b}+\tfrac{1}{2}\mathbf{c}}.\\ \overrightarrow{DC}&=\overrightarrow{DB}+\overrightarrow{BC}\\ &=\tfrac{4}{3}\overrightarrow{NB}+\overrightarrow{BC}\\ &=\tfrac{4}{3}(\overrightarrow{NC}+\overrightarrow{CB})+\overrightarrow{BC}\\ &=\tfrac{4}{3}(\tfrac{1}{2}\overrightarrow{AC}+\overrightarrow{CB})+\overrightarrow{BC}\\ &=\tfrac{4}{3}\left(\tfrac{1}{2}(\mathbf{b}+\mathbf{c})-\mathbf{c}\right)+\mathbf{c}\\ &=\underline{\tfrac{2}{3}\mathbf{b}+\tfrac{1}{3}\mathbf{c}}.\end{aligned}$$

$\overrightarrow{AM}$ is parallel to $\overrightarrow{DC}$ because $\overrightarrow{DC} = \frac{2}{3}\overrightarrow{AM}$ and therefore in the ratio 3 : 2. I appreciate that that will probably mean nothing to you (because I have not explained anything), but I did my best.

Handling Data: Probability & Frequency Density

- The beginning of this chapter is mostly obvious. If you roll a dice 300 times you would *expect* to get 50 sixes ($P(6) \times 300 = \frac{1}{6} \times 300$). This is not certain to happen, but it would be your best estimate.

- Two events are said to be *mutually exclusive* if they both can't happen at the same time. For example being a genius and being thick are mutually exclusive. Rolling a 5 and rolling a 6 when rolling a dice one time are mutually exclusive. Taking A Level Maths and A Level Physics are *not* mutually exclusive because you could take both. If two events, A and B, are mutually exclusive then $P(A \text{ or } B) = P(A) + P(B)$.

- Two events are *independent* if the outcome of one does not affect the other. For example if you have bag with 3 red and 5 green balls in it and you take a ball out and replace it and then take another out, these are independent events. However if the first ball is not replaced then these are *not* independent, because whether you picked a red or a green first affects the probabilities of the second ball being red or green. If two events are independent then $P(A \text{ and } B) = P(A) \times P(B)$.

- If you have one event followed by another (by rolling two dice, or picking two balls from a bag, or going to work twice, etc.) you can represent these on a tree diagram. On each individual branch the probabilities sum to one. You *multiply along the branches* and *sum at the end*.

- Histograms are usually used to represent *continuous data*. The *area* of the bars represents the frequencies, not the heights. For data of constant class width it is therefore good enough to plot the frequencies as the heights.

- However in the exam you can expect unequal class width, so you need the concept of *frequency density* which you plot as the heights. It is defined

$$\text{Frequency density} = \frac{\text{Frequency}}{\text{Class Width}}.$$

  However this has a nasty habit of generating horrible numbers so it can sometimes be scaled. Therefore in general

$$\text{Frequency density} = \alpha \times \frac{\text{Frequency}}{\text{Class Width}}$$

  where α is a constant to be determined.

- In the questions what you need to do is find a *link* between the graph given and the table given. You might find that for the class $\boxed{30 \leqslant x < 35 \mid 28}$ The height of the bar is 56. Putting the numbers into the above formula we discover $56 = \alpha \times \frac{28}{5}$, so $\alpha = 10$. We can then use this constant to help complete the table or the graph.