MATHEMATICS

Duration: Three hours   Maxim... 150

➤ **Read the following instructions carefully**

1. This question paper contains TWO SECTIONS: 'A' and 'B'

2. Section A consists of two questions of the multiple choice type. Question 1 consists of Twenty Five sub-questions of One mark each and Questions 2 consists of Twenty Five sub-questions of Two marks each.

3. Answer Section A only on the special machine-gradable Objective Response Sheet (ORS), Questions of Section A will not be graded if answered anywhere else.

4. Answer problems of section B in the answer-book.

5. Write your name, registration number and the name of the centre at the specified locations on the right half of the ORS for section A.

6. Using a soft HB pencil darken the appropriate bubble under each digit of your registration number.

7. The objective response sheet will be collected back after 120 minutes have expired from the start of the examination. In case you finish section A before the expiry of 120 minutes, you may start answering Section B.

8. Questions of Section A are to be answered by darkening the appropriate bubble (marked A, B, C and D) using a soft HB pencil against the question on the left hand side of the Objective Response Sheet.

9. In case you wish to change an answer, erase the old answer completely using a good soft eraser.

10. There is no negative marking.

11. Section B consists of Twenty questions of Five marks each. Any fifteen out of them have to be answered. If more number of questions are attempted, score off the answers not be evaluated, else only the first fifteen un scored answers will be considered strictly.

12. In all the 5 marks questions, clearly show the steps. The symbols R and C denote the set of all real and complex numbers respectively. Vector quantities are denoted by bold letters.

SECTION A (75 Marks)

1. –MA. This question consists of Twenty Five sub-questions (1.1-1.25) of One...

correct. Answer each sub-qu... darkening the appropriate bubble... Objective Response Sheet (ORS) us... HB pencil. Do not use the ORS for a... rough work. You may like to use the Answer Book for any rough work, if needed.

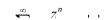1.1 The eigen values of a 3×3 real matrix P are 1, –2, 3. Then

(a) $P^{-1} = \frac{1}{6}\left(5I + 2P - P^2\right)$

(b) $P^{-1} = \frac{1}{6}\left(5I - 2P + P^2\right)$

(c) $P^{-1} = \frac{1}{6}\left(5I - 2P - P^2\right)$

(d) $P^{-1} = \frac{1}{6}\left(5I + 2P + P^2\right)$

1.2 Let $T : C^n \to C^n$ be a linear operator having n distinct eigen values. Then
(a) T is invertible
(b) T is invertible as well as diagonalizable
(c) T is not diagonalizable
(d) T is diagonalizable

1.3 Let U be a 3×3 complex Hermitian matrix which is Unitary. Then the distinct eigen values of U are
(a) $\pm i$
(b) $1 \pm i$
(c) ± 1
(d) $\frac{1}{2}\left(1 \pm i\right)$

1.4 The function sin z is analytic in
(a) $C \cup \{\infty\}$
(b) C express on the negative real axis
(c) $C - \{0\}$
(d) C

$\sum_{}^{\infty} z^n$ ...

(b) Uniformly and absolutely convergent

(c) Absolutely convergent but not uniformly convergent

(d) Convergent by not uniformly convergent

1.6   If $f(z) = z^3$, then it

(a) Has an essential singularity at $z = \infty$

(b) Has a pole of order 3 at $z = \infty$

(c) Has a pole of order 3 at $z = 0$

(d) Is analytic at $z = \infty$

1.7   A uniformly continuous function is

(a) Measurable

(b) Not Measurable

(c) Measurable and simple

(d) Integral and simple

1.8   Which of the following pair of functions is not a linearly independent pair of solutions of $y'' + 9y = 0$?

(a) $\sin 3x, \sin 3x - \cos 3x$

(b) $\sin 3x + \cos 3x, 3\sin x - 4\sin^3 x$

(c) $\sin 3x, \sin 3x \cos 3x$

(d) $\sin 3x + \cos 3x, 4\cos^3 x - 3\cos x$

1.9   Determine the type of the following differential equation

$$\frac{d^2 y}{dx^2} + \sin(x + y) = \sin x$$

(a) Linear, homogeneous

(b) Nonlinear, homogeneous

(c) Linear, non homogeneous

(d) Nonlinear, non homogeneous

1.10 Which of the following is not an integrating factor of $xdy - ydx = 0$?

(a) $\dfrac{1}{x^2}$

(b) $\dfrac{1}{x^2 + y^2}$

(c) $\dfrac{1}{xy}$

(d) $\dfrac{x}{y}$

1.11 Let G be a group of order 49. Then

(a) G is abelian

1.12 The polyno___ ___ $x^5 + 5$ is

(a) Irreducible ___

(b) Irreducible ove___

(c) Irreducible over Q

(d) Not irreducible Q

Where Q denote s the fi___ ___al number.

1.13 Given a nontrivial nor med line___ the non triviality of its dual sp___ assured by

(a) The Hahn-Banach Theorem

(b) The Principle of Uniform bound ness

(c) The Open Mapping Theorem

(d) The closed Graph theorem

1.14 If Δ and ∇ are the forward and the backward difference operators respectively, then $\Delta - \nabla$ is equal to

(a) $-\Delta\nabla$

(b) $\Delta\nabla$

(c) $\Delta + \nabla$

(d) $\dfrac{\Delta}{\nabla}$

1.15 One root of the equation $e^x - 3x^2 = 0$ lies in the interval $(3,4)$. The least number of iterations of the bisection method so that $|\text{error}| \leq 10^{-3}$ are

(a) 10

(b) 8

(c) 6

(d) 4

1.16 If (r, θ, φ) is a harmonic function in a domain D, where (r, θ, φ) are spherical polar co-ordinates, then so is

(a) $\dfrac{1}{r} f(r, \theta, \varphi)$

(b) $\dfrac{1}{r^2} f\left(\dfrac{1}{r}, \theta, \varphi\right)$

(c) $\dfrac{1}{r^2} f\left(\dfrac{1}{r^2}, \theta, \varphi\right)$

(d) $\dfrac{1}{r} f\left(\dfrac{1}{r}, \theta, \varphi\right)$

1.17 The solution of the initial value problem

(a) x

(b) $\dfrac{x^2}{2}$

(c) 2x

(d) 2t

1.18 In the motion of a two particle system, if two particles are connected by a rigid weightless rod of constant length, then the number of degrees of freedom of the system is

(a) 2

(b) 3

(c) 5

(d) 6

1.19 Consider a planet of mass m orbiting around the sun under the inverse square law of attraction $\dfrac{\mu m}{r^2}, \mu > 0$. If the position of the planet at time t is given by the polar co-ordinates (r, θ) then the Lagrangian L of the system us given by

(a) $\dfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \dfrac{\mu m}{r}$

(b) $\dfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \dfrac{\mu m}{r}$

(c) $\dfrac{1}{2}m\left(\dot{r}^2 + \dot{\theta}^2\right) + \dfrac{\mu m}{r}$

(d) $\dfrac{1}{2}m\left(\dot{r}^2 + \dot{\theta}^2\right) - \dfrac{\mu m}{r}$

1.20 The following statement is false

(a) Any product of compact spaces is compact

(b) Any product of Hausdorff spaces is Hausdorff

(c) Any product of connected spaces is connected

(d) Any product of mertizable spaces mertizable

1.21 The random variable X has a t–distribution with v degree of freedom. Then the probability distribution of X^2 is

(a) Chi-square distribution with 1 degree of freedom

(b) Chi-square distribution with v degrees of freedom

(d) F-distribution with (v,1) degrees of freedom

1.22 Let $S_1 = \left\{(x, y) \qquad {}^2 \le 1\right\}$ and $S_2 = \left\{(x, y) \in R^2 : y \le \right.$

(a) S_1 and S_2 both are con

(b) S_1 is a convex set but convex set

(c) S_2 is a convex set but S_1 is not a set

(d) Neither S_1 nor S_2 is a convex set

1.23 Let T be the matrix (occurring in a typical transformation problem) given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then

(a) Rank T = 4 and T is unimodular

(b) Rank T = 4 and T is not unimodular

(c) Rank T = 3 and T is unimodular

(d) Rank T = 3 and T is not unimodular

1.24 Consider the primal problem (LP)

max $4x_1 + 3x_2$

$x_1 + x_2 \le 8$

$2x_1 + x_2 \le 10$

$x_1 \ge 0, x_2 \ge 10$

Together with its dual (LD). Then

(a) (LP) and (LD) both are infeasible

(b) (LP) and (LD) both are feasible

(c) (LP) is feasible but (LD) is infeasible

(d) (LP) is infeasible but (LD) is feasible

1.25 The initial value problem corresponding to the integral equation

$$y(x) = 1 + \int_0^x y(t)\,dt \text{ is}$$

(a) $y' - y = 0, \quad y(0) = 1$

(b) $y' + y = 0, \quad y(0) = 0$

(c) $y' - y = 0, \quad y(0) = 0$

(d) $y' + y = 0, \quad y(0) = 1$

2. –MA. This question consists of Twenty

Answer each sub-question by darkening the appropriate bubble on the Objective Response Sheet (ORS) using a soft HB pencil. Do not use the ORS for any rough work. You may like to use the Answer Book for any rough work, if needed.

2.1 Let A be an n × n complex matrix whose characteristic polynomial is given by
$$f(t) = t^n + c_{n-1} t^{n-1} + \ldots + c_1 t + c_0.$$
Then
(a) $\det(A) = c_{n-1}$

(b) $\det(A) = c_0$

(c) $\det(A) = (-1)^n c_{n-1}$

(d) $\det(A) = (-1)^n c_0$

2.2 Let A be any n × n non-singular complex matrix and let $B = (\overline{A})^t$, where $(\overline{A})^t$ is the conjugate transpose of A. If λ is an Eigen value of B, then
(a) λ is real and $\lambda < 0$
(b) λ is real and $\lambda \le 0$
(c) λ is real and $\lambda \ge 0$
(d) λ is real and $\lambda > 0$

2.3 Let $T : C^n \to C^n$ be a linear operator rank n – 2. Then
(a) 0 is not an eigen value of T
(b) 0 must be an eigen value of T
(c) 1 can never be an eigen value of T
(d) 1 must be an eigen value

2.4 The fixed points of $f(z) = \dfrac{2iz + 5}{z - 2i}$ are
(a) $1 \pm i$
(b) $1 \pm 2i$
(c) $2i \pm 1$
(d) $i \pm 1$

2.5 The function $f(z) = |z|^2$ is
(a) Differentiable everywhere
(b) Differentiable only at the origin
(c) Not differentiable anywhere
(d) Differentiable on real x–axis

2.6 The connected subsets of the real line with the usual topology are

2.7 Let $f : [a, \quad$ bounded function where $-\infty < \quad$ Then f is Riemann integra$\quad$ nly if f is continuous every wh$\quad$ except on
(a) The empty set
(b) A set of measure zero
(c) A finite number of points
(d) A count ably finite number of point

2.8 The general solution of the differential equation
$$\frac{dy}{dx} + \tan y \tan x = \cos x \sec y \text{ is}$$
(a) $2 \sin y = (x + c - \sin x \cos x) \sec x$

(b) $\sin y = (x + c) \cos x$

(c) $\cos y = (x + c) \sin x$

(d) $\sec y = (x + c) \cos x$

2.9 The eigen values of the Sturm Liouville system
$$y'' + \lambda y = 0, 0 \le x \le \pi$$
$$y(0) = 0, \ y'(\pi) = 0 \text{ are}$$
(a) $\dfrac{n^2}{4}$

(b) $\dfrac{(2n-1)^2 \pi^2}{4}$

(c) $\dfrac{(2n-1)^2}{4}$

(d) $\dfrac{n^2 \pi^2}{4}$

2.10 The differential equation whose linearly independent solutions are $\cos 2x, \sin 2x$ and e^{-x} is
(a) $(D^3 + D^2 + 4D + 4) y = 0$

(b) $(D^3 - D^2 + 4D - 4) y = 0$

(c) $(D^3 + D^2 - 4D - 4) y = 0$

(d) $(D^3 - D^2 - 4D + 4) y = 0$

Where $D = \dfrac{d}{dx}$.

(a) 1
(b) 2
(c) 3
(d) 4

2.12 Let G be finite group of order 200. Then the number of subgroups of G of order 25 is
(a) 1
(b) 4
(c) 5
(d) 10

2.13 If p is prime, and Z_{p^4} denote the ring of integers modulo p^4, then the number of maximal ideals in Z_{p^4} is
(a) 4
(b) 2
(c) 3
(d) 1

2.14 All norms on a nor med vector space X are equivalent provided
(a) X is reflexive
(b) X is complete
(c) X is finite dimensional
(d) X is an inner product space

2.15 The space l_p is a Hilbert space if and only if
(a) p > 1
(b) p = even
(c) $p = \infty$
(d) p = 2

2.16 The least squares approximation of first degree to the function $f(x) = \sin x$ over the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is
(a) $\dfrac{24x}{\pi^3}$
(b) $\dfrac{24x}{\pi^2}$
(c) $\dfrac{24x}{\pi}$
(d) 24 x

2.17 The order of the numerical differentiation formula

(b) 3
(c) 4
(d) 1

2.18 The method
$$y_n + 1 = y_n + \frac{1}{4}\left(k_1 + 3k_2\right),$$
$$k_1 = hf\left(x_n, y_n\right)$$
$$k_2 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2}{3}k_1\right)$$
Is used to solve the initial value problem
$$y' = f(x, y) = -10y, \quad y(0) = 1$$
The method will produce stable results if the step size h satisfies
(a) $0.2 < h < 0.5$
(b) $0 < h < 0.5$
(c) $0 < h < 1$
(d) $0 < h < 0.2$

2.19 The general integral of the partial differential equation
$$(y + zx)z_x - (x + yz)z_y = x^2 - y^2 \text{ is}$$
(a) $F\left(x^2 + y^2 + z^2, xy + z\right) = 0$
(b) $F\left(x^2 + y^2 - z^2, xy + z\right) = 0$
(c) $F\left(x^2 - y^2 - z^2, xy + z\right) = 0$
(d) $F\left(x^2 + y^2 + z^2, xy - z\right) = 0$
Where F is an arbitrary function.

2.20 The differential equation governing the damped motion of a certain coil spring of unit mass under the action of an external force is given by
$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 24x = 30\cos\omega t,$$
Where $x(t)$ is the displacement from the equilibrium position at time t and ω is a constant. The reason ant frequency when the forcing function is in resonance with the system is
(a) $\sqrt{\dfrac{2}{\pi}}$
(b) $\dfrac{\pi}{2}$
(c) 2π
(d) $\dfrac{2}{\pi}$

(c) Lindelof

(d) Separable

2.22 Let X be the indiscrete space and Y a T_0 space. If $f : X \to Y$ is continuous, then

(a) X must be a one-point space

(b) Y must be discrete

(c) f must be a constant

(d) Y must be a one-point space

2.23 Let (X,Y) be the co-ordinates of a point chosen at random inside the disc $x^2 + y^2 \le r^2$ where $r > 0$. The probability that $Y > mX$ is

(a) $\dfrac{1}{2^r}$

(b) $\dfrac{1}{2}$

(c) $\dfrac{1}{2^m}$

(d) $\dfrac{1}{2^{m+r}}$

2.24 Let (X,Y) be a two-dimensional random variable such that

$E(X) = E(Y) = 3, \operatorname{var}(X) = \operatorname{var}(Y) = 1$

and $\operatorname{cov}(X,Y) = \dfrac{1}{2}$.

Then $P(|X - Y| > 6)$ is

(a) Less than $\dfrac{1}{6}$

(b) Equal to $\dfrac{1}{2}$

(c) Equal to $\dfrac{1}{3}$

(d) Greater than $\dfrac{1}{2}$

2.25 Let Z* denote the optimal value of LPP

Max $Z = 4x_1 + 6x_2 + 2x_3$

Such that

$3x_1 + 2x_2 + x_3 = 12$

$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

Then

(a) $10 \le Z* \le 20$

(b) $20 < Z* \le 30$

(c) $30 < Z* \le 40$

(d) $Z* > 40$

This section consists of ___ questions of Five marks each. Any Fi___ them have to be answered on the Answe___ ___ed.

$(15 \times 5 = 75)$

3. Let $T : V \to V$ be a line___ ___ation on a vector space V ov___ ___ K satisfying the property $Tx = 0$___ $x_1, x_2, \dots, x_n$ are linearly inde___ elements in V, show that $Tx_1, Tx_2, \dots$ are also linearly independent elements i___ V, show that $Tx_1, Tx_2, \dots, Tx_n$ are also linearly independent.

4. Let $T : V \to V$ be a linear operator on a finite dimensional vector space V over a field K and let p(t) be the minimal polynomial of T. If T is diagonalizable, show that

$p(t) = (t - \lambda_1)(t - \lambda_2) \dots (t - \lambda_r)$

For some distinct scalars $\lambda_1, \lambda_2, \dots, \lambda_r$.

5. Suppose $z = a$ is an isolated singularity of $f(z)$. Prove that $f(z)$ cannot be bounded in a neighborhood of $z = a$.

6. Evaluate $\displaystyle\int_{-\infty}^{\infty} \dfrac{x \sin \pi x}{x^2 + 2x + 5} dx$ using the method of residues.

7. The function f is defined on $[0,1]$ as follows $f(x) = x \sin \dfrac{1}{x}, \; x \ne 0$

$f(0) = 0$

Find the (Lebesgue) measure of the set $\{ x \mid f(x) \ge 0 \}$.

8. Consider two metric spaces $(R, d_1), (R, d_2)$ where

$d_1(y,z) = |y - z|$ and

$d_2(y,z) = \left| \dfrac{y}{1+|y|} - \dfrac{z}{1+|z|} \right|$.

Let the functions $f, f_n : [0, \infty] \to R$ be defined by

$f(x) = x, \; f_n(x) = x\left(1 + \dfrac{1}{n}\right),$ for

$0 \le x < \infty$.

does not converge uniformly to f on $[0,\infty]$ where R has metric d_1.

9. (a) Find the general solution of the differential equation
$$\frac{d^2y}{dx^2} + 4y = \sec^2 2x$$
Using the method of variation of parameters.
(b) Construct Green's function for the boundary value problem,
$$y'' + y = -x, \qquad y(0) = y(\pi) = 0 \quad \text{if it exists.}$$

10. Show that the alternating group An, $n \geq 3$ is generated by all cycles of length 3.

11. Let R be a commutative principal ideal domain with identity $1 \neq 0$ and let P be a non-zero prime ideal of R. Show that P is a maximal ideal of R.

12. In an inner product space X, fix $b \in X$ and define $f(x) = <x, b>$ where $<x, b>$ is the inner product of x with b. Show that f is a continuous linear functional and $\|f\| = \|b\|$.

13. Find the value of p such that the integration method
$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] + ph^3[f''(x_0) + f''(x_1)]$$
where $x_1 = x_0 + h$ provides exact result for highest degree polynomial. Find also the order of the method and the error term.

14. Set up the Gauss-Siedel iteration scheme in matrix form to solve the system of equations
$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix} x = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$$
Is this method convergent? If yes, find its rate of convergence.

15. Find the region in which the particle differential equation
$$u_{xx} - yu_{xy} + xu_x + yu_y + u = 0$$
Is hyperbolic and reduce it to a canonical form.

16. Determine the curve joining two points,

17. Let X be a with identity e and let $p : X \to R$ be satisfying
(i) $p(x) \geq 0$ for $p(x) = 0$ if $x = e$
(ii) $p(xy) \leq p(x) + p(y)$, $\in X$
(iii) $p(x^{-1}) = p(x)$, for all
Define $d(x, y) = p(x^{-1}y)$. Show a metric on X.

18. The number N of the persons getting injured in a bomb blast at a busy market place is a random variable having a Poisson distribution with parameter $\lambda (\geq 1)$. A person injured in the explosion may either suffer a minor injury requiring first aid or suffer a major injury requiring hospitalization. Let the number of persons with minor injury be N_1 and the conditional distribution of N_1 given N be
$$P\left(N_1 = \frac{i}{N}\right) = \frac{1}{N}, \ t = 1, 2....., N$$
Find the expect number of persons requiring hospitalization in a bomb blast.

19. Let $X_1, X_2, ..., X_{200}$ be identically and independently distributed random variables each with mean 0 and variance 1. Show that
$$\sqrt{\frac{\pi}{2}} P\left(|X_1X_2 + X_3X_4 + ... + X_{199}X_{200}| < 10\right)$$
is approximately equal to $\int_0^1 e^{\frac{-x^2}{2}}\,dx$.

20. The following table gives the scores on some scale of four ability groups taught by three different teaching methods

Ability Group	Teaching Method		
	A	B	C
1	5	9	4
2	8	7	2
3	12	15	7
4	7	11	9

Test whether or not the teaching methods are equally effective. You may use appropriate values from the following percentile values of F-distribution for your test
$$F_{....} = 10.92 \quad F_{....} = 9.78$$