Pre-Calculus 12

Resource Exam B

Scoring Guide

Calculator Permitted

1. A food sample contains 300 bacteria. The doubling time for bacteria left at room temperature is 20 minutes. How many minutes will it take to reach an unsafe level of 100 000 bacteria?

   Solve algebraically using logarithms. Answer must be written as a decimal accurate to at least 2 decimal places. **(4 marks)**

P—B10/10.5—M—MC—LOGS

SOLUTION

$$\overset{\frac{1}{2}\textbf{ mark}}{\overset{\downarrow}{100\,000}} = \overset{\frac{1}{2}\textbf{ mark}}{\overset{\downarrow}{300(2)}}^{\frac{t}{20}} \leftarrow \tfrac{1}{2}\textbf{ mark}$$

$$\frac{1000}{3} = 2^{\frac{t}{20}} \qquad \leftarrow \tfrac{1}{2}\textbf{ mark}$$

$$\log\left(\frac{1000}{3}\right) = \frac{t}{20}\log 2 \qquad \leftarrow \textbf{1 mark}$$

$$\frac{20\log\left(\frac{1000}{3}\right)}{\log 2} = t \qquad \leftarrow \tfrac{1}{2}\textbf{ mark}$$

$$167.62 = t \text{ minutes} \qquad \leftarrow \tfrac{1}{2}\textbf{ mark}$$

ALTERNATE SOLUTION

$$\underset{\frac{1}{2}\text{ mark}}{\overset{}{}}\quad\underset{}{}$$

$$\overset{\frac{1}{2}\,\textbf{mark}\ \downarrow}{100\,000} = \overset{\frac{1}{2}\,\textbf{mark}\ \downarrow}{300}(2)^{\frac{t}{20}} \leftarrow \tfrac{1}{2}\ \textbf{mark}$$

$$\frac{1000}{3} = 2^{\frac{t}{20}} \qquad \leftarrow \tfrac{1}{2}\ \textbf{mark}$$

$$\log_2\left(\frac{1000}{3}\right) = \frac{t}{20} \qquad \leftarrow \textbf{1 mark}$$

$$\frac{20\log\left(\frac{1000}{3}\right)}{\log 2} = t \qquad \leftarrow \tfrac{1}{2}\ \textbf{mark}$$

$$167.62 \text{ minutes} = t \qquad \leftarrow \tfrac{1}{2}\ \textbf{mark}$$

2. Given $\sin\alpha = \frac{1}{5}$, where α is in quadrant I and $\cos\beta = \frac{2}{3}$ where β is in quadrant IV, determine the exact value of $\sin(\alpha - \beta)$. **(4 ma**

P—A6/6.7—H—MN—TRIG

SOLUTION

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \underset{\frac{1}{2}\,\mathbf{mk}}{\underset{\uparrow}{\left(\frac{1}{5}\right)}}\underset{\frac{1}{2}\,\mathbf{mk}}{\underset{\uparrow}{\left(\frac{2}{3}\right)}} - \underset{1\,\mathbf{mk}}{\underset{\uparrow}{\left(\frac{2\sqrt{6}}{5}\right)}}\underset{1\,\mathbf{mk}}{\underset{\uparrow}{\left(\frac{-\sqrt{5}}{3}\right)}}$$

$$\left.\begin{aligned} &= \frac{2}{15} + \frac{2\sqrt{30}}{15} \\ &= \frac{2 + 2\sqrt{30}}{15} \end{aligned}\right\} \leftarrow \textbf{1 mark}$$

3. Prove algebraically:

$$\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \sec\theta\csc\theta - \cot\theta$$

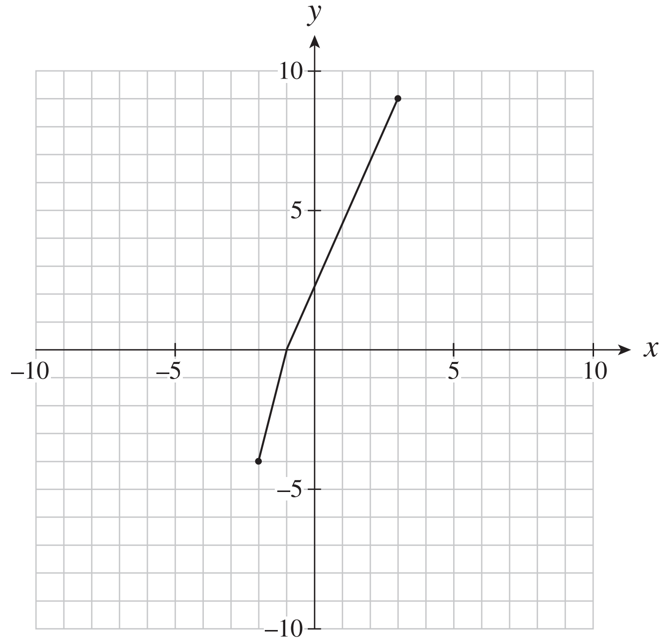
Q—A6/6.6—H—MC—TRIG

SOLUTION

LEFT SIDE	RIGHT SIDE	
$\frac{\cos\theta}{1-\sin\theta}$	$\sec\theta + \sec\theta\csc\theta - \cot\theta$	
	$= \frac{1}{\cos\theta} + \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$	$\leftarrow$ **1 mark**
	$= \frac{\sin\theta + 1 - \cos^2\theta}{\sin\theta\cos\theta}$	$\leftarrow$ **$\frac{1}{2}$ mark**
	$= \frac{\sin\theta + \sin^2\theta}{\sin\theta\cos\theta}$	$\leftarrow$ **$\frac{1}{2}$ mark**
	$= \frac{\cancel{\sin\theta}(1+\sin\theta)}{\cancel{\sin\theta}\cos\theta}$	$\leftarrow$ **$\frac{1}{2}$ mark**
	$= \frac{1+\sin\theta}{\cos\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta}$	$\leftarrow$ **$\frac{1}{2}$ mark**
	$= \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}$	$\leftarrow$ **$\frac{1}{2}$ mark**
	$= \frac{\cos^2\theta}{\cancel{\cos\theta}(1-\sin\theta)}$	
	$= \frac{\cos\theta}{1-\sin\theta}$	$\leftarrow$ **$\frac{1}{2}$ mark**

4. The graph of $y = f(x)$ is sketched below. Determine the domain and range of $y = \sqrt{f(x)}$ and explain how this was determined. **(4**

P—B13/13.4—M—WR—RADS



SOLUTION

For $y = \sqrt{f(x)}$ domain : $[-1,\ 3]$ $\leftarrow$ **1 mark**

range: $[0,\ 3]$ $\leftarrow$ **1 mark**

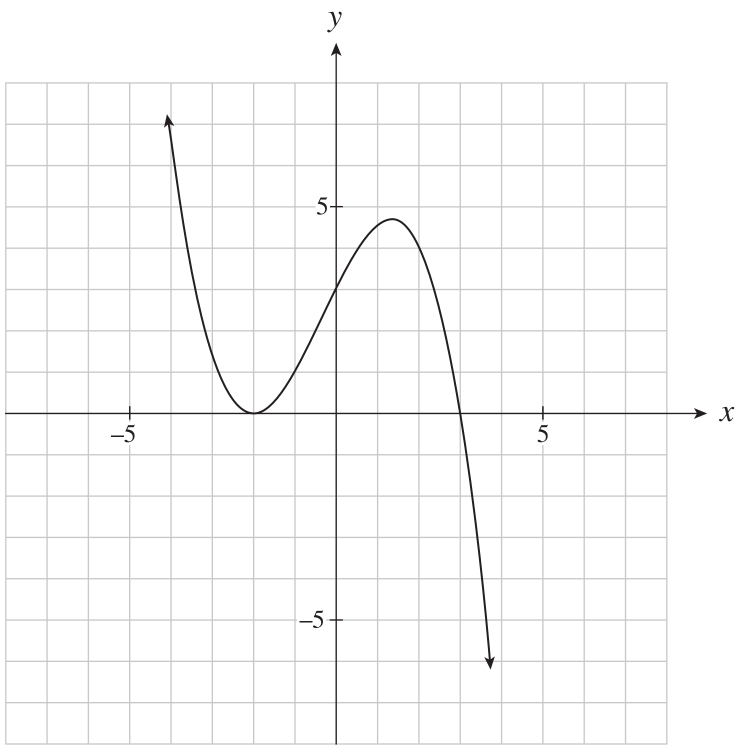
To determine the domain of $y = \sqrt{f(x)}$, only values of $f(x)$ that are positive can be considered which is where the range of $y = f(x)$ is positive. This occurs in the restricted domain $[-1,\ 3]$. **(1 mark)**

To determine the range of $y = \sqrt{f(x)}$, the lowest $f(x)$ value in the restricted domain is 0 and the largest value is 9. The square roots of these values are 0 and 3. That determines the range $[0,\ 3]$. **(1 mark)**

Calculator NOT Permitted

1. Determine the equation for the cubic polynomial given function below.
   Leave answer in factored form. **(4 marks)**

P—B12/12.2—M—WR—POLY



SOLUTION

$y = a(x+2)^2(x-3)$ $\leftarrow$ **2 marks**

passing through $(0,\ 3)$

$3 = a(0+2)^2(0-3)$ $\leftarrow$ **1 mark**

$3 = a(-12)$

$-\frac{1}{4} = a$ $\leftarrow$ **$\frac{1}{2}$ mark**

$y = -\frac{1}{4}(x+2)^2(x-3)$ $\leftarrow$ **$\frac{1}{2}$ mark**

2. Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

   Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

P—B6, B9/6.x—M—WR—TRANS

Scoring Rationale:

Concepts and Connections	***Curricular Reference*:** **B6 Demonstrate an understanding of inverses of relations** **B9 Graph and analyze exponential and logarithmic functions**
Problem Solving and Reasoning	**Students may:** **1) interchange $x \leftrightarrow y$ values in a table of values** **2) introduce logarithms**
Procedures	**The student was able to accurately show asymptotes, domain, range and the equation of the inverse**
Representation and Communication	**The reader is able to easily understand the process used because the work is clear, complete and organized with reference to the $(x, y) \leftrightarrow (y, x)$ relation**

Criteria:

- **Consider the characteristics of $f(x)$ and its inverse and the relationship between the two functions**

Pre-Calculus Mathematics 12 Scoring Rubric

	1 Below Expectations	2 Minimal	3 Proficient	4 Excellent
Snapshot	*Does not meet basic requirements of the problem*	*Partially solves the problem and meets some basic requirements but solution is incomplete or flawed*	*Solution is reasonable and complete for most parts of the task. All requirements met (may be minor flaws)*	*Solution is well-developed and justified. Thoroughly satisfies requirements; may be insightful or innovative*
Concepts and Connections [CN] *Recognizes the mathematics needed; explanation shows understanding of concepts*	• Does not recognize the mathematics; shows little/no understanding (may misunderstand)	• Recognizes/applies some concepts needed; shows partial understanding (often vague/incomplete)	• Recognizes/applies concepts needed; shows understanding of most relevant concepts	• Recognizes/applies concepts needed (may make insightful connections); shows thorough understanding
Problem-solving and reasoning [PS] [R] [V] [ME] *Uses appropriate strategies to solve the problem* *Verifies and justifies that results are reasonable*	• Does not use appropriate strategies • Does not verify results or solutions	• Uses appropriate strategies for some parts • Attempts to verify or justify results or solutions but is not fully successful	• Uses appropriate strategies for all parts • Verifies and justifies results or solutions (may be imprecise)	• Selects and uses highly effective, and often innovative, strategies • Verifies and justifies results or solutions with precision
Procedures [ME] [T] *shows accuracy and precision (e.g., in recording, substitutions, calculations, units, and symbols); efficient*	• Limited accuracy; major errors or omissions	• Follows procedures with partial accuracy; some errors or omissions	• Follows procedures accurately with minor errors or omissions	• Follows procedures accurately; very few if any minor errors/omissions; highly efficient
Representation and Communication [C] [V] *Clear, complete, organized using words, pictures and/or numbers* *Includes appropriate graphics; representations (e.g., charts, tables, graphs, diagrams; sketches)*	• Unclear; confusing and/or incomplete • Omits required graphics or representations and/or does not construct them appropriately; many omissions; serious flaws	• Presents parts of the process and solution; parts are omitted or unclear • Constructs most required graphics and/or representations; parts are omitted or inappropriate	• Presents process and solution clearly • Work is generally clear; easy to follow • Constructs required graphics and/or representations appropriately; may have minor errors or flaws	• Presents process and solution clearly and effectively • Work is detailed, precise and logically organized • Constructs required graphics and/or representations effectively and accurately

Code 0	Code NR
• Data simply recopied form the question • Picture, work or solution does not relate to the problem • Incorrect solution with no work shown • Everything erased	• No response (answer page is blank)

Sample B: Written-Response Question 2 (calculator NOT permitted) Exemplar

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

$f(x) = 2^x + 1$

$f^{-1}(x) \Rightarrow x = 2^y + 1$

$x - 1 = 2^y$

$\log_2(x-1) = y$

There ralationship is that it is reflected over the $y=x$

inverse $= \log_2(x-1)$
V.A $x = 1$

$D: \{x \mid x > 1, x \in \mathbb{R}\}$

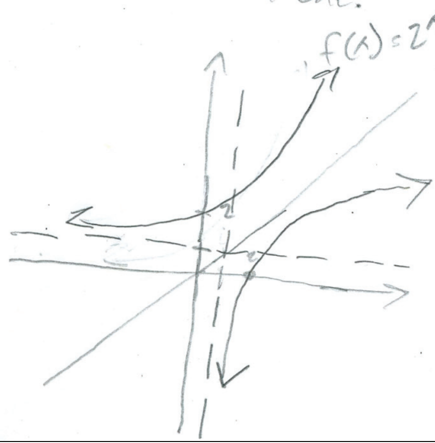
$R: \{y \mid y \in \mathbb{R}\}$

Regular $= 2^x + 1$
H.A $y = 1$

$D: \{x \mid x \in \mathbb{R}\}$

$R: \{y \mid y > 1, y \in \mathbb{R}\}$

∴ There relationship is the opposite switching the Domain and Range and the vertical asymptote for a Horizontal one.



Exemplar #1: Score 4

- recognizes concepts needed
- strategy is highly effective
- no errors or omissions
- presents solutions clearly

Sample B: Written-Response Question 2 (calculator NOT permitted) Exemplar

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

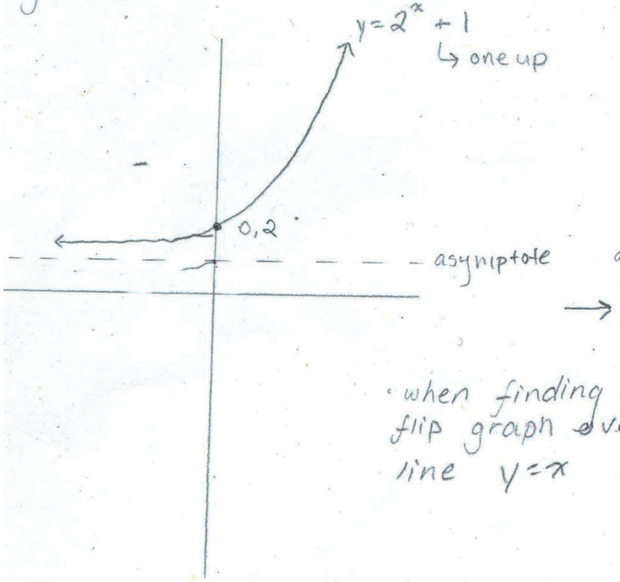
Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

- the inverse of $y = 2^x + 1$:

$$\left.\begin{aligned} y - 1 &= 2^x \\ x - 1 &= 2^y \\ \log_2 x - 1 &= y \end{aligned}\right\} \text{algebraic}$$

relationships:
- the original shift of one up becomes a shift one right
- y values + x values switch
- graph is flipped over line $y = x$
- the inverse equation of graphs with equation of a base to the power 'x' become log of the original base to the x value.

graphical:



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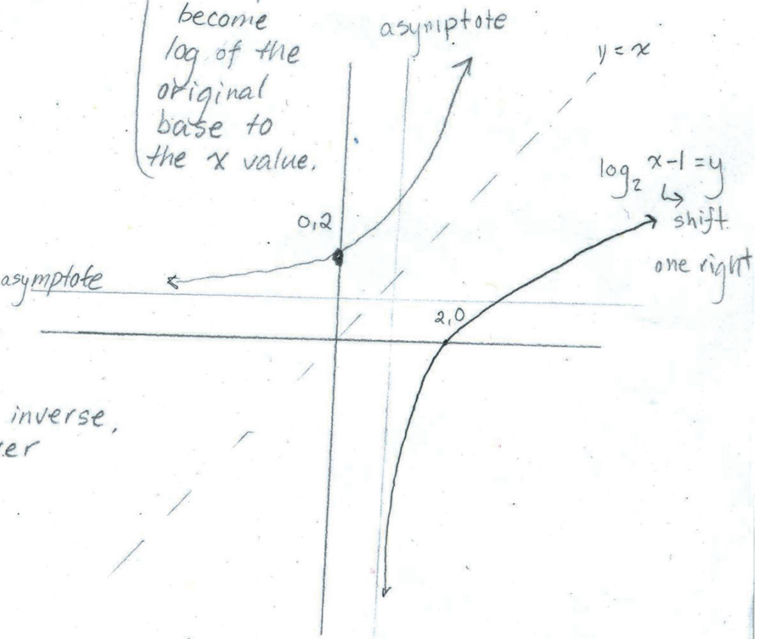


- when finding inverse, flip graph over line $y = x$

Exemplar #2: Score 4

- shows thorough understanding
- uses highly effective strategies
- highly efficient with one minor omission (equation of asymptotes)
- work is detailed, precise and logically organized

Sample B: Written-Response Question 2 (calculator NOT permitted) Exemplar

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

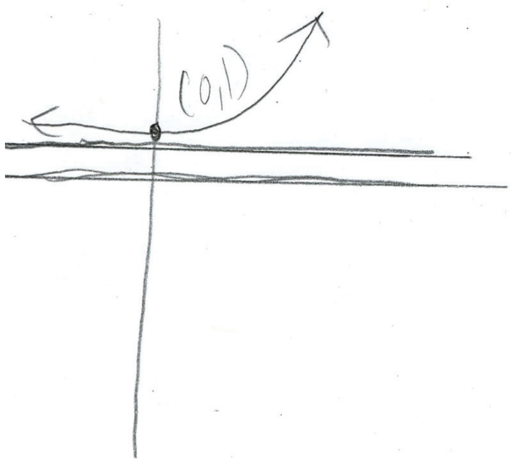
$y = 2^x + 1$

$y - 1 = 2^x$

$\Rightarrow x - 1 = 2^y$

$\log_2(x-1) = y$

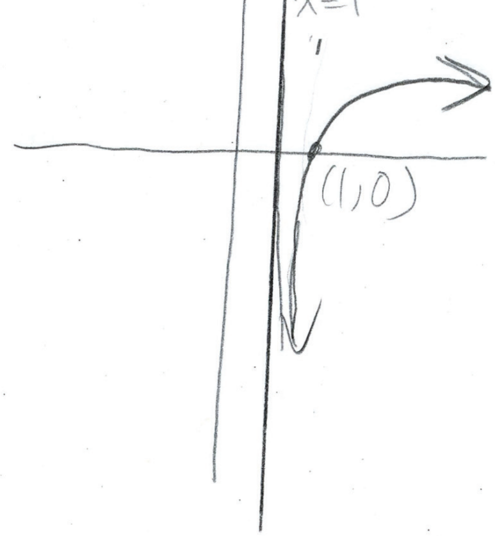
$y = 2^x + 1$



$\log_2(x-1) = y$

asymptote: $x - 1 = 0$

$x = 1$



relationship: Inverse

reflect over $y = x$

Exemplar #3: Score 3

- shows understanding of most relevant concepts
- uses appropriate algebraic strategies for all parts
- follows procedures accurately with minor errors and omissions (missing $(x, y) \rightarrow (y, x)$ relationship
- presentation (graph) has minor flaws

Sample B: Written-Response Question 2 (calculator NOT permitted) Exempla

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language.

(4 marks)

$f(x) = 2^x + 1$

$y = 2^x + 1$

calculator: $\dfrac{\log(x-1)}{\log(2)}$

$x = 2^y + 1$

$x - 1 = 2^y$

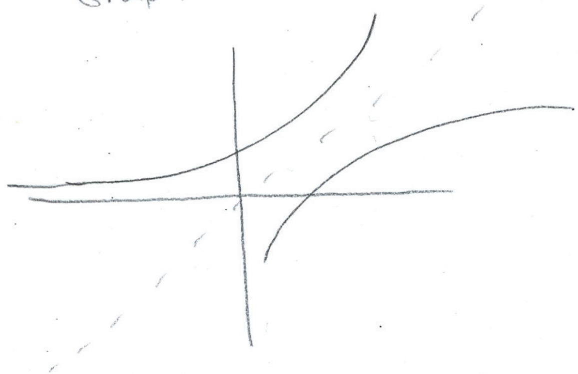
$\log_2(x-1) = y$

$y = \log_2(x-1)$

$f(x) = \log_2(x-1)$

Graph:

$y=x$



the relationship is that the inverse function is reflected over the $y=x$. However, the inverse does not exist until $x>1$. because $\log_2(x-1)$ cannot have units within the brakets be zero or below.

Exemplar #4: Score 3

- shows understanding of most relevant concepts
- uses appropriate algebraic strategies for all parts
- follows procedures accurately with minor errors and omissions
- work is generally clear, easy to follow with errors on graph

Sample B: Written-Response Question 2 (calculator NOT permitted) Exempla

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

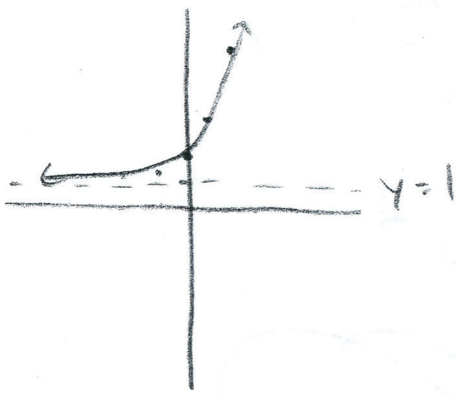
Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

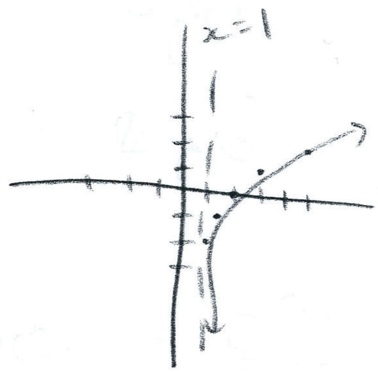
$f(x) = 2^x + 1$

x	y
-2	$1\frac{1}{4}$
-1	$1\frac{1}{2}$
0	2
1	3
2	5

$f(x)^{-1} = 2^x + 1$

x	y
$1\frac{1}{4}$	-2
$1\frac{1}{2}$	-1
2	0
3	1
5	2





Exemplar #5: Score 2

- recognizes some concepts needed; shows partial understanding
- uses appropriate strategies for some parts
- follows procedures with major omissions
- incomplete

Sample B: Written-Response Question 2 (calculator NOT permitted) Exemplar

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

inv:

$y = 2^x + 1$

$x = 2^y + 1$

$x - 1 = 2^y$

$\log(x-1) = y \log 2$

$f^{-1}(x)$ $y = \dfrac{\log(x-1)}{\log 2}$

The graphs of both $f(x) = 2^x + 1$ and its inverse are symmetrical on $y = x$.

The y-intercept on $f(x) = 2^x + 1$ becomes the x-intercept of this function's inverse.

Exemplar #6: Score 2

- recognizes some concepts
- uses appropriate strategies for some parts
- follows procedures with some omissions (graphical characteristics)
- incomplete

Sample B: Written-Response Question 2 (calculator NOT permitted) Exemplar

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

$f(x) = 2^x + 1$

INV: $x = 2^y + 1$

$x - 1 = 2^y$

$\log_2(x-1) = y$

$f^{-1}(x) = \boxed{y = \log_2(x-1)}$

Exemplar #7: Score 1

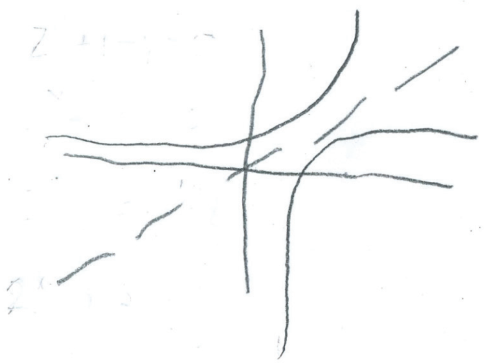
- shows partial understanding
- uses appropriate strategies for some parts
- major omissions
- incomplete

Sample B: Written-Response Question 2 (calculator NOT permitted) Exemplar

Explain the relationship between the exponential function $f(x) = 2^x + 1$ and its inverse.

Provide an answer that includes an algebraic analysis and describes graphical characteristics. You will be evaluated on the concepts expressed, the organization and accuracy of your work, and your use of language. **(4 marks)**

$y = 2^x + 1$ and its inverse $x = 2^y + 1$ are related in that they are reflections of each otheover the line $y = x$.



Exemplar #8: Score 1

- shows little or no understanding of logarithmic connections
- uses appropriate strategies for some parts
- major omissions
- incomplete