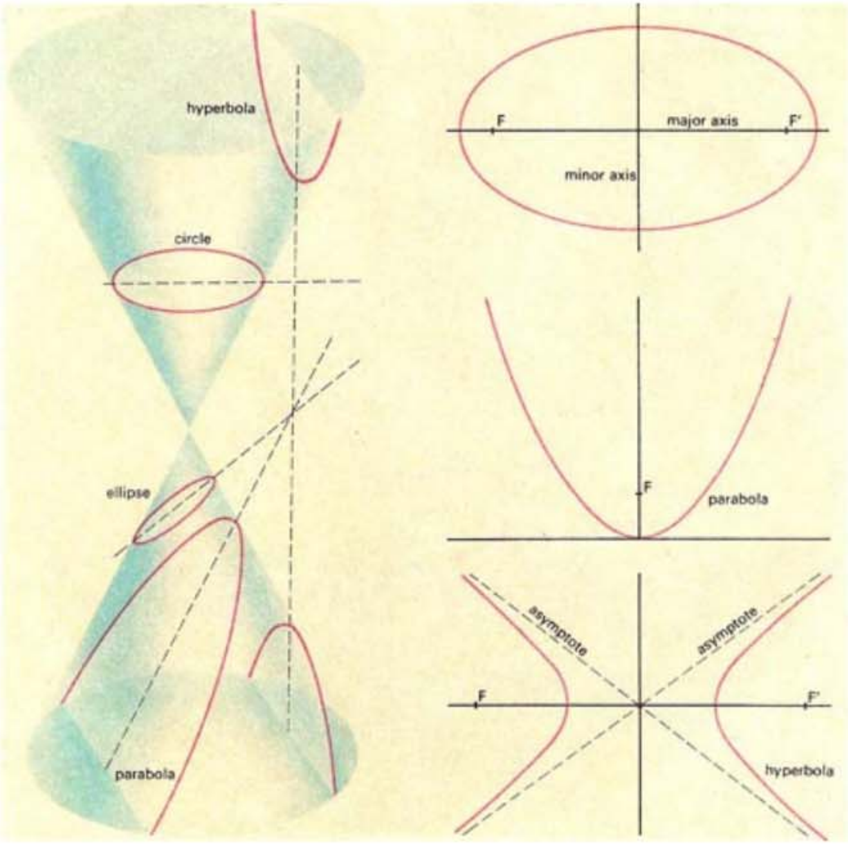
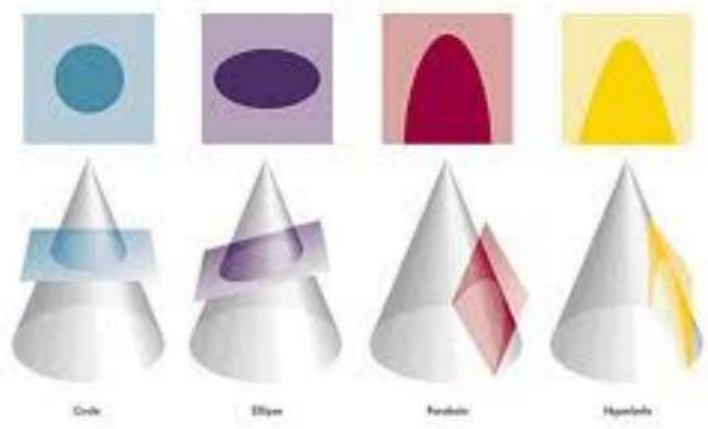
Conic sections

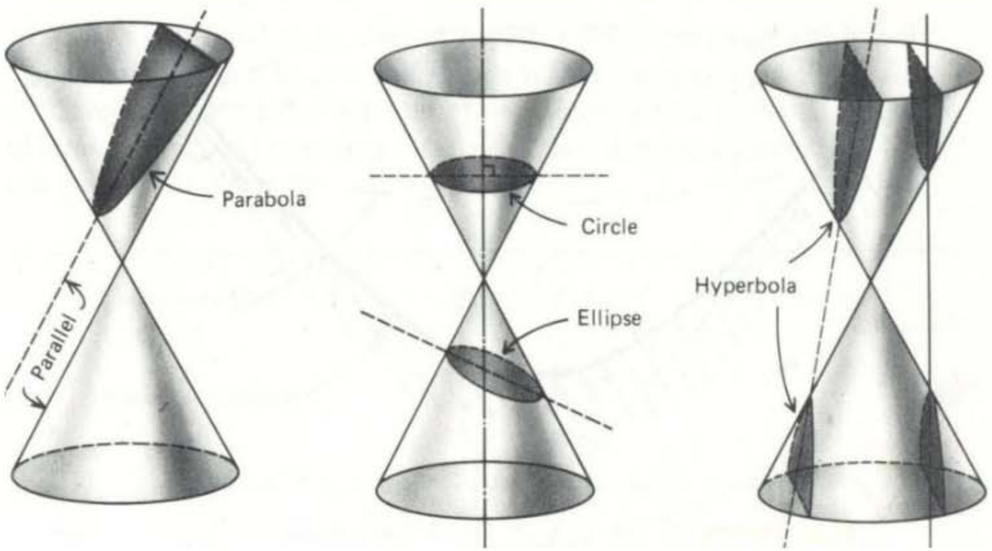
Specifications

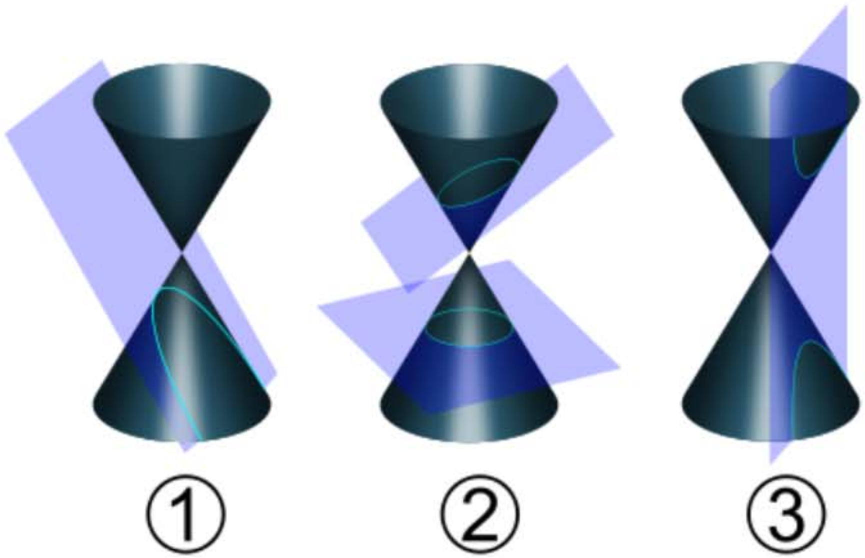
Graphs of parabolas, ellipses and hyperbolas with equations $y^2 = 4ax$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $xy=c^2$.	Sketching the graphs. Finding points of intersection with the coordinate axes or other straight lines. Candidates will be expected to interpret the geometrical implication of equal roots, distinct real roots or no real roots. Knowledge of the effects on these equations of single transformations of these graphs involving translations, stretches parallel to the $x-$ or $y-$axes, and reflections in the line $y=x$. Including the use of the equations of the asymptotes of the hyperbolas given in the formulae booklet..

Cones and sections









Transforming graphs

Notation: Explicit / Implicit equations of a curve

- An explicit equation of a curve can be written: $y = f(x)$
  (y is expressed in terms of x)

  Examples:

- An implicit equation of a curve can be written: $f(x,y)=0$
  (the "left side" of the equation is an expression involving x and y)

  Examples:

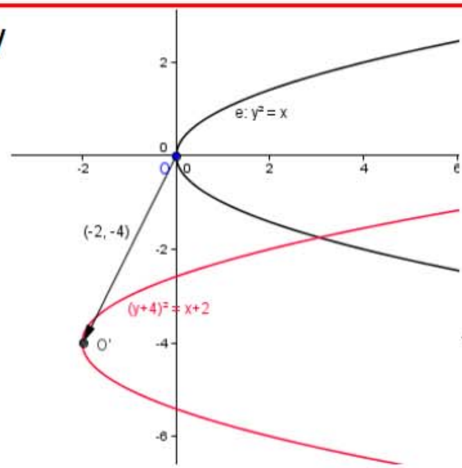
Translation of a curve.

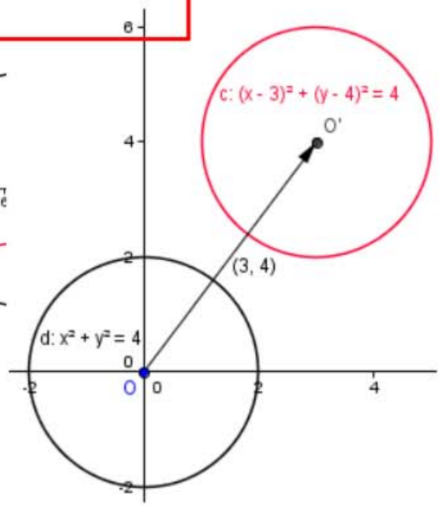
Consider a curve C with equation $f(x, y) = 0$.

The curve C', translation of the curve C by a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ has equation $f((x-a),(y-b)) = 0$

(Substitute x and y by (x-a) and (y-b) respectively

Examples:





Have a go:

1 Find the equation of the curve resulting from translating the curve with equation $\frac{x^2}{2} + \frac{y^2}{3} = 1$ through $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

2 The following curves are translated through the vectors indicated. Find the equations of the new curves.

(a) $\frac{x^2}{4} - \frac{y^2}{5} = 1$, $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$,

(b) $5x^2 + 7y^2 = 12$, $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$,

(c) $x^2 - y^2 = 3$, $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$,

(d) $(x - 1)(y + 2) = 3$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$,

(e) $y^2 = x - 3$, $\begin{bmatrix} -3 \\ -1 \end{bmatrix}$,

(f) $(x - 1)^2 - (y + 3)^2 = 7$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

3 The curve with equation $xy = 4$ is transformed into the curve with equation $(x - 5)(y + 4) = 4$. Describe geometrically the transformation that has taken place.

4 The curve with equation $x^2 - y^2 = 1$ is translated onto the curve with equation $x^2 - 6x - y^2 + 2y = k$. Find the value of k and find the vector of the translation.

Stretch of a curve.

The curve C has equation $f(x, y) = 0$.

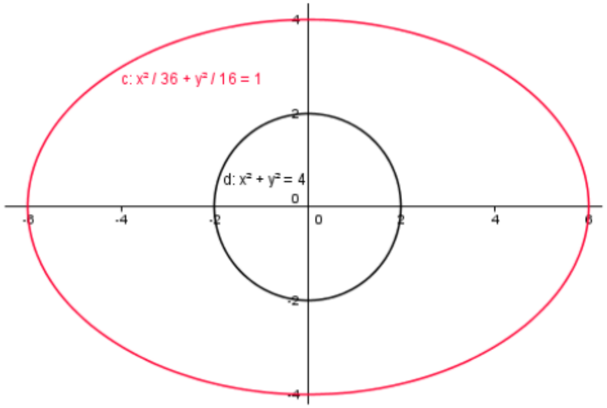
Consider a strecth by a factor a in the x-direction

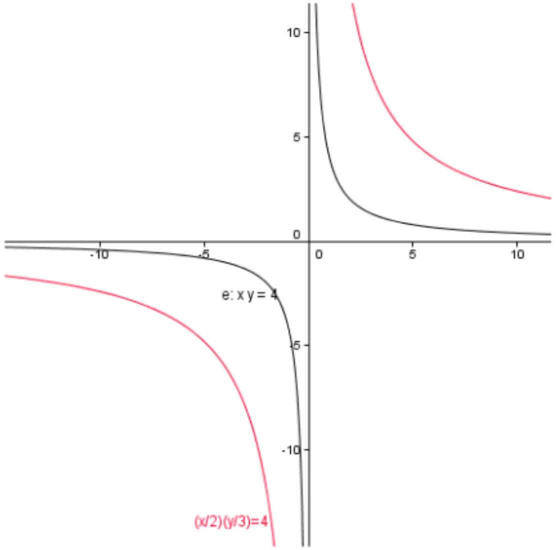
and a stretch by a factor b in the y-direction

The curve C' image of C through these stretches has equation $f((\frac{x}{a}), (\frac{y}{b})) = 0$

Substitute x and y by $\frac{x}{a}$ *and* $\frac{y}{b}$ *respectively*

Examples:





Reflection the line $y = x$.

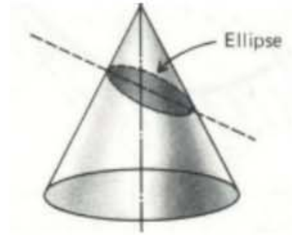
The curve C has equation $f(x, y) = 0$.

The curve C', reflection of C in $y = x$ has equation $f(x', y') = 0$

where $x' = y$ *and* $y' = x$

(Replace x by y and y by x in the equation)

The ellipse

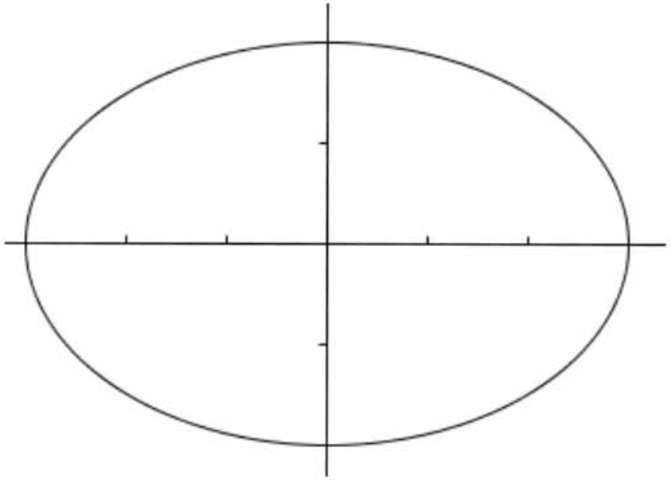


An ellipse can be seen as the stretch of a circle $x^2 + y^2 = 1$

The standard equation for an ellipse is : $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where a and b are positive values

Intersection with the x-axis and y-axis



Intersection with straight lines

To find the coordinates of the points of intersection

between $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and* $y = mx + c$,

solve the equations simultaneously (using substitution)

To consider how a straight line intersects a parabola, ellipse or hyperbola:

1. form a quadratic equation (usually in terms of x) of the form $ax^2 + bx + c = 0$;
2. when $b^2 - 4ac < 0$ there are no points of intersection;
3. when $b^2 - 4ac > 0$ there are two distinct points of intersection;
4. when $b^2 - 4ac = 0$ there is a single point of intersection. The line is a tangent to the curve at that point.

Example:

a)Sketch the ellipse, E, with equation $\frac{x^2}{4}+\frac{y^2}{25}=1$

Indicating where the ellipse crosses the axes.

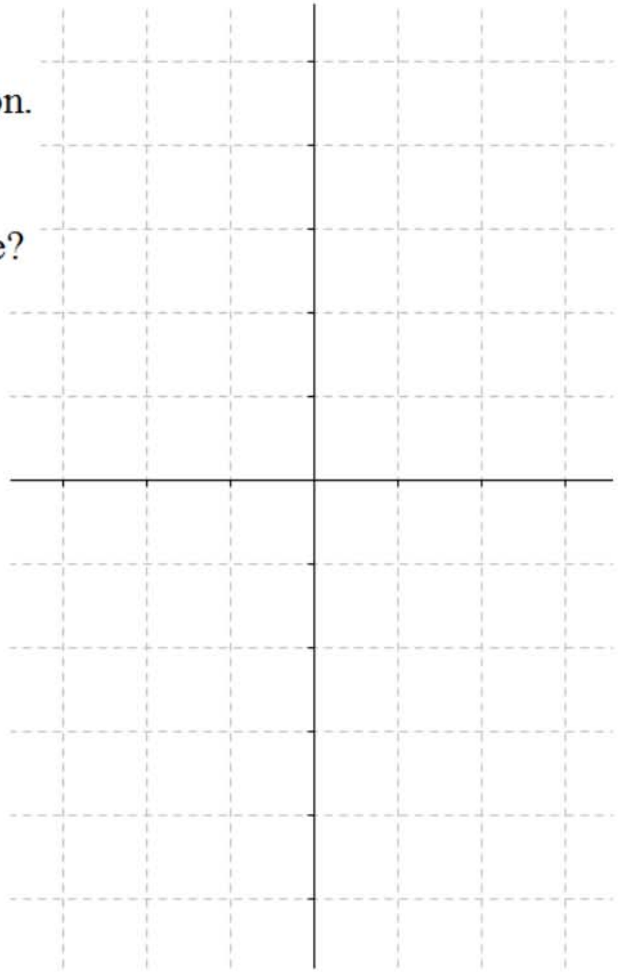
b) Work out the coordinates of the points of intersection with the line $y=x+2$

c) The ellipse E is translated by a vector $\begin{pmatrix}-1\\2\end{pmatrix}$. Write the equation of the new ellipse.

d) The ellipse E is stretch by a factor 2 in the x-direction an factor $\frac{1}{5}$ in the y-direction.

Give the equation of this new ellipse.

e) The ellipse E is reflected in the line $y=x$, what is the equation of the image ellipse?



Exercises:

1 Sketch the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$, showing its points of intersection with the axes.

Find the points of intersection of this ellipse with the lines:

a) $y=9x$ b) $y+x=2$

2 Sketch the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$, showing its points of intersection with the axes.

Find the points of intersection (if any) of this ellipse with the lines:

a) $y=4x$ b) $y=2$ c) $4y+x=20$

3 Find the points of intersection of the ellipse $\frac{x^2}{25}+\frac{y^2}{81}=1$ with the lines:

a) $y=9x$ b) $9y+x=12$

In Questions **4** to **7** sketch the ellipse before and after the transformation given. Write down also the equation of the ellipse after the transformation.

4 $\frac{x^2}{25}+\frac{y^2}{9}=1$ The ellipse is transformed by a reflection in the line $y=x$.

5 $\frac{x^2}{9}+\frac{y^2}{4}=1$ The ellipse is transformed by a translation $\begin{pmatrix}3\\-5\end{pmatrix}$.

6 $\frac{x^2}{36}+\frac{y^2}{25}=1$ The ellipse is transformed by a stretch parallel to the x-axis, scale factor 2.

7 $\frac{x^2}{25}+\frac{y^2}{36}=1$ The ellipse is transformed by a stretch parallel to the y-axis, scale factor 4.

8 The ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ is transformed by a reflection in the line $y=x$ followed by a stretch parallel to the y-axis, scale factor 3.

Find the equation of the new ellipse.

9 The ellipse $\frac{x^2}{36}+\frac{y^2}{49}=1$ is transformed by a reflection in the line $y=x$, followed by a stretch parallel to the y-axis, scale factor 5, and then by a stretch parallel to the x-axis, scale factor 2.

Find the equation of the new ellipse.

1 a) $\left(\pm\frac{10}{\sqrt{2029}}, \pm\frac{90}{\sqrt{2029}}\right)$ b) $(0, 2), \left(\frac{100}{29}, -\frac{42}{29}\right)$ **2** a) $\left(\pm\frac{4}{\sqrt{65}}, \pm\frac{16}{\sqrt{65}}\right)$ b) $(0, 2)$ c) No roots

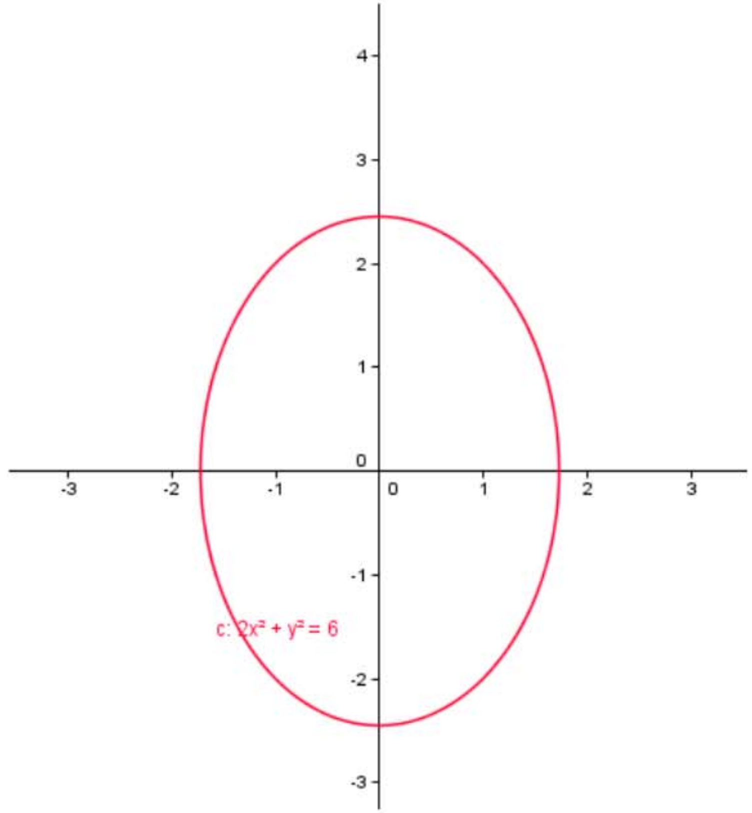
3 a) $\left(\pm\frac{5}{\sqrt{26}}, \pm\frac{45}{\sqrt{26}}\right)$ b) $(-4.890, 1.877), (4.981, 0.780)$ **4** $\frac{y^2}{25}+\frac{x^2}{4}=1$ **5** $\frac{(x-3)^2}{9}+\frac{(y+5)^2}{4}=1$

6 $\frac{x^2}{144}+\frac{y^2}{25}=1$ **7** $\frac{x^2}{25}+\frac{y^2}{64}=1$ **8** $\frac{x^2}{225}+\frac{y^2}{4}=1$ **9** $\frac{x^2}{900}+\frac{y^2}{196}=1$

Using a parameter

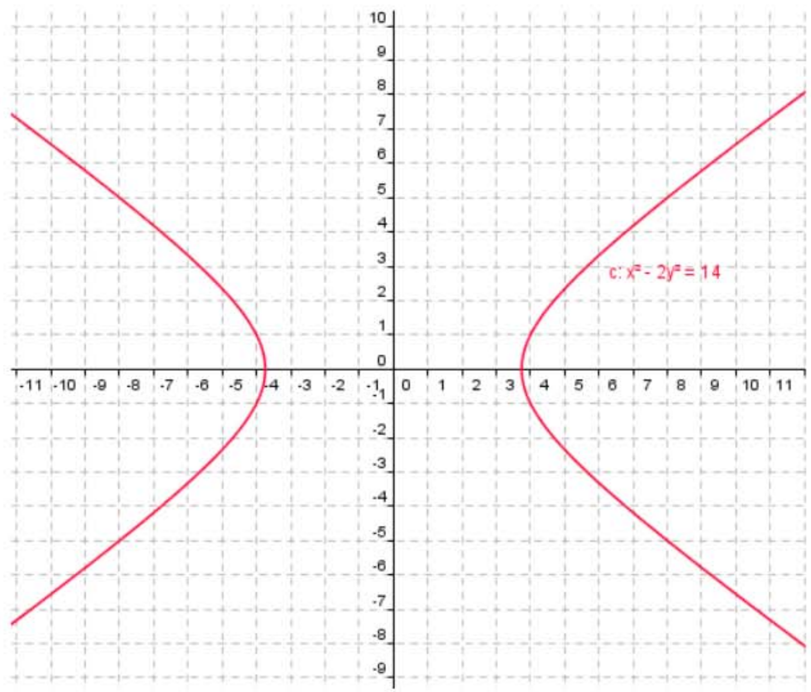
Find the possible values of the constant m so that the line $y = mx + 3$ is a tangent to the ellipse $2x^2 + y^2 = 6$.

Plot the tangents when you have found their equations

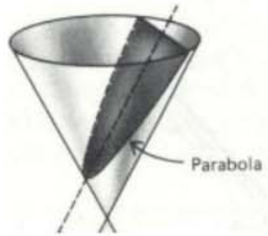


Find the possible values of m such that the line $y = mx - 7$ is a tangent to the hyperbola with equation $x^2 - 2y^2 = 14$.

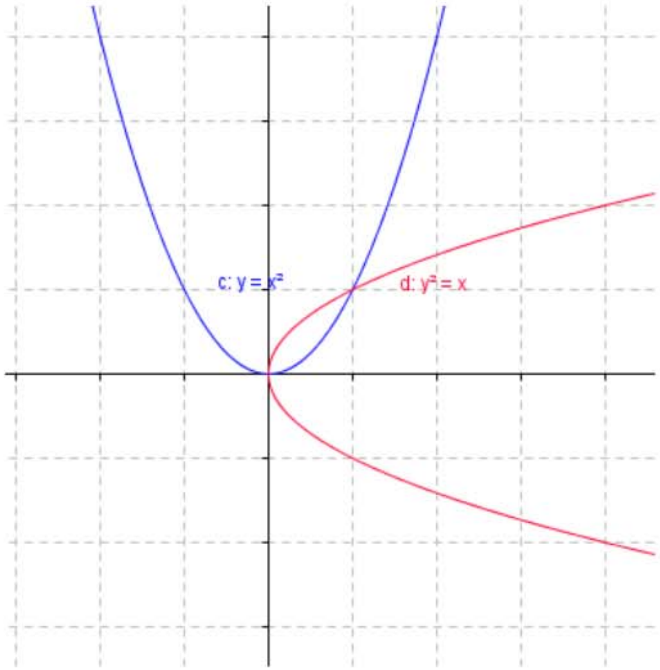
Plot the tangents



The parabola



The simplest equations for a parabola are $y^2 = x$ or $x^2 = y$



The standard equation for a parabola is $y^2 = 4ax$ where "a" is a given number.

The vertex of this parabola is (0,0)

The x-axis is an axis of symmetry

Intersections with lines

1 Find the coordinates of any points of intersection of the parabola $y^2 = 20x$ with the line:

a) $y = 5x$ b) $2y = x + 20$

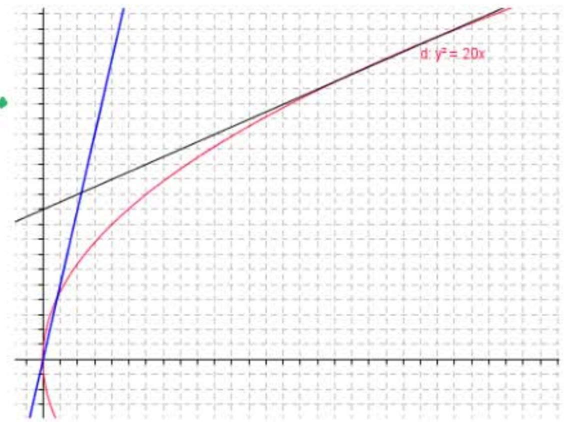
2 Find the coordinates of any points where the parabola $y^2 = 32x$ intersects the line:

a) $y = 4x$ b) $3y = x + 72$ c) $y = 4x + 5$

3 Find the coordinates of any points where the parabola $y^2 = 60x$ intersects the line:

a) $y = 5x$ b) $y = 12x$

c) $y = 6x + 5$ d) $y = x + 15$



1 a) $(0, 0)$, $(\frac{4}{5}, 4)$ b) $(20, 20)$ **2** a) $(0, 0)$, $(2, 8)$ b) $(72, 48)$ c) No points **3** a) $(0, 0)$, $(\frac{12}{5}, 12)$ b) $(0, 0)$, $(\frac{5}{12}, 5)$
3 c) No points d) $(15, 30)$

Transforming parabolas - exercises

In Questions **1** to **5**, sketch the parabola before and after the transformation.

1 $y^2 = 16x$ The parabola is transformed by a reflection in the line $y = x$.

2 $y^2 = 24x$ The parabola is transformed by a translation $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

3 $y^2 = 20x$ The parabola is transformed by a stretch parallel to the x-axis, scale factor 4.

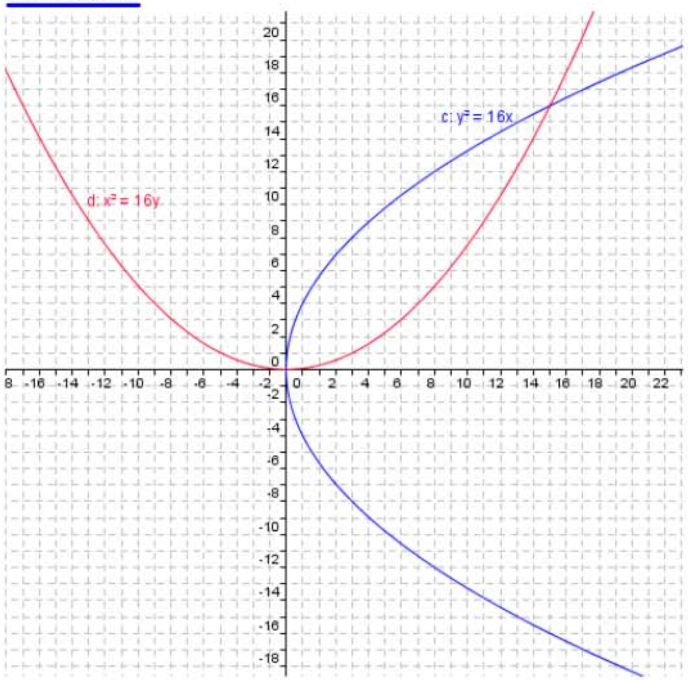
4 $y^2 = 28x$ The parabola is transformed by a stretch parallel to the y-axis, scale factor 3.

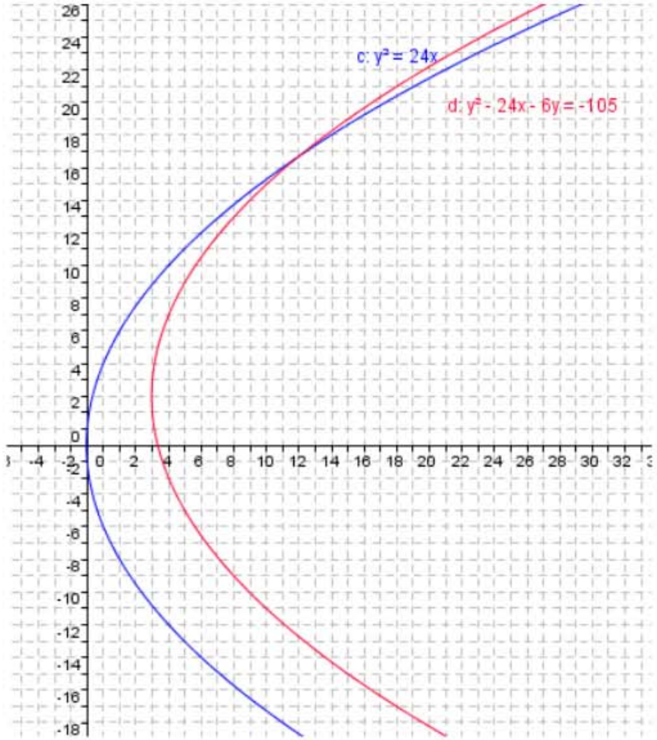
5 $y^2 = 12x$ The parabola is transformed by a translation $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

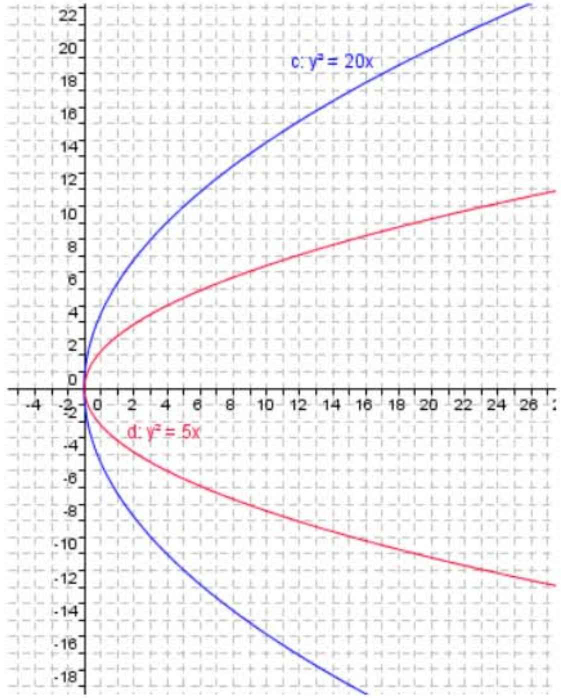
6 The parabola $y^2 = 16x$ is transformed by a stretch parallel to the y-axis, scale factor 2, followed by a reflection in the line $y = x$. Find the equation of the new parabola.

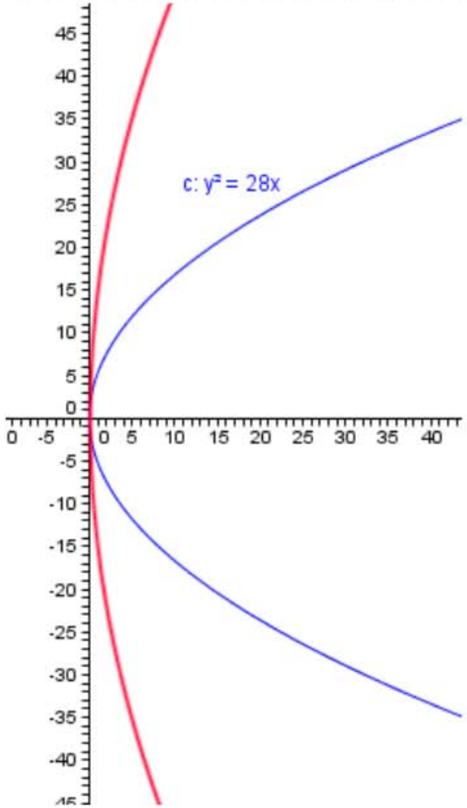
7 The parabola $y^2 = 36x$ is transformed by a reflection in the line $y = x$ followed by a stretch parallel to the x-axis, scale factor 7. Find the equation of the new parabola.

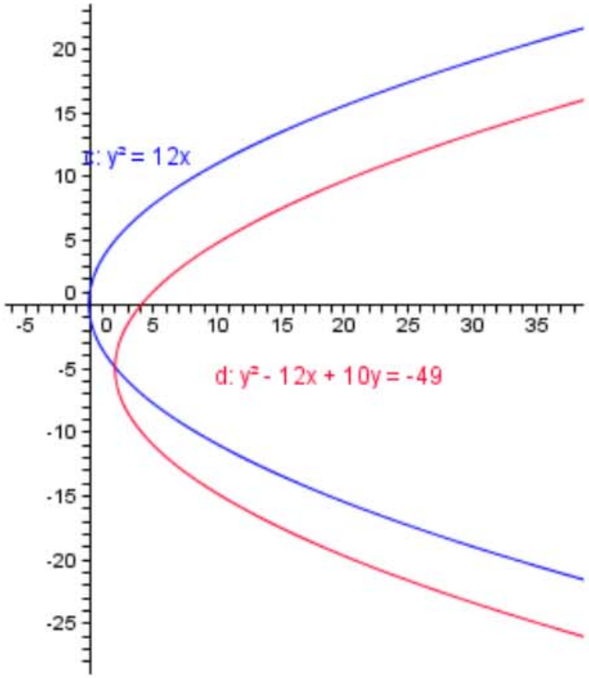
Answers





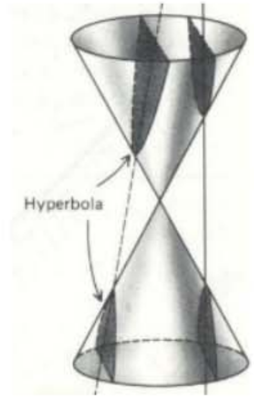






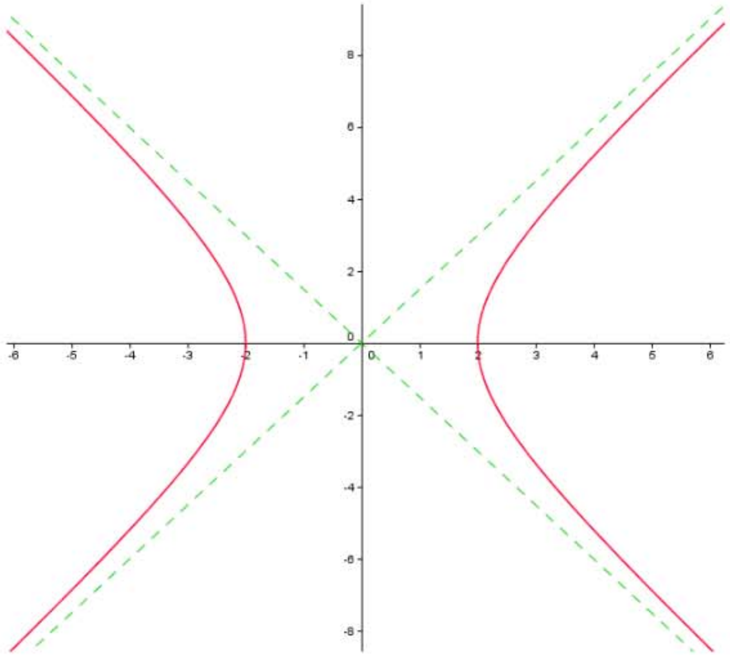
6 $x^2 = 8y$ **7** $x^2 = 1764y$

The Hyperbola



The standard equation for a hyperbola is : $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

General shape:



Intersection with the axes:

Asymptotes

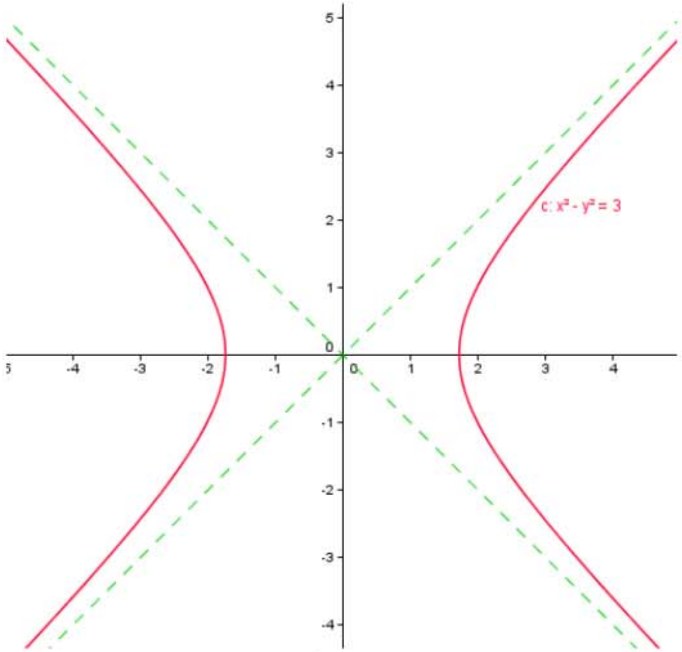
Summary

The equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ represents a hyperbola with centre at the origin cutting the x-axis when $x = \pm a$.
It does not intersect the y-axis and its asymptotes have equations $y = \pm\dfrac{b}{a}x$. When $b = a$ the hyperbola is said to be a rectangular hyperbola.

Rectangular hyperbola

A hyperbola H, with equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

is said to be rectangular when $a = b$



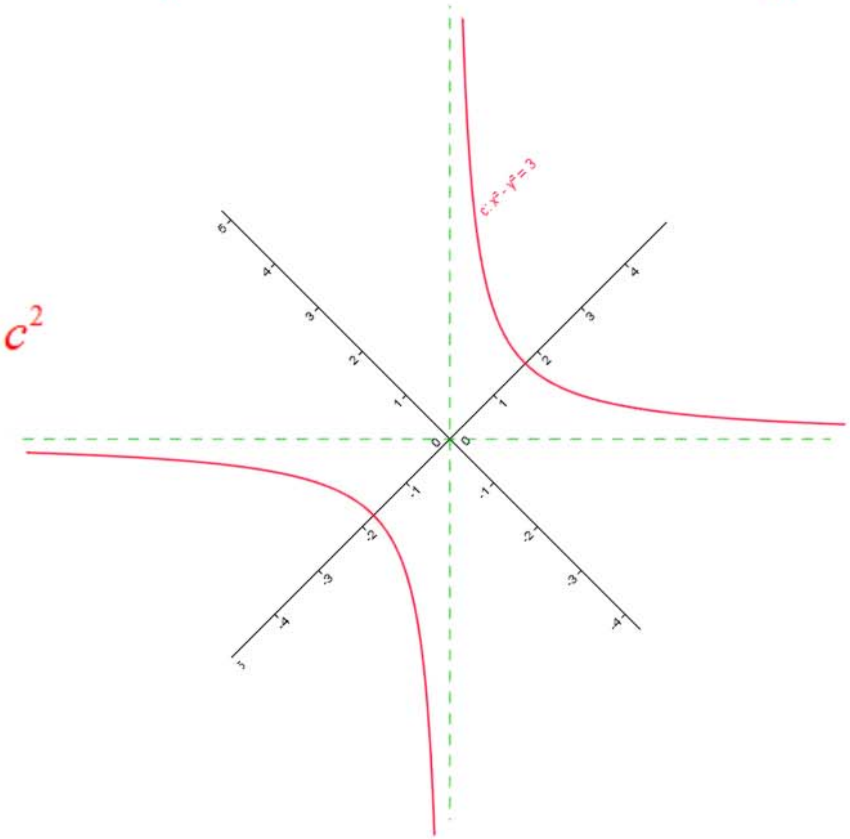
This gives:

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1 \;\; or \;\; x^2 - y^2 = a^2$$

the equations of the asymptotes are : $y = x$ and $y = -x$

Rotate this hyperbola 45° and you have :

Another expression for a rectangular hyperbola, $xy = c^2$



Exercises:

1 Find the points of intersection of the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$ with the lines:

a) $9y = x$ b) $y + x = 12$

2 Find the points of intersection of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ with the lines:

a) $4y = x$ b) $4y + x = 2$

3 Find the points of intersection of the hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$ with the lines:

a) $5y = x$ b) $4y + 5x = 1$

In Questions **4** to **7** sketch the hyperbola before and after the transformation given. Also write down the equation of the hyperbola after the transformation.

4 $\frac{x^2}{16} - \frac{y^2}{4} = 1$ The hyperbola is transformed by a reflection in the line $y = x$.

5 $\frac{x^2}{16} - \frac{y^2}{81} = 1$ The hyperbola is transformed by a translation $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

6 $\frac{x^2}{9} - \frac{y^2}{49} = 1$ The hyperbola is transformed by a stretch parallel to the x-axis, scale factor 4.

7 $\frac{x^2}{16} - \frac{y^2}{75} = 1$ The hyperbola is transformed by a stretch parallel to the y-axis, scale factor 5.

8 The hyperbola $\frac{x^2}{32} - \frac{y^2}{25} = 1$ is transformed by a reflection in the line $y = x$, followed by a stretch parallel to the y-axis, scale factor 2. Find the equation of the new hyperbola.

9 The hyperbola $\frac{x^2}{32} - \frac{y^2}{9} = 1$ is transformed by a reflection in the line $y = x$, followed by a stretch parallel to the x-axis, scale factor 7. Find the equation of the new hyperbola.

1 a) $\left(\pm\frac{90}{\sqrt{299}}, \pm\frac{10}{\sqrt{299}}\right)$ b) $(9.00, 3.00), (19.6, -7.6)$ **2** a) $\left(\pm\frac{12}{\sqrt{35}}, \pm\frac{3}{\sqrt{35}}\right)$ b) $(2, 0), \left(-\frac{74}{35}, \frac{36}{35}\right)$
3 a) $\left(\pm\frac{100}{\sqrt{609}}, \pm\frac{20}{\sqrt{609}}\right)$ b) $\left(\frac{401}{10}, -\frac{399}{8}\right)$ **4** $\frac{y^2}{16} - \frac{x^2}{4} = 1$ **5** $\frac{(x-4)^2}{16} - \frac{(y-3)^2}{81} = 1$ **6** $\frac{x^2}{144} - \frac{y^2}{49} = 1$
7 $\frac{x^2}{16} - \frac{y^2}{1875} = 1$ **8** $\frac{y^2}{128} - \frac{x^2}{25} = 1$ **9** $\frac{y^2}{32} - \frac{x^2}{441} = 1$

Key point summary

1 A reflection in the line $y = x$ maps (x, y) onto (x', y'),

where $\begin{matrix} x' = y \\ y' = x \end{matrix}$.

The equation of the new curve after a reflection in the line $y = x$ is obtained by interchanging x and y in the original equation.

2 A parabola with its vertex at the origin and its axis along the x-axis will have an equation of the form $y^2 = kx$, where k is a constant.

3 A stretch of scale factor c in the x-direction and scale factor d in the y-direction, maps (x, y) onto (x', y'),

where $\begin{matrix} x' = cx \\ y' = dy \end{matrix}$.

The equation of the new curve is obtained by replacing x by $\left(\frac{x}{c}\right)$ and y by $\left(\frac{y}{d}\right)$ in the original equation.

4 The general equation of an ellipse with its centre at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

It cuts the x-axis when $x = \pm a$ and cuts the y-axis when $y = \pm b$.

5 An equation of the form $xy = c^2$ represents a hyperbola with the coordinate axes as asymptotes. It is often referred to as a rectangular hyperbola.

It can be rotated through 45° to give an equation of the form $x^2 - y^2 = k$.

6 The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a hyperbola with centre at the origin cutting the x-axis when $x = \pm a$.

It does not intersect the y-axis and its asymptotes have equations $y = \pm\frac{b}{a}x$. When $b = a$ the hyperbola is said to be a rectangular hyperbola.

7 A translation with vector $\begin{bmatrix} c \\ d \end{bmatrix}$ maps (x, y) onto (x', y'),

where $\begin{matrix} x' = x + c \\ y' = y + d \end{matrix}$.

The equation of the new curve is obtained by replacing x by $(x - c)$ and y by $(y - d)$ in the original equation.

8 To consider how a straight line intersects a parabola, ellipse or hyperbola:

1 form a quadratic equation (usually in terms of x) of the form $ax^2 + bx + c = 0$;

2 when $b^2 - 4ac < 0$ there are no points of intersection;

3 when $b^2 - 4ac > 0$ there are two distinct points of intersection;

4 when $b^2 - 4ac = 0$ there is a single point of intersection. The line is a tangent to the curve at that point.