# Solutionbank M3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 1**

**Question:**

A circular flywheel of diameter 7 cm is rotating about the axis through its centre and perpendicular to its plane with constant angular speed 1000 revolutions per minute. Find, in $\text{m s}^{-1}$ to 3 significant figures, the speed of a point on the rim of the flywheel.

**[E]**

**Solution:**

diameter = 7 cm

$\therefore$ radius = 3.5 cm

i.e. $r$ = 0.035 m

Find the radius in $m$.

angular speed = 1000 revs/ min

i.e. $\omega = \dfrac{1000 \times 2\pi}{60}$ radians/second

Change the units of angular speed to radians per second.

Using $v = r\omega$

$\therefore\ v = 0.035 \times \dfrac{1000 \times 2\pi}{60}\ \text{m s}^{-1}$

$= 3.67\ \text{m s}^{-1}$ (3 s.f.)

Use the formula connecting linear speed and angular speed.

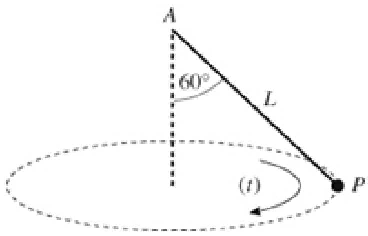
So the speed is $3.67\ \text{ms}^{-1}$ (3 s.f.)

Quote answer to required accuracy.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 2**

**Question:**

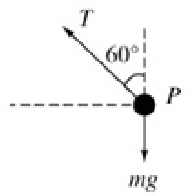



A particle $P$ of mass $m$ is attached to one end of a light string. The other end of the string is attached to a fixed point $A$. The particle moves in a horizontal circle with constant angular speed $\omega$ and with the string inclined at an angle of $60°$ to the vertical, as shown in the diagram above. The length of the string is $L$.

**a** Show that the tension in the string is $2mg$.

**b** Find $\omega$ in terms of $g$ and $L$. **[E]**

**Solution:**




Draw a diagram showing the forces acting an $P$.

**a** R(↑)

Resolve vertically

$T\cos 60° - mg = 0$

$\therefore T = \dfrac{mg}{\cos 60°}$

i.e. $T = 2mg$

Make $T$ the subject of the formula.

**b** Resolve R(←)

Using the equation of motion:

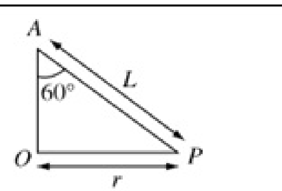
$T\sin 60 = ma$

$= mr\omega^2$

Use Newton's 2nd Law $F = ma$ with $a = r\omega^2$, where $r$ is the radius of the circular motion.

But $r = L\sin 60°$

$\therefore T\sin 60° = mL\sin 60°\,\omega^2$




From the triangle $AOP$

$r = L\sin 60°$.

$\therefore \omega^2 = \dfrac{T}{mL}$

$= \dfrac{2mg}{mL}$

$= \dfrac{2g}{L}$

$\therefore \omega = \sqrt{\dfrac{2g}{L}}$

Substitute the value for $T$ from part **a**.

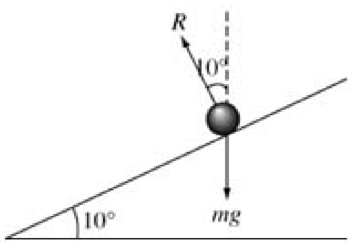
# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 3**

**Question:**

A car moves round a bend which is banked at a constant angle of $10°$ to the horizontal. When the car is travelling at a constant speed of $18\text{ m s}^{-1}$, there is no sideways frictional force on the car. The car is modelled as a particle moving in a horizontal circle of radius $r$ metres. Calculate the value of $r$. **[E]**

**Solution:**




Draw a diagram showing the forces acting on the car – its weight $mg$ and the normal reaction $R$.

R(↑)

$R\cos 10° - mg = 0$

$$\therefore R = \frac{mg}{\cos 10°} \quad ①$$

Using $F = ma$ horizontally gives

Use Newton's Second Law.

$$R\sin 10° = \frac{mv^2}{r} = \frac{m \times 18^2}{r}$$

Substituting for $R$

Use the result in equation ①.

$$\frac{mg}{\cos 10°} \times \sin 10° = \frac{m \times 18^2}{r}$$

$$\therefore r = \frac{18^2}{g \times \tan 10°}$$

$$\frac{\sin 10°}{\cos 10°} = \tan 10°$$

$= 187$ (3 s.f.) or $190$ (2 s.f.)

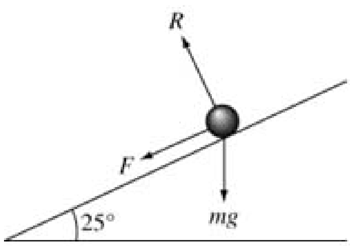
# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 4**

**Question:**

A cyclist is travelling around a circular track which is banked at $25°$ to the horizontal. The coefficient of friction between the cycle's tyres and the track is 0.6. The cyclist moves with constant speed in a horizontal circle of radius 40 m, without the tyres slipping.
Find the maximum speed of the cyclist. **[E]**

**Solution:**




Draw a diagram showing the weight $mg$, the normal reaction $R$ and the friction acting down the slope for maximum speed.

R(↑)

$R\cos 25° - F\sin 25° - mg = 0$ ①

R(←)

$R\sin 25° + F\cos 25° = \frac{mv^2}{40}$ ②

Use $F = ma$ with $a = \frac{v^2}{r}$ in a horizontal direction.

Also $F = 0.6R$

$\therefore R(\cos 25° - 0.6\sin 25°) = mg$

i.e. $R = \frac{mg}{\cos 25° - 0.6\sin 25°}$

The maximum friction is $\mu R$ for no slipping.

Make $R$ the subject of the formula in equation ①.

Also

$R(\sin 25° + 0.6\cos 25°) = \frac{mv^2}{40}$

$\therefore \frac{mg(\sin 25° + 0.6\cos 25°)}{\cos 25° - 0.6\sin 25°} = \frac{mv^2}{40}$

Substitute into equation ②.

$\therefore v^2 = 40g\frac{(\sin 25° + 0.6\cos 25°)}{(\cos 25° - 0.6\sin 25°)}$

$= 580$

$\therefore v = 24.1$ or $24$

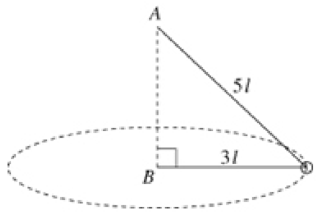
So maximum speed is $24$ m s$^{-1}$ (2 s.f.)

Make $v^2$ the subject of the formula and obtain $v$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 5**

**Question:**




A light inextensible string of length $8l$ has its ends fixed to two points $A$ and $B$, where $A$ is vertically above $B$. A small smooth ring of mass $m$ is threaded on the string. The ring is moving with constant speed in a horizontal circle with centre $B$ and radius $3l$, as shown in the diagram. Find

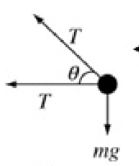
**a** the tension in the string,

**b** the speed of the ring.

**c** State briefly in what way your solutions might no longer be valid if the ring were firmly attached to the string. **[E]**

**Solution:**

**a**



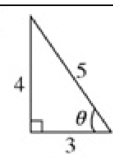

Draw a diagram showing the weight of the ring and the tensions of equal magnitude $T$.

R(↑)

$T\sin\theta - mg = 0$

$\therefore T = \frac{mg}{\sin\theta}$

$= \frac{5mg}{4}$




$\theta$ is an angle in a 3-4-5 triangle.

**b** R(←)

$T + T\cos\theta = \frac{mv^2}{3l}$

Use Newton's 2nd Law with $a = \frac{v^2}{3l}$. $v$ is the speed of the ring.

$\therefore T\left(1 + \frac{3}{5}\right) = \frac{mv^2}{3l}$

$\cos\theta = \frac{3}{5}$ from 3-4-5 triangle.

i.e. $\frac{5mg}{4} \times \frac{8}{5} = \frac{mv^2}{3l}$

Substitute value for $T$ from part **a**.

$\therefore v^2 = 6gl$

$v = \sqrt{6gl}$

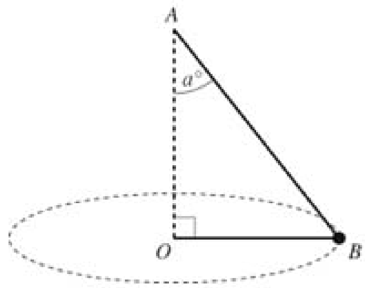
Make $v$ the subject of the formula.

**c** The tensions could not be assumed to have the same magnitude.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 6**

**Question:**



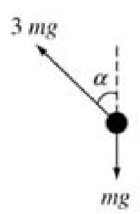

A metal ball $B$ of mass $m$ is attached to one end of a light inextensible string. The other end of the string is attached to a fixed point $A$. The ball $B$ moves in a horizontal circle with centre $O$ vertically below $A$, as shown in the diagram. The string makes a constant angle $\alpha°$ with the downward vertical and $B$ moves with constant angular speed $\sqrt{(2gk)}$, where $k$ is a constant. The tension in the string is $3mg$. By modelling $B$ as a particle, find

**a** the value of $\alpha$,

**b** the length of the string. **[E]**

**Solution:**

**a**




Draw a diagram showing the forces acting on the metal ball.

R(↑)

$3mg\cos\alpha - mg = 0$

$\therefore \cos\alpha = \frac{1}{3}$

$\therefore \alpha = 70.5°$

Resolve vertically and make $\alpha$ the subject of the equation.

**b** R(←)

$3mg\sin\alpha = mr\omega^2$

$= mr \times 2gk$

But $r = l\sin\alpha$

$\therefore 3mg = ml \times 2gk$

$\therefore l = \frac{3}{2k}$

Use Newton's 2nd Law $F = ma$ with $a = r\omega^2$.

Use $\Delta AOB$ to express the radius $r$ in terms of the length of the string $l$ and the angle $\alpha$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

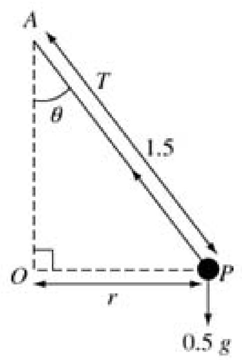
**Review Exercise 2**
**Exercise A, Question 7**

**Question:**

A particle $P$ of mass 0.5 kg is attached to one end of a light inextensible string of length 1.5 m. The other end of the string is attached to a fixed point $A$. The particle is moving, with the string taut, in a horizontal circle with centre $O$ vertically below $A$. The particle is moving with constant angular speed $2.7 \text{ rad s}^{-1}$. Find

**a** the tension in the string,
**b** the angle, to the nearest degree, that $AP$ makes with the downward vertical. **[E]**

**Solution:**




Draw a diagram. Let the tension in the string be $T$ and let the string make an angle $\theta$ with the vertical.

**a** $r = 1.5\sin\theta$

Let $OP = r$.

R(←)

$T\sin\theta = mr\omega^2$

$= 0.5 \times 1.5\sin\theta \times 2.7^2$

$\therefore T = 5.4675 \text{ N}$

$= 5.5 \text{ N (2 s.f.)}$

Use Newton's 2nd Law $F = ma$ with $a = r\omega^2$.

**b** R(↑)

Resolve vertically.

$T\cos\theta - 0.5g = 0$

$\therefore T\cos\theta = 0.5g$

Substitute the value for $T$ found in part **a**.

$\therefore \cos\theta = \frac{0.5g}{5.4675}$

$= 0.896$

$\therefore \theta = 26°$ (nearest degree)

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

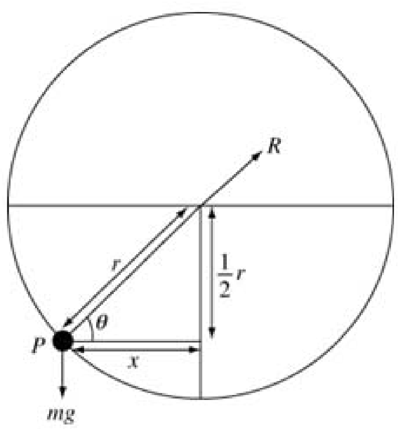
**Review Exercise 2**
**Exercise A, Question 8**

**Question:**

A particle $P$ of mass $m$ moves on the smooth inner surface of a spherical bowl of internal radius $r$. The particle moves with constant angular speed in a horizontal circle, which is at a depth $\frac{1}{2}r$ below the centre of the bowl.

**a** Find the normal reaction of the bowl on $P$.
**b** Find the time for $P$ to complete one revolution of its circular path. **[E]**

**Solution:**




Draw a diagram showing the forces acting an the particle $P$. $\theta$ is the angle between the normal reaction and the horizontal.

**a** $\sin\theta = \dfrac{\frac{1}{2}r}{r} = \dfrac{1}{2}$

$\therefore \theta = 30°$

R(↑)

Then $R\sin\theta - mg = 0$

$\therefore R = \dfrac{mg}{\sin\theta} = 2mg$

Find $\theta$ and use this to find $R$.

**b** R(→)

$R\cos\theta = mx\omega^2$

$= m(r\cos\theta)\omega^2$

As $R = 2mg$

$2mg\cos\theta = mr\cos\theta\omega^2$

$\therefore 2g = r\omega^2$

$\omega = \sqrt{\dfrac{2g}{r}}$

Use Newton's 2nd Law '$F = ma$' with $a = x\omega^2$ $x = \dfrac{\sqrt{3}}{2}r$ but you do not need to find this.

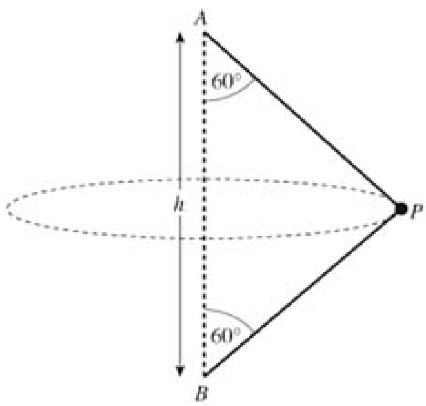
Time to complete one revolution $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{r}{2g}}$

You should learn this formula.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 9**

**Question:**




A particle $P$ of mass $m$ is attached to two light inextensible strings. The other ends of the string are attached to fixed points $A$ and $B$. The point $A$ is a distance $h$ vertically above $B$. The system rotates about the line $AB$ with constant angular speed $\omega$. Both strings are taut and inclined at $60°$ to $AB$, as shown in the diagram. The particle moves in a circle of radius $r$.

**a** Show that $r = \frac{\sqrt{3}}{2}h$.

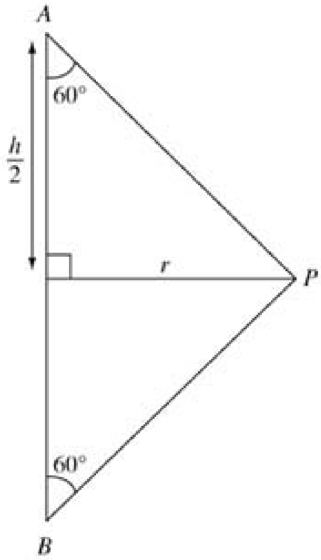
**b** Find, in terms of $m$, $g$, $h$ and $\omega$, the tension in $AP$ and the tension in $BP$.

The time taken for $P$ to complete one circle is $T$.

**c** Show that $T < \pi\sqrt{\left(\frac{2h}{g}\right)}$. **[E]**

**Solution:**

**a**



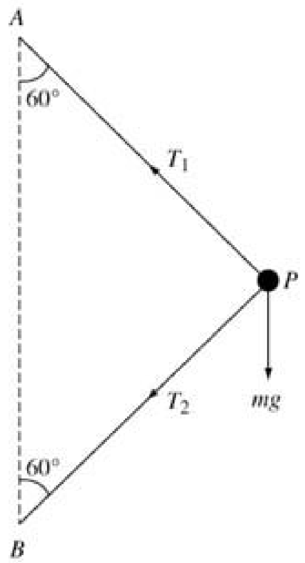

Divide the equilateral triangle into two right angled triangles. Use trigonometry to express $r$ in terms of $h$.

$$\tan 60^\circ = \frac{r}{\frac{h}{2}}$$

$$\therefore r = \frac{h}{2} \times \tan 60^\circ$$

$$\text{i.e. } r = \frac{\sqrt{3}h}{2}$$

**b**




Draw another diagram showing the forces acting on $P$.
Let $T_1$ be the tension in $AP$ and $T_2$ be the tension in $BP$.

R(↑)

$T_1 \cos 60° - T_2 \cos 60° - mg = 0$

$\therefore \frac{1}{2}T_1 - \frac{1}{2}T_2 = mg$ ①

R(←)

$T_1 \sin 60° + T_2 \sin 60° = mr\omega^2$

$\therefore \frac{\sqrt{3}}{2}T_1 + \frac{\sqrt{3}}{2}T_2 = m\frac{\sqrt{3}}{2}h\omega^2$

i.e. $\frac{1}{2}T_1 + \frac{1}{2}T_2 = \frac{1}{2}mh\omega^2$ ②

Use equation of motion $F = ma$ with $a = r\omega^2$.

Adding ① and ②

$T_1 = mg + \frac{1}{2}mh\omega^2$

Subtracting ② − ①

$T_2 = \frac{1}{2}mh\omega^2 - mg$

Solve the simultaneous equations to find the two tensions.

**c** $T_2 > 0 \Rightarrow \omega > \sqrt{\frac{2g}{h}}$

$T_1 > 0$ for all values of $\omega$ so consider the condition $T_2 > 0$.

As $T = \frac{2\pi}{\omega}, T < 2\pi\sqrt{\frac{h}{2g}}$

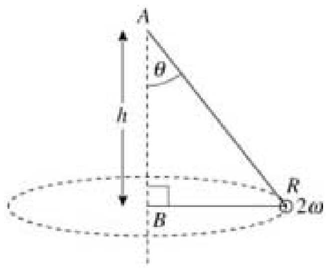
i.e. $T < \pi\sqrt{\frac{2h}{g}}$

Use $T = \frac{2\pi}{\omega}$ to find the time to complete one circle.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 10**

**Question:**




One end of a light inextensible string is attached to a fixed point $A$. The other end of the string is attached to a fixed point $B$, vertically below $A$, where $AB = h$. A small smooth ring $R$ of mass $m$ is threaded on the string. The ring $R$ moves in a horizontal circle with centre $B$, as shown in the diagram. The upper section of the string makes a constant angle $\theta$ with the downward vertical and $R$ moves with constant angular speed $\omega$. The ring is modelled as a particle.

**a** Show that $\omega^2 = \frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right)$.

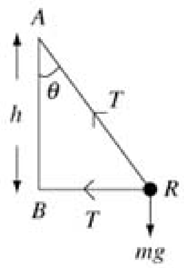
**b** Deduce that $\omega > \sqrt{\frac{2g}{h}}$.

Given that $\omega = \sqrt{\frac{3g}{h}}$,

**c** find, in terms of $m$ and $g$, the tension in the string.

**Solution:**

**a**




Draw a diagram showing the forces acting on the ring.

R(↑)

$T\cos\theta - mg = 0$

$\therefore\ T\cos\theta = mg$ ①

Resolve vertically.

R(←)

$T + T\sin\theta = mr\omega^2$

But $r = h\tan\theta$

Resolve horizontally using $F = ma$ with $a = r\omega^2$, where $r = h\tan\theta$.

$\therefore \dfrac{mg}{\cos\theta}(1+\sin\theta) = mh\dfrac{\sin\theta}{\cos\theta}\omega^2$

$\therefore \omega^2 = \dfrac{g}{h}\left(\dfrac{1+\sin\theta}{\sin\theta}\right)$

Show each line of working as there is a printed answer.

**b** As $\omega^2 = \dfrac{g}{h}\left(\dfrac{1}{\sin\theta}+1\right)$ and $\sin\theta < 1$ so that $\dfrac{1}{\sin\theta} > 1$

$\therefore \omega^2 > \dfrac{g}{h}\times 2$

So $\omega > \sqrt{\dfrac{2g}{h}}$

Express $\omega^2$ in this way, as it is clear that when $\sin\theta$ is a maximum, $\omega^2$ is a minimum.

**c** Given $\omega = \sqrt{\dfrac{3g}{h}}$

Then $\dfrac{g}{h}\left(\dfrac{1+\sin\theta}{\sin\theta}\right) = \dfrac{3g}{h}$

Use the expression obtained in part **a** to find the angle $\theta$.

$\therefore 1+\sin\theta = 3\sin\theta \Rightarrow \sin\theta = \dfrac{1}{2}$

$\therefore \cos\theta = \dfrac{\sqrt{3}}{2}$

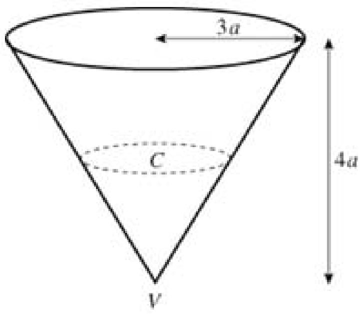
and $T = \dfrac{2\sqrt{3}}{3}mg$ or $1.15mg$.

From equation ① $T = \dfrac{mg}{\cos\theta}$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

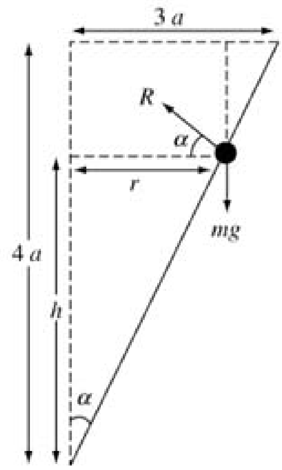
**Review Exercise 2**
**Exercise A, Question 11**

**Question:**




A hollow cone, of base radius $3a$ and height $4a$, is fixed with its axis vertical and vertex $V$ downwards, as shown in the diagram. A particle moves in a horizontal circle with centre $C$, on the smooth inner surface of the cone with constant angular speed $\sqrt{\frac{8g}{9a}}$. Find the height of $C$ above $V$. **[E]**

**Solution:**




$\tan\alpha = \frac{3a}{4a} = \frac{3}{4}$

$\tan\alpha = \frac{r}{h}$

Draw a diagram showing angle $\alpha$ between the slant side and the axis of the cone.

Also show the radius $r$ and the height $h$.

R(↑)

$R\sin\alpha - mg = 0$

$\therefore R\sin\alpha = mg$ ①

R(←)

$R\cos\alpha = mr\omega^2$

$R\cos\alpha = mr \times \frac{8g}{9a}$ ②

Use the equation of motion $F = ma$ with $a = r\omega^2$.

Divide equation ① by equation ②

Then $\tan\alpha = mg \div \frac{8mrg}{9a}$

$= \frac{9a}{8r}$

$\therefore \frac{9a}{8r} = \frac{3}{4} \Rightarrow r = \frac{3a}{2}$

$\therefore h = \frac{r}{\tan\alpha} = \frac{3a}{2} \times \frac{4}{3}$

$= 2a$

The height of $C$ above $V$ is $2a$.

Use the result that $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$.

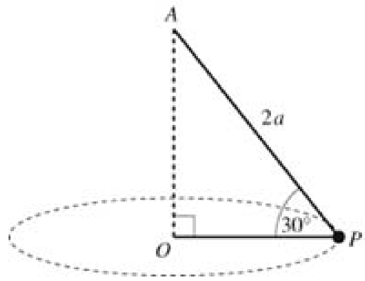
Use $\tan\alpha = \frac{3}{4}$ to obtain $r$.

Use $\tan\alpha = \frac{r}{h}$ to obtain $h$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 12**

**Question:**




A particle $P$ of mass $m$ is attached to one end of a light inextensible string of length $2a$. The other end of the string is fixed to a point $A$ which is vertically above the point $O$ on a smooth horizontal table. The particle $P$ remains in contact with the surface of the table and moves in a circle with centre $O$ and with angular speed $\sqrt{\frac{kg}{3a}}$, where $k$ is a constant. Throughout the motion the string remains taut and $\angle APO = 30°$, as shown in the diagram.

**a** Show that the tension in the string is $\frac{2kmg}{3}$.

**b** Find, in terms of $m$, g and $k$, the normal reaction between $P$ and the table.

**c** Deduce the range of possible values of $k$.

The angular speed of $P$ is changed to $\sqrt{\left(\frac{2g}{a}\right)}$. The particle $P$ now moves in a horizontal circle above the table. The centre of this circle is $X$.

**d** Show that $X$ is the mid-point of $OA$. **[E]**

**Solution:**

**a** R(←)

Use equation of motion

$T\cos 30° = m(2a\cos 30°)\left(\frac{kg}{3a}\right)$

$\therefore T = m \times 2a \times \frac{kg}{3a}$

$T = \frac{2kmg}{3}$

Let the tension in the string be $T$
Use $F = ma$ with $a = r\omega^2$,
noting that $r = 2a\cos 30°$.

**b** R(↑)

$R + T\sin 30° - mg = 0$

$\therefore R = mg - \frac{2kmg}{3} \times \frac{1}{2}$

$= mg\left(1 - \frac{k}{3}\right)$

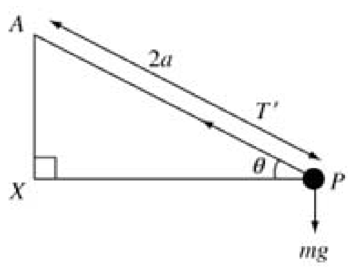
Let the normal reaction be $R$ and resolve vertically.

**c** As $R > 0$, $1 - \frac{k}{3} > 0$

$\therefore k < 3$

Use the condition $R > 0$ to find $k$.

**d**




Let the new tension be $T'$ and let $AP$ make an angle $\theta$ with the horizontal.

$PX = 2a\cos\theta$

R(←)

$T'\cos\theta = m \times 2a\cos\theta \times \left(\frac{2g}{a}\right)$

$\therefore T' = 4mg$

Use $F = ma$ for the new horizontal circular motion.

R(↑)

$T'\sin\theta - mg = 0$

$\therefore \sin\theta = \frac{mg}{T'} = \frac{mg}{4mg} = \frac{1}{4}$

Find the expression for $\theta$.

As $AX = 2a\sin\theta$

$AX = 2a \times \frac{1}{4} = \frac{1}{2}a$

But $AO = 2a\sin 30° = a$

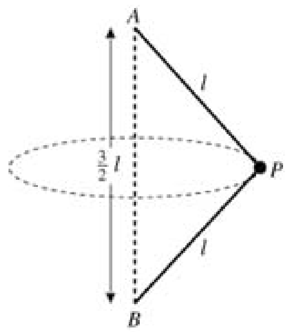
$\therefore AX = \frac{1}{2}AO$ as required.

Find the lengths $AX$ and $AO$ to show the result which is asked.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 13**

**Question:**



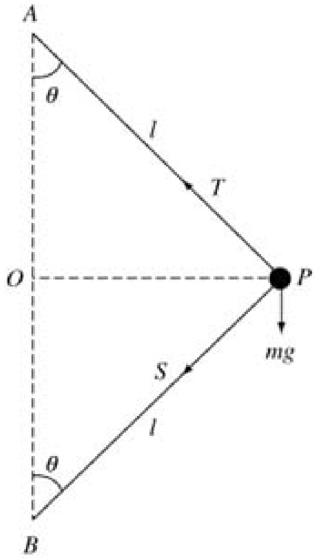

A particle $P$ of mass $m$ is attached to the ends of two light inextensible strings $AP$ and $BP$ each of length $l$. The ends $A$ and $B$ are attached to fixed points, with $A$ vertically above $B$ and $AB = \frac{3}{2}l$, as shown in the diagram above. The particle $P$ moves in a horizontal circle with constant angular speed $\omega$. The centre of the circle is the mid-point of $AB$ and both strings remain taut.

**a** Show that the tension in $AP$ is $\frac{1}{6}m(3l\omega^2 + 4g)$.

**b** Find, in terms of $m$, $l$, $\omega$ and g, an expression for the tension in $BP$.

**c** Deduce that $\omega^2 \geq \frac{4g}{3l}$. **[E]**

**Solution:**

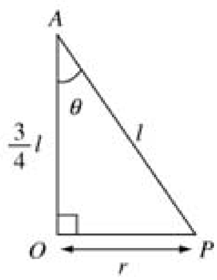



Let the tension in $AP$ be $T$ and in $BP$ be $S$.
Draw a diagram showing the forces acting.

**a** R(↑)

$T\cos\theta - S\cos\theta - mg = 0$

$\therefore T - S = \frac{mg}{\cos\theta} = \frac{4mg}{3}$ ①




From triangle $APO$ $\cos\theta = \frac{3}{4}$

R(←)

$T\sin\theta + S\sin\theta = mr\omega^2$

$= ml\sin\theta\omega^2$

$\therefore T + S = ml\omega^2$ ②

Also from triangle $APO$ $r = l\sin\theta$.

Adding equations ① and ② gives

$2T = \frac{4}{3}mg + ml\omega^2$

$\therefore T = \frac{1}{6}m(3l\omega^2 + 4g)$

Solve the simultaneous equation to obtain $T$ first.

**b** Subtracting ② − ① gives

$2S = ml\omega^2 - \frac{4mg}{3}$

$\therefore S = \frac{1}{6}m(3l\omega^2 - 4g)$

Also use these equations to find $S$.

**c** As $S \geq 0$, $\omega^2 \geq \frac{4g}{3l}$ as required.

As string $BP$ is taut it is under tension and $S \geq 0$

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 14**

**Question:**

A rough disc rotates in a horizontal plane with constant angular velocity $\omega$ about a fixed vertical axis. A particle $P$ of mass $m$ lies on the disc at a distance $\frac{4}{3}a$ from the axis. The coefficient of friction between $P$ and the disc is $\frac{3}{5}$. Given that $P$ remains at rest relative to the disc,

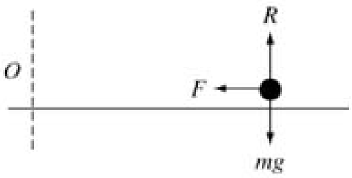
**a** prove that $\omega^2 \leq \frac{9g}{20a}$.

The particle is now connected to the axis by a horizontal light elastic string of natural length $a$ and modulus of elasticity $2mg$. The disc again rotates with constant angular velocity $\omega$ about the axis and $P$ remains at rest relative to the disc at a distance $\frac{4}{3}a$ from the axis.

**b** Find the greatest and least possible values of $\omega^2$. **[E]**

**Solution:**

**a**




R(↑)

$R - mg = 0 \quad \therefore\ R = mg$

R(←)

Use $F = ma$ with $a = r\omega^2$ for circular motion.

$F = mr\omega^2$

$= m\left(\frac{4}{3}a\right)\omega^2$

The radius of the circular motion is $\frac{4a}{3}$.

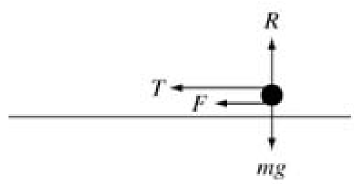
As $P$ remains at rest $F \le \mu R$

$\therefore\ m\left(\frac{4}{3}a\right)\omega^2 \le \frac{3}{5}mg$

Substitute $R = mg$ and $F = m\left(\frac{4}{3}a\right)\omega^2$ into $F \le \mu R$.

$\therefore\ \omega^2 \le \frac{9g}{20a}$

**b**




Draw a diagram showing friction acting towards the centre of the motion.
Case **i** maximum value for $\omega$:

Case **i** minimum value for $\omega$:

$T = \frac{2mg}{a} \times \frac{a}{3}$

Use Hooke's Law $T = \frac{\lambda e}{l}$ with $\lambda = 2mg$, $l = a$ and $e = \frac{4a}{3} - a = \frac{a}{3}$.

R(←)

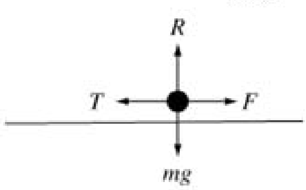
$T + F = mr\omega_{max}^2$

$F = \mu R = \frac{3}{5}mg$, as in **a**.

$\therefore\ \frac{2mg}{3} + \frac{3}{5}mg = m \times \frac{4a}{3}\omega_{max}^2$

$\therefore\ \omega_{max}^2 = \frac{19g}{20a}$

This is the greatest possible value of $\omega^2$.




Draw a diagram showing friction acting away from the centre of the circular motion.

Case **ii** minimum value for $\omega$:

R(←)

$T - F = mr\omega_{min}^2$

$\therefore\ \frac{2mg}{3} - \frac{3mg}{5} = m \times \frac{4a}{3}\omega_{min}^2$

$\therefore\ \omega_{min}^2 = \frac{g}{20a}$

This is the least possible value of $\omega^2$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 15**

**Question:**

One end of a light inextensible string of length $l$ is attached to a particle $P$ of mass $m$. The other end is attached to a fixed point $A$. The particle is hanging freely at rest with the string vertical when it is projected horizontally with speed $\sqrt{\frac{5gl}{2}}$.

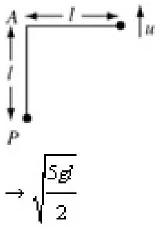
**a** Find the speed of $P$ when the string is horizontal.

When the string is horizontal it comes into contact with a small smooth fixed peg which is at the point $B$, where $AB$ is horizontal, and $AB < l$. Given that the particle then describes a complete semi-circle with centre $B$,

**b** find the least possible value of the length $AB$. **[E]**

**Solution:**

**a**




Let $u$ be the speed of $P$ when the string is horizontal.

Conservation of energy:

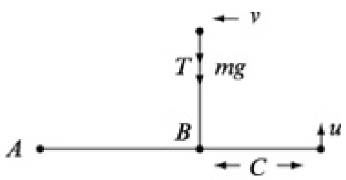
Using loss of kinetic energy = gain in potential energy.

$$\frac{1}{2}m\left[\frac{5gl}{2}-u^2\right]=mgl$$

$$\therefore u^2=\frac{5gl}{2}-2gl=\frac{gl}{2}$$

$$\therefore u=\sqrt{\frac{gl}{2}}$$

**b**




Let the particle move in a semi-circle of radius $r$.

Conservation of energy

$$\frac{1}{2}m(u^2-v^2)=mgr$$

$$\therefore v^2=u^2-2gr \quad ①$$

Write down an equation of motion $F=ma$ when the particle is at the highest point.

$$R(\downarrow): T+mg=\frac{mv^2}{r}$$

$$\therefore T=m\frac{(u^2-2gr)}{r}-mg$$

Substitute from equation ①.

$$=\frac{mu^2}{r}-3mg$$

Use the value of $u$ from part **a**.

$$=\frac{mgl}{2r}-3mg$$

As $T\geq 0 \Rightarrow \frac{mgl}{2r}\geq 3mg$

$$\therefore \frac{l}{6}\geq r$$

As the string does not go slack $T\geq 0$. Use this condition to find the least possible value of the length $AB$.

$\therefore$ least value of $AB$ is $l-\frac{l}{6}=\frac{5l}{6}$

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 16**

**Question:**

One end of a light inextensible string of length $l$ is attached to a fixed point $A$. The other end is attached to a particle $P$ of mass $m$ which is hanging freely at rest at point $B$. The particle $P$ is projected horizontally from $B$ with speed $\sqrt{3gl}$. When $AP$ makes an angle $\theta$ with the downward vertical and the string remains taut, the tension in the string is $T$.

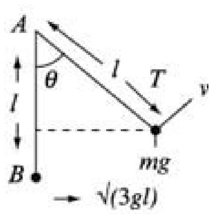
**a** Show that $T = mg(1+3\cos\theta)$.

**b** Find the speed of $P$ at the instant when the string becomes slack.

**c** Find the maximum height above the level of $B$ reached by $P$. **[E]**

**Solution:**

**a**




Use conservation of energy:

$$\frac{1}{2}m(u^2 - v^2) = mgl(1-\cos\theta)$$

$$\therefore v^2 = u^2 - 2gl(1-\cos\theta)$$

$$= 3gl - 2gl + 2gl\cos\theta$$

$$\text{i.e. } v^2 = gl + 2gl\cos\theta \quad ①$$

Use loss in kinetic energy = gain in potential energy.

Make $v^2$ the subject of the formula.

Resolve along the string

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$= \frac{mgl + 2mgl\cos\theta}{l}$$

$$\therefore T = mg(1+3\cos\theta)$$

Use Newton's 2nd Law $F = ma$ with $a = \frac{v^2}{r}$ and $r = l$.

Substitute $v^2$ from equation ①.

**b** Put $T = 0$

Then $1+3\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{3}$

$$\therefore v^2 = gl + 2gl\left(-\frac{1}{3}\right)$$

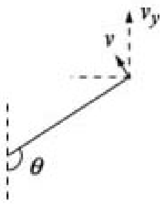
$$= \frac{gl}{3}$$

$$\therefore v = \sqrt{\frac{gl}{3}}$$

When the string becomes slack, $T = 0$.

Solve to find $\cos\theta$ and substitute into equation ① to give $v^2$.

**c**




The particle now moves as a projectile, under gravity. Maximum height is achieved when the vertical component of the velocity is zero.

$$v_y = v\sin\theta = \sqrt{\frac{gl}{3}} \times \frac{2\sqrt{2}}{3}$$

Consider vertical motion

$u = v_y, v = 0, s = h, a = -g$

use $v^2 = u^2 - 2gh$

$$\therefore h = \frac{v_y^2}{2g} = \frac{gl}{3} \times \frac{8}{9} \times \frac{1}{2g}$$

This is the height above the point at which the string becomes slack.

$$\text{i.e. } h = \frac{4l}{27}$$

$$\therefore H = l(1-\cos\theta) + \frac{4l}{27} = \frac{4l}{3} + \frac{4l}{27}$$

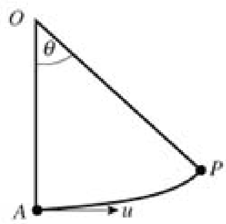
i.e. maximum height above $B$ is $\frac{40l}{27}$.

You need to add $l(1-\cos\theta)$ to obtain the height above $B$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 17**

**Question:**




A particle of mass $m$ is attached to one end of a light inextensible string of length $l$. The other end of the string is attached to a fixed point $O$. The particle is hanging at the point $A$, which is vertically below $O$. It is projected horizontally with speed $u$. When the particle is at the point $P$, $\angle AOP = \theta$, as shown in the diagram. The string oscillates through an angle $\alpha$ on either side of $OA$ where $\cos\alpha = \frac{2}{3}$.

**a** Find $u$ in terms of g and $l$.
When $\angle AOP = \theta$, the tension in the string is $T$.

**b** Show that $T = \frac{mg}{3}(9\cos\theta - 4)$.

**c** Find the range of values of $T$. **[E]**

**Solution:**

**a** Using conservation of energy

$$\frac{1}{2}mu^2 = mgl(1-\cos\alpha)$$

$$= \frac{mgl}{3}$$

$$\therefore u^2 = \frac{2}{3}gl \text{ and } u = \sqrt{\frac{2gl}{3}}$$

> Use loss of kinetic energy = gain in potential energy and substitute $\cos\alpha = \frac{7}{3}$.

**b** Resolve along the string

$$T - mg\cos\theta = \frac{mv^2}{l} \quad ①$$

> Use Newton's 2nd Law $F = ma$ with $a = \frac{v^2}{r}$ and $r = l$.

Conservation of energy:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgl(1-\cos\theta) \quad ②$$

> Use loss of K.E. = gain in P.E.

$$\therefore T = mg\cos\theta + \frac{mu^2}{l} - 2mg(1-\cos\theta)$$

$$= 3mg\cos\theta + \frac{2mg}{3} - 2mg$$

$$= \frac{mg}{3}(9\cos\theta - 4)$$

> Eliminate $v^2$ from equations ① and ② and substitute the value for $u^2$ from part **a**.

**c** Maximum value of $T$ is when $\theta = 0$

$$T_{max} = \frac{5mg}{3}$$

> When $\theta = 0$, $\cos\theta = 1$ which is the maximum value that $\cos\theta$ can take.

Minimum value of $T$ is when

$$\cos\theta = \frac{2}{3}$$

$$T_{min} = \frac{2mg}{3}$$

> $\cos\theta = \frac{2}{3}$ is the minimum value that $\cos\theta$ can take.

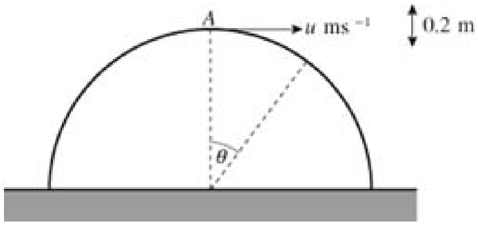
$$\therefore \frac{2mg}{3} \le T \le \frac{5mg}{3}$$

> State the range of values of $T$, as requested.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 18**

**Question:**




A smooth solid hemisphere, of radius 0.8 m and centre $O$, is fixed with its plane face on a horizontal table. A particle of mass 0.5 kg is projected horizontally with speed $u$ m s$^{-1}$ from the highest point $A$ of the hemisphere. The particle leaves the hemisphere at the point $B$, which is a vertical distance of 0.2 m below the level of $A$. The speed of the particle at $B$ is $v$ m s$^{-1}$ and the angle between $OA$ and $OB$ is $\theta$, as shown in the diagram.

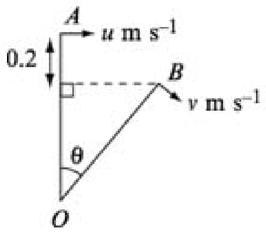
**a** Find the value of $\cos\theta$.

**b** Show that $v^2 = 5.88$.

**c** Find the value of $u$. **[E]**

**Solution:**

**a**




Draw a diagram indicating the lengths given in the question.

As $AO$ is a radius, $AO = 0.8\text{ m}$

$AN = 0.2\text{ m}$

But $\therefore\ ON = 0.8 - 0.2$

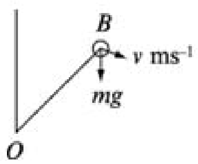
$= 0.6\text{ m}$

$OB$ is a radius, $OB = 0.8\text{ m}$

$\therefore\ \cos\theta = \dfrac{0.6}{0.8} = \dfrac{3}{4}$ or $0.75$

Identify the lengths of sides of $\Delta OBN$ where $BN$ is perpendicular to $OA$.

**b**




As the particle leaves the hemisphere at $B$ the normal reaction $R = 0$.

Resolve along the radius $OB$:

$mg\cos\theta = \dfrac{mv^2}{l} = \dfrac{mv^2}{0.8}$

$\therefore\ v^2 = 0.8g\cos\theta = 0.6g$

$= 5.88\ (3\text{ s.f.})$

Use Newton's 2nd Law $F = ma$ with $a = \dfrac{v^2}{l}$.

This is the answer printed on the question so give 3 s.f.

**c** $\dfrac{1}{2}m \times 5.88 - \dfrac{1}{2}mu^2 = mg \times 0.2$

$\therefore\ u^2 = 5.88 - 0.4g$

$\therefore\ u^2 = 0.2g$

$u = 1.4$

Use gain in kinetic energy = loss in potential energy.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 19**

**Question:**

A smooth solid sphere, with centre $O$ and radius $a$, is fixed to the upper surface of a horizontal table. A particle $P$ is placed on the surface of the sphere at a point $A$, where $OA$ makes an angle $\alpha$ with the upward vertical, and $0 < \alpha < \frac{\pi}{2}$. The particle is released from rest. When $OP$ makes an angle $\theta$ with the upward vertical, and $P$ is still on the surface of the sphere, the speed of $P$ is $v$.

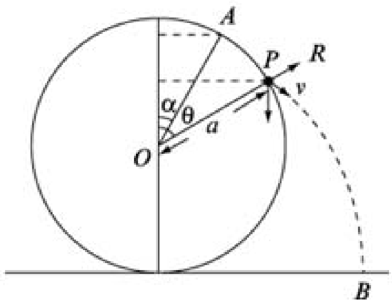
**a** Show that $v^2 = 2ga(\cos\alpha - \cos\theta)$.

Given that $\cos\alpha = \frac{3}{4}$, find

**b** the value of $\theta$ when $P$ loses contact with the sphere.

**c** the speed of $P$ as it hits the table. **[E]**

**Solution:**




Draw a diagram.

**a** $\frac{1}{2}mv^2 = mg(a\cos\alpha - a\cos\theta)$

$\therefore v^2 = 2ga(\cos\alpha - \cos\theta)$

Use gain in kinetic energy = loss in potential energy.

**b** Resolve along the radius

$mg\cos\theta - R = \frac{mv^2}{a}$ and $R = 0$

$\therefore g\cos\theta = 2g(\cos\alpha - \cos\theta)$

$= 2g\left(\frac{3}{4} - \cos\theta\right)$

$\therefore 3g\cos\theta = \frac{3g}{2} \Rightarrow \cos\theta = \frac{1}{2}$

i.e. $\theta = 60°$

When $P$ losses contact with the sphere the normal reaction $R = 0$.

Use Newton's 2nd Law.

Substitute $\cos\alpha = \frac{3}{4}$ and find the value of $\cos\theta$.

**c** From $A$ to $B$

$\frac{1}{2}m\omega^2 = mg(a + a\cos\alpha)$

$\therefore \omega^2 = 2ga\left(1 + \frac{3}{4}\right)$

$\therefore \omega = \left(\frac{7ga}{2}\right)^{\frac{1}{2}}$

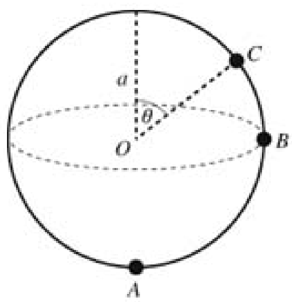
You may use gain in kinetic energy = loss in potential energy.

There are alternative methods involving projectiles, but this is the shortest method.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 20**

**Question:**




The diagram shows a fixed hollow sphere of internal radius $a$ and centre $O$. A particle $P$ of mass $m$ is projected horizontally from the lowest point $A$ of a sphere with speed $\sqrt{\left(\frac{7}{2}ag\right)}$. It moves in a vertical circle, centre $O$, on the smooth inner surface of the sphere. The particle passes through the point $B$, which is in the same horizontal plane as $O$. It leaves the surface of the sphere at the point $C$, where $OC$ makes an angle $\theta$ with the upward vertical.

**a** Find, in terms of $m$ and g, the normal reaction between $P$ and the surface of the sphere at $B$.

**b** Show that $\theta = 60°$.

After leaving the surface of the sphere, $P$ meets it again at the point $A$.

**c** Find, in terms of $a$ and g, the time $P$ takes to travel from $C$ to $A$. **[E]**

**Solution:**

**a** Conservation of energy:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga$$

Use loss in K.E. = gain in P.E.

i.e. $\frac{1}{2}m \times \frac{7ag}{2} - mga = \frac{1}{2}mv^2$ or $v^2 = \frac{3}{2}ga$

Resolve along radius ←

$$R = \frac{mv^2}{a}$$

When the particle is at $B$, the radius is horizontal use $F = ma$.

i.e. $R = \frac{3}{2}mg$

**b** $\frac{1}{2}m \times \frac{7ga}{2} - \frac{1}{2}mV^2 = mga(1+\cos\theta)$ ①

Loss in K.E. = gain in P.E.

Resolving along radius

$$mg\cos\theta = \frac{mV^2}{a} \quad ②$$

The normal reaction is zero at this point $C$, as the particle leaves the sphere.

Eliminate $V^2$ from equations ① and ②

$$ag\cos\theta = \frac{7ga}{2} - 2ga(1+\cos\theta)$$

Solve to find $\cos\theta$.

$$\therefore 3ga\cos\theta = \frac{3ga}{2} \Rightarrow \cos\theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

**c** Consider motion in horizontal direction.

Distance $= a\sin 60^\circ$

$V_x = V\cos 60^\circ$

The particle moves as a projectile.

$$\therefore \text{Time } t = \frac{a\sin 60^\circ}{V\cos 60^\circ}$$

But $V^2 = ag\cos 60^\circ = \frac{ag}{2}$

$$\therefore t = a\tan 60^\circ \div \sqrt{\frac{ag}{2}}$$

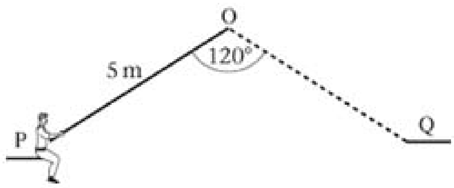
$$= \sqrt{\frac{6a}{g}}$$

This comes from equation ② in part **b**.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 21**

**Question:**




A trapeze artiste of mass 60 kg is attached to the end $A$ of a light inextensible rope $OA$ of length 5 m. The artiste must swing in an arc of a vertical circle, centre $O$, from a platform $P$ to another platform $Q$, where $PQ$ is horizontal. The other end of the rope is attached to the fixed point $O$ which lies in the vertical plane containing $PQ$, with $\angle POQ = 120°$ and $OP = OQ = 5\text{ m}$, as shown in the diagram.

As part of her act, the artiste projects herself from $P$ with speed $\sqrt{15}\text{ m s}^{-1}$ in a direction perpendicular to the rope $OA$ and in the plane $POQ$. She moves in a circular arc towards $Q$. At the lowest point of her path she catches a ball of mass $m$ kg which is travelling towards her with speed $3\text{ m s}^{-1}$ and parallel to $QP$. After catching the ball, she comes to rest at the point $Q$.

By modelling the artiste and the ball as particles and ignoring her air resistance, find

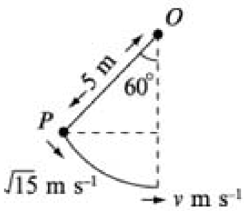
**a** the speed of the artiste immediately before she catches the ball,

**b** the value of $m$,

**c** the tension in the rope immediately after she catches the ball. **[E]**

**Solution:**

**a**




$$\frac{1}{2}mv^2 - \frac{1}{2}m(\sqrt{15})^2$$
$$= mg \times 5(1-\cos 60°)$$

Gain in kinetic energy = loss in potential energy.

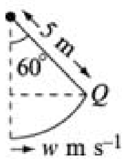
$$\therefore v^2 = 15 + 10g \times \frac{1}{2}$$

i.e. $v = 8$

Speed is $8 \text{ m s}^{-1}$.

**b** Let her velocity after catching the ball be $\omega \text{ m s}^{-1}$.

Consider motion after catching the ball.




$$\therefore \frac{1}{2}M\omega^2 - 0 = Mg5(1-\cos 60°)$$
$$\therefore \omega^2 = 49$$

i.e. $\omega = 7$

Loss in K.E. = Gain in P.E.

Then use C.L.M. when she catches the ball:

| | (60) | (m) |
|---|---|---|
| Before impact | $\rightarrow 8 \text{ m s}^{-1}$ | $\leftarrow 3 \text{ m s}^{-1}$ |
| After impact | (60 + $m$) $\rightarrow 7 \text{ m s}^{-1}$ | |

Use conservation of linear momentum at the instant when she catches the ball.

$$\therefore 60 \times 8 - 3 \times m = (60 + m) \times 7$$

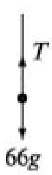
i.e. $480 - 3m = 420 + 7m$

i.e. $60 = 10m$

i.e. $m = 6$

The $3 \text{ m s}^{-1}$ is in the opposite direction from the $8 \text{ m s}^{-1}$, so use a minus sign.

**c**




Use Newton's 2nd Law $F = ma$ where $a = \frac{v^2}{r}$.

R(↑)

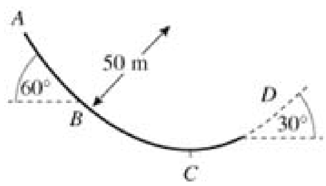
$$T - 66g = 66 \times \frac{7^2}{5}$$

i.e. $T = 132g$ or 1300 N (2 s.f.)

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 22**

**Question:**




The diagram represents the path of a skier of mass 70 kg moving on a ski-slope $ABCD$. The path lies in a vertical plane. From $A$ to $B$, the path is modelled as a straight line inclined at $60°$ to the horizontal. From $B$ to $D$, the path is modelled as an arc of a vertical circle of radius 50 m. The lowest point of the arc $BD$ is $C$.
At $B$, the skier is moving downwards with speed $20 \text{ m s}^{-1}$. At $D$, the path is inclined at $30°$ to the horizontal and the skier is moving upwards. By modelling the slope as smooth and the skier as a particle, find

**a** the speed of the skier at $C$,
**b** the normal reaction of the slope on the skier at $C$,
**c** the speed of the skier at $D$,
**d** the change in the normal reaction of the slope on the skier as she passes $B$.

The model is refined to allow for the influence of friction on the motion of the skier.
**e** State briefly, with a reason, how the answer to part **b** would be affected by using such a model. (No further calculations are expected.) **[E]**

**Solution:**

**a** Let the speed at $C$ be $v$ m s$^{-1}$

$$\frac{1}{2}mv^2 - \frac{1}{2}m \times 20^2 = mg \times 50(1-\cos 60^\circ)$$

Gain in K.E. = loss of P.E.

$$v^2 = 20^2 + 50g \Rightarrow v^2 = 890$$

$\therefore v = 30$ (2 s.f.) $\therefore$ Speed is 30 m s$^{-1}$.

**b** $\uparrow$ at $C$: $R - mg = \frac{m \times 890}{50}$

$\therefore R = 1900$ (2 s.f.)

Normal reaction is 1900 Newtons.

Use $F = ma$ where $a = \frac{v^2}{r}$.

**c** Consider motion $C$ to $D$. Let speed of skier at $D$ be $\omega$ m s$^{-1}$.

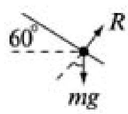
Then $\frac{1}{2}m \times 890 - \frac{1}{2}m\omega^2 = mg \times 50(1-\cos 30^\circ)$

Use loss of K.E. = gain in P.E.

$$\therefore \omega^2 = 890 - 100g(1-\cos 30^\circ)$$
$$= 759$$
$$\therefore \omega = 28 \text{ (to 2 s.f.)}$$

$\therefore$ Speed of $D$ is 28 m s$^{-1}$.

**d** Before:




$R = mg\cos 60^\circ$

After:

$$R - mg\cos 60^\circ = \frac{m \times 20^2}{50}$$

$$\text{i.e.: } R = mg\cos 60^\circ + \frac{m \times 20^2}{50}$$

Resolve perpendicular to the slope at $B$ just before the circular motion and just as circular motion begins.

$\therefore$ Change in $R$ is $\frac{70 \times 20^2}{50} = 560$ N

So change in normal reaction is 560 N.

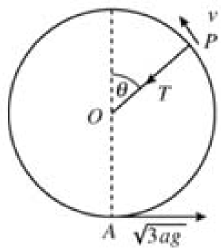
**e** Lower speed at $C \Rightarrow$ the normal reaction is reduced.

Starting 'lower speed' gives the reason for the reduction in normal reaction.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 23**

**Question:**




A particle $P$ of mass $m$ is attached to one end of a light inextensible string of length $a$. The other end of the string is attached to a point $O$. The point $A$ is vertically below $O$, and $OA = a$. The particle is projected horizontally from $A$ with speed $\sqrt{(3ag)}$. When $OP$ makes an angle $\theta$ with the upward vertical through $O$ and the string is still taut, the tension in the string is $T$ and the speed of $P$ is $v$, as shown in the diagram.

**a** Find, in terms of $a$, $g$ and $\theta$, an expression for $v^2$.

**b** Show that $T = (1-3\cos\theta)mg$.

The string becomes slack when $P$ is at the point $B$.

**c** Find, in terms of $a$, the vertical height of $B$ above $A$.

After the string becomes slack, the highest point reached by $P$ is $C$.

**d** Find, in terms of $a$, the vertical height of $C$ above $B$. **[E]**

**Solution:**

**a** Conservation of energy:

$$\frac{1}{2}m3ag - \frac{1}{2}mv^2 = mga(1+\cos\theta)$$

$$\therefore v^2 = ag(1-2\cos\theta)$$

Use loss of K.E. = gain in P.E.

**b** Resolve ↙ along radius:

$$T + mg\cos\theta = \frac{mv^2}{a}$$

$$\therefore T = (1-3\cos\theta)mg$$

Use Newton's 2nd Law $F = ma$ with $a = \frac{v^2}{r}$.

**c** Use $T = 0$, then $\cos\theta = \frac{1}{3}$

String becomes slack when $T = 0$.

$$\therefore \text{height above } A = a + \frac{1}{3}a$$

$$= \frac{4}{3}a$$

Substitute into $h = a(1+\cos\theta)$.

**d** At point $B$, $v^2 = \frac{1}{3}ag$

Consider vertical motion

$$(v\sin\theta)^2 = 2gh$$

As $\cos\theta = \frac{1}{3}, \cos^2\theta = \frac{1}{9}$ and $\sin^2\theta = \frac{8}{9}$

$$\therefore \frac{1}{3}ag \times \frac{8}{9} = 2gh$$

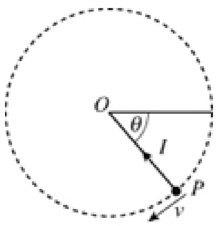
$$\therefore h = \frac{4}{27}a \text{ or } 0.148a$$

This method considers motion under gravity but the solution could be found using energy.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 24**

**Question:**




A particle $P$ of mass $m$ is attached to one end of a light inextensible string of length $a$. The other end of the string is fixed at a point $O$. The particle is held with the string taut and $OP$ horizontal. It is then projected vertically downwards with speed $u$, where $u^2 = \frac{3}{2}ga$. When $OP$ has turned through an angle $\theta$ and the string is still taut, the speed of $P$ is $v$ and the tension in the string is $T$, as shown in the diagram above.

**a** Find an expression for $v^2$ in terms of $a$, g and $\theta$.

**b** Find an expression for $T$ in terms of $m$, g and $\theta$.

**c** Prove that the string becomes slack when $\theta = 210°$.

**d** State, with a reason, whether $P$ would complete a vertical circle if the string were replaced by a light rod.

After the string becomes slack, $P$ moves freely under gravity and is at the same level as $O$ when it is at the point $A$.

**e** Explain briefly why the speed of $P$ at $A$ is $\sqrt{\left(\frac{3}{2}ga\right)}$.

The direction of motion of $P$ at $A$ makes an angle $\phi$ with the horizontal.

**f** Find $\phi$. **[E]**

**Solution:**

**a** $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga\sin\theta$

$\therefore\ v^2 = u^2 + 2ga\sin\theta$

$= \frac{3}{2}ga + 2ga\sin\theta \quad ①$

Gain in K.E. = loss in P.E. from conservation of energy.

**b** Radial equation:

$T - mg\sin\theta = \frac{mv^2}{a}$

$T = mg\sin\theta + \frac{3}{2}mg + 2mg\sin\theta$

$= \frac{3mg}{2}(1 + 2\sin\theta)$

Use Newton's 2nd Law in the direction of the radius.

**c** Put $T = 0$, then $\sin\theta = -\frac{1}{2}$ so $\theta = 210°$

When the string is *slack* $T = 0$.

**d** Set $v = 0$ in ①

Then $\sin\theta = -\frac{3}{4}$, so not a complete circle.

i.e. $P$ would not complete vertical circle.

To complete the circle $v \neq 0$ before reaching the top point.

**e** Consider motion at start and at $A$:
no change in potential energy $\Rightarrow$
no change in kinetic energy

so $v = u = \sqrt{\frac{3}{2}ga}$.

The particle began its motion at the same level as $O$ and thus at the same level as $A$.

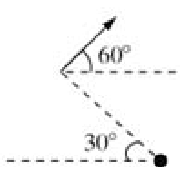
**f** When the string becomes slack

$v^2 = \frac{3}{2}ga + 2ga\left(-\frac{1}{2}\right) = \frac{1}{2}ga$

Its horizontal component of velocity

is $\sqrt{\frac{1}{2}ga}\cos 60°$

Substitute $\sin\theta = -\frac{1}{2}$ from **c** into the equation obtained in part **a**.




When $P$ reaches point $A$, horizontal component of velocity is $\sqrt{\frac{3}{2}ga}\cos\phi$

$\therefore \sqrt{\frac{3ga}{2}}\cos\phi = \sqrt{\frac{1}{2}ga}\cos 60°$

i.e. $\cos\phi = \frac{\cos 60°}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$

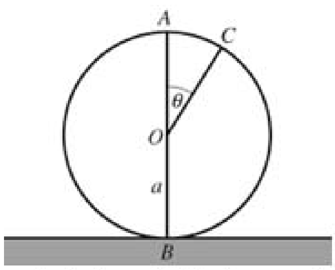
i.e. $\phi = 73.2°$ (3 s.f.)

There are a number of possible methods but conservation of horizontal component of velocity is direct and short.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 25**

**Question:**




A particle is at the highest point $A$ on the outer surface of a fixed smooth sphere of radius $a$ and centre $O$. The lowest point $B$ of the sphere is fixed to a horizontal plane. The particle is projected horizontally from $A$ with speed $u$, where $u < \sqrt{(ag)}$. The particle leaves the sphere at the point $C$, where $OC$ makes an angle $\theta$ with the upward vertical, as shown in the diagram above.

**a** Find an expression for $\cos\theta$ in terms of $u$, $g$ and $a$.

The particle strikes the plane with speed $\sqrt{\left(\frac{9ag}{2}\right)}$.

**b** Find, to the nearest degree, the value of $\theta$. **[E]**

**Solution:**

**a** $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga(1-\cos\theta)$ ← Gain in K.E. = Loss in P.E.

Resolve in radical direction:

$-R + mg\cos\theta = \frac{mv^2}{a}$ ① ← Use equation on motion $F = ma$.

When particle leaves the sphere $R = 0$

$\therefore mg\cos\theta = \frac{mu^2}{a} + 2mg(1-\cos\theta)$ ← Put $R = 0$ and eliminate $v^2$.

$\therefore 3\cos\theta = \frac{u^2}{ag} + 2$ ← Make $\cos\theta$ the subject of the formula.

i.e. $\cos\theta = \frac{2}{3} + \frac{u^2}{3ag}$

**b** Conservation of energy between $C$ and the ground

$\frac{1}{2}m\left(\frac{9ag}{2}\right) - \frac{1}{2}mv^2 = mga(1+\cos\theta)$ ← Energy is the most direct method, though you could use projectiles.

Putting $R = 0$ into equation ① gives

$v^2 = ag\cos\theta$ ← Find an expression for the velocity when the particle leaves the sphere.

$\therefore \frac{9}{4}ag - \frac{1}{2}ag\cos\theta = ag + ag\cos\theta$

$\therefore \frac{5}{4}ag = \frac{3}{2}ag\cos\theta$ ← Substitute $v^2$ into the energy equation to find $\cos\theta$.

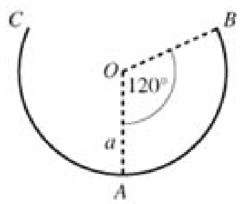
$\therefore \cos\theta = \frac{5}{6}$

So $\theta = 34°$ (nearest degree) ← Give $\theta$ to the nearest degree.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 26**

**Question:**




Part of a hollow spherical shell, centre $O$ and radius $a$, is removed to form a bowl with a plane circular rim. The bowl is fixed with the circular rim uppermost and horizontal. The point $A$ is the lowest point of the bowl. The point $B$ is on the rim of the bowl and $\angle AOB = 120°$, as shown in the diagram above. A smooth small marble of mass $m$ is placed inside the bowl at $A$ and given an initial horizontal speed $u$. The direction of motion of the marble lies in the vertical plane $AOB$. The marble stays in contact with the bowl until it reaches $B$. When the marble reaches $B$, its speed is $v$.

**a** Find an expression for $v^2$.

**b** For the case when $u^2 = 6ga$, find the normal reaction of the bowl on the marble as the marble reaches $B$.

**c** Find the least possible value of $u$ for the marble to reach $B$.

The point $C$ is the other point on the rim of the bowl lying in the vertical plane $OAB$.

**d** Find the value of $u$ which will enable the marble to leave the bowl at $B$ and meet it again at the point $C$. **[E]**

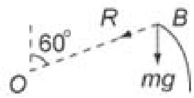
**Solution:**

**a** $\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga(1 + \cos 60°)$

$\therefore v^2 = u^2 - 3ga$ ①

Loss in K.E. = Gain in P.E.

**b** Resolve in a radial direction:




Use equation of motion: $F = ma$ with $a = \frac{v^2}{r}$.

$R + mg\cos 60° = \frac{mv^2}{a}$ ②

When $u^2 = 6ga, v^2 = 3ga$

Use equation ①.

$\therefore R = 3mg - \frac{mg}{2}$

i.e. $R = \frac{5mg}{2}$

Substitute into equation ②.

**c** $R = 0$ at point B $\Rightarrow \frac{mg}{2} = \frac{mv^2}{a}$

$\Rightarrow v^2 = \frac{ag}{2}$

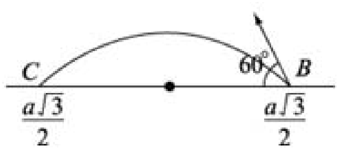
The least value for $u$ will enable particle to teach $B$, but to leave the surface at $B$, so $R = 0$.

$\therefore \frac{ag}{2} = u^2 - 3ga$

Use equation ①.

i.e. $u^2 = \frac{7ga}{2} \Rightarrow u = \sqrt{\frac{7ga}{2}}$

**d**




Consider motion of a projectile under gravity.

$(\rightarrow) B$ to $C: v\cos 60° \times t = a\sqrt{3}$

$\therefore t = \frac{2a\sqrt{3}}{v}$

First consider horizontal component of motion.

$(\uparrow) B$ to $C: 0 = v\sin 60° t - \frac{1}{2}gt^2$

$\Rightarrow t = \frac{2v\sin 60°}{g} = \frac{v\sqrt{3}}{g}$

Then consider the vertical component.

$\therefore \frac{2a\sqrt{3}}{v} = \frac{v\sqrt{3}}{g} \Rightarrow v^2 = 2ag$

Equate the two expressions for $t$ and find $v^2$.

$\therefore$ From ①, $u^2 = 5ag$, i.e. $u = \sqrt{5ag}$

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 27**

**Question:**

A uniform solid is formed by rotating the region enclosed between the curve with equation $y=\sqrt{x}$, the $x$-axis and the line $x=4$, through one complete revolution about the $x$-axis. Find the distance of the centre of mass of the solid from the origin $O$. **[E]**

**Solution:**

$$\bar{x}=\frac{\pi\int y^2 x\,dx}{\pi\int y^2\,dx}$$

You should learn this formula.

Substitute $y=\sqrt{x}$ into the formula.

$$=\frac{\pi\int_0^4 x^2\,dx}{\pi\int_0^4 x\,dx}$$

Integrate the numerator and the denominator.

$$=\frac{\left[\frac{1}{3}\pi x^3\right]_0^4}{\left[\frac{1}{2}\pi x^2\right]_0^4}$$

Evaluate and simplify.

$$=\frac{64}{3}\div 8$$

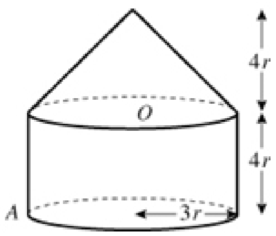
$$=\frac{8}{3}$$

$\therefore$ The centre of mass of the solid is at a distance $\frac{8}{3}$ from $O$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 28**

**Question:**




A toy is formed by joining a uniform solid right circular cone, of base radius $3r$ and height $4r$, to a uniform solid cylinder, also of radius $3r$ and height $4r$. The cone and the cylinder are made from the same material, and the plane face of the cone coincides with a plane face of the cylinder, as shown in the diagram. The centre of this plane face is $O$.

**a** Find the distance of the centre of mass of the toy from $O$.

The point $A$ lies on the edge of the plane face of the cylinder which forms the base of the toy. The toy is suspended from $A$ and hangs in equilibrium.

**b** Find, in degrees to one decimal place, the angle between the axis of symmetry of the toy and the vertical.

The toy is placed with the curved surface of the cone on horizontal ground.

**c** Determine whether the toy will topple. **[E]**

**Solution:**

**a**

| Shape | Mass | Mass ratio | Distance of centre of mass from $O$ |
|---|---|---|---|
| Cylinder | $36\pi\rho r^3$ | 3 | $2r$ |
| Cone | $12\pi\rho r^3$ | 1 | $-r$ |
| Toy | $48\pi\rho r^3$ | 4 | $\bar{x}$ |

Let $\rho$ be the mass per unit volume.

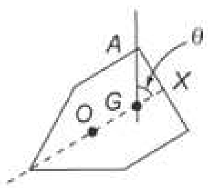
Set out table – downward from $O$ is positive here.

$\circlearrowleft M(O): 3\times 2r + 1\times(-r) = 4\bar{x}$

$x = \frac{5r}{4}$

Centre of mass is $\frac{5r}{4}$ below point $O$ on the figure.

**b**




In $\Delta AGX$, $AX$ is the radius $3r$ of the cylinder and $X$ is the centre of the base of the cylinder.

$AG$ is vertical.
In $\Delta AGX$,

$GX = OX - OG$.

$$\tan\theta = \frac{AX}{GX}$$

$$\therefore \tan\theta = \frac{3r}{4r - \bar{x}}$$

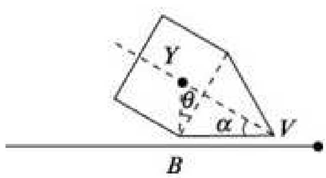
$$= \frac{3r}{4r - \frac{5r}{4}}$$

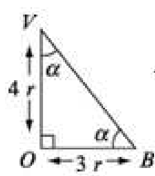
$$= \frac{12}{11}$$

$\therefore \theta = 47.5°$ (1 d.p.)

The question asked for the angle to 1 decimal place.

**c**




The toy will not topple if $G$ (the centre of mass) is vertically above a point between $B$ and $V$.

$\tan\alpha = \frac{3r}{4r}$ from $\Delta VOB$

$\tan\alpha = \frac{OY}{OB} = \frac{OY}{3r}$ from $\Delta BOY$

Let $Y$ be the point directly above $B$ on the axis of symmetry of the toy.

$\therefore \frac{OY}{3r} = \frac{3}{4} \Rightarrow OY = \frac{9r}{4}$

But $OG = \frac{5r}{4}$ so $OG < OY$

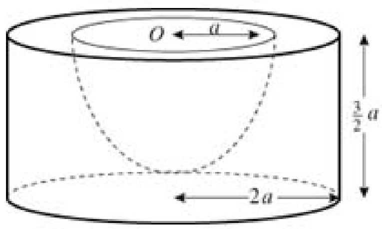
This means that the toy will not topple.

Then if $VG < VY$ toy will not topple.
i.e. $OG < OY$ – no toppling.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 29**

**Question:**




A uniform solid cylinder has radius $2a$ and height $\frac{3}{2}a$. A hemisphere of radius $a$ is removed from the cylinder. The plane face of the hemisphere coincides with the upper plane face of the cylinder, and the centre $O$ of the hemisphere is also the center of this plane face, as shown in the diagram above. The remaining solid is $S$.

**a** Find the distance of the centre of mass of $S$ from $O$.

The lower plane face of $S$ rests in equilibrium on a desk lid which is inclined at an angle $\theta$ to the horizontal. Assuming that the lid is sufficiently rough to prevent $S$ from slipping, and that $S$ is on the point of toppling when $\theta = \alpha$,

**b** find the value of $\alpha$.

Given instead that the coefficient of friction between $S$ and the lid is 0.8, and that $S$ is on the point of sliding down the slide when $\theta = \beta$,

**c** find the value of $\beta$. **[E]**

**Solution:**

**a**

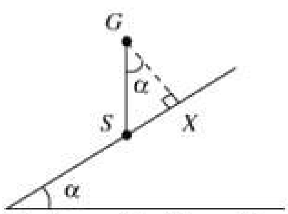
| Shape | Mass | Mass ratios | Distance of centre of mass from $O$ |
|---|---|---|---|
| Cylinder | $\pi\rho(2a)^2\left(\frac{3}{2}a\right)$ | 6 | $\frac{3}{4}a$ |
| Hemi-sphere | $\frac{2}{3}\pi\rho a^3$ | $\frac{2}{3}$ | $\frac{3}{8}a$ |
| Remainder | $\pi\rho\left[6a^3-\frac{2}{3}a^3\right]$ | $\frac{16}{3}$ | $\bar{x}$ |

Draw a table giving mass ratios and distances.

$$\circlearrowleft M(O): 6\times\frac{3}{4}a-\frac{2}{3}\times\frac{3}{8}a = \frac{16}{3}\bar{x}$$
$$\therefore \frac{9}{2}a-\frac{1}{4}a=\frac{16}{3}\bar{x}$$
$$\therefore \bar{x}=\frac{51a}{64} \text{ or } 0.797a\ (3\text{ s.f.})$$

Take moments and find $\bar{x}$, the distance of the centre of mass from $O$.

**b**




Draw a diagram showing $G$, the position of the centre of mass, above $S$.

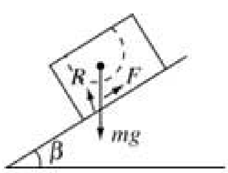
On the point of toppling: $G$ is above $S$ – the lowest point on the bottom circular face.

$$\tan\alpha = \frac{SX}{XG}=\frac{2a}{\frac{3}{2}a-\bar{x}}=\frac{2a}{\frac{45a}{64}}$$

Let $X$ be the centre of the base of the cylinder.

$$\therefore \tan\alpha = \frac{128}{45}\Rightarrow \alpha = 70.6^\circ$$

**c**




Draw a diagram showing the forces acting on the solid.

$R(\nwarrow): R-mg\cos\beta = 0$
$\therefore R = mg\cos\beta$
$R(\nearrow): F-mg\sin\beta = 0$
$\therefore F = mg\sin\beta$

Resolve perpendicular to and parallel to the plane

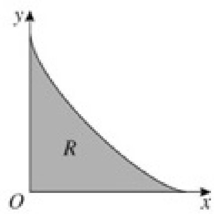
Using $F=\mu R: mg\sin\beta = 0.8mg\cos\beta$
$\therefore \tan\beta = 0.8$
So $\beta = 38.7^\circ$

Use the condition for sliding that $F=\mu R$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 30**

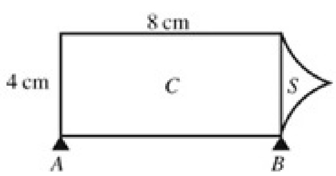
**Question:**




The shaded region $R$ is bounded by part of the curve with equation $y = \frac{1}{2}(x-2)^2$, the $x$-axis and the $y$-axis, as shown above. The unit of length on both axis is 1 cm. A uniform solid $S$ is made by rotating $R$ through $360°$ about the $x$-axis. Using integration,

**a** calculate the volume of the solid $S$, leaving your answer in terms of $\pi$,

**b** show that the centre of mass of $S$ is $\frac{1}{3}$ cm from its plane face.




A tool is modelled as having two components, a solid uniform cylinder $C$ and the solid $S$. The diameter of $C$ is 4 cm and the length of $C$ is 8 cm. One end of $C$ coincides with the plane face of $S$. The components are made of different materials. The weight of $C$ is $10W$ newtons and the weight of $S$ is $2W$ newtons. The tool lies in equilibrium with its axis of symmetry horizontal on two smooth supports $A$ and $B$, which are at the ends of the cylinder, as shown above.

**c** Find the magnitude of the force of the support $A$ on the tool. **[E]**

**Solution:**

**a** $V = \pi \int y^2 \, dx$

$= \pi \int_0^2 \frac{1}{4}(x-2)^4 \, dx$

Substitute $y = \frac{1}{2}(x-2)^2$ into equation.

$= \frac{\pi}{4}\left[\frac{1}{5}(x-2)^5\right]_0^2$

$= \frac{\pi}{4}\left[0 - \left(\frac{-32}{5}\right)\right]$

Use limits 0 and 2 as $y = 0 \Rightarrow x = 2$.

$= \frac{32\pi}{20} = \frac{8\pi}{5}$

**b** $\bar{x} = \dfrac{\pi \int xy^2 \, dx}{\pi \int y^2 \, dx}$

Use the standard formula.

$= \dfrac{\pi \int_0^2 \frac{1}{4} x(x-2)^4 \, dx}{\frac{8\pi}{5}}$

Use a substitution $u = x - 2, \frac{du}{dx} = 1$ and $x = (u+2)$

$= \frac{5}{8\pi} \times \frac{\pi}{4} \int_{-2}^0 (u+2)u^4 \, du$

Change the limits of integration.

$= \frac{5}{32} \int_{-2}^0 u^5 + 2u^4 \, du$

$= \frac{5}{32}\left[\frac{1}{6}u^6 + \frac{2}{5}u^5\right]_{-2}^0$

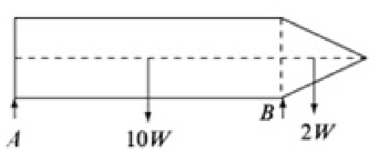
$= \frac{5}{32}\left[0 - \left(\frac{64}{6} - \frac{64}{5}\right)\right]$

$= \frac{5}{32} \times \frac{64}{30}$

$\therefore \bar{x} = \frac{1}{3}$

The centre of mass lies on the axis of symmetry at a distance $\frac{1}{3}$ cm from the plane base.

**c**




Let the reaction at $A$ be $A$, and at $B$ be $B$.

$\circlearrowleft M(B): A \times 8 + 2W \times \frac{1}{3} = 10W \times 4$

$\therefore A = \dfrac{\left(40 - \frac{2}{3}\right)W}{8} = \frac{118W}{24}$

Take moments about $B$ to eliminate the force $B$.

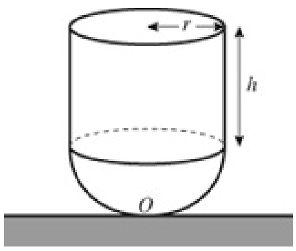
$= \frac{59W}{12}$

This is the force of the support $A$ on the tool.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 31**

**Question:**




A child's toy consists of a uniform solid hemisphere attached to a uniform solid cylinder. The plane face of the hemisphere coincides with the plane face of the cylinder, as shown in the diagram above. The cylinder and the hemisphere each have radius $r$ and the height of the cylinder is $h$. The material of the hemisphere is six times as dense as the material of the cylinder. The toy rests in equilibrium on a horizontal plane with the cylinder above the hemisphere and the axis of the cylinder vertical.

**a** Show that the distance $d$ of the centre of mass of the toy from its lowest point $O$ is given by $d = \dfrac{h^2 + 2hr + 5r^2}{2(h + 4r)}$

When the toy is placed with any point of the curved surface of the hemisphere resting on the plane it will remain in equilibrium.

**b** Find $h$ in terms of $r$. **[E]**

**Solution:**

**a**

| Shape | Mass | Mass ratio | Distance of centre of mass from $O$ |
|---|---|---|---|
| Hemisphere | $\frac{2}{3}\pi r^3 6\rho$ | $4r$ | $\frac{5r}{8}$ |
| Cylinder | $\pi r^2 h\rho$ | $h$ | $\frac{h}{2}+r$ |
| Toy | $\pi r^2\rho(4r+h)$ | $4r+h$ | $\bar{x}$ |

Draw a table showing masses and position of centre of mass.

$$\circlearrowleft M(O)\ 4r\times\frac{5r}{8}+h\left(\frac{h}{2}+r\right)=(4r+h)\bar{x}$$

$$\therefore \bar{x}=\frac{\frac{5}{2}r^2+\frac{h^2}{2}+rh}{4r+h}$$

$$\text{So } d=\frac{h^2+2hr+5r^2}{2(h+4r)}$$

Use moments equation and make $\bar{x}$ the subject of the formula.

**b** As $\bar{x}=r$

$$\therefore h^2+2hr+5r^2=2r(h+4r)$$

$$\text{i.e. } h^2+2hr+5r^2=2rh+8r^2$$

$$\therefore h^2=3r^2$$

$$\text{i.e. } h=\sqrt{3}r$$

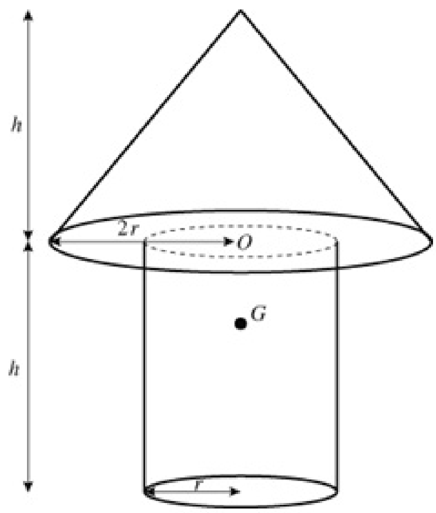
The centre of mass must be at the centre of the hemisphere, so $\bar{x}=r$.

Solve the resulting equation to give $h$ in terms of $r$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 32**

**Question:**




A model tree is made by joining a uniform solid cylinder to a uniform solid cone made of the same material. The centre $O$ of the base of the cone is also the centre of one end of the cylinder, as shown in the diagram. The radius of the cylinder is $r$ and the radius of the base of the cone is $2r$. The height of the cone and the height of the cylinder are each $h$. The centre of mass of the model is at the point $G$.

**a** Show that $OG = \frac{1}{14}h$.

The model stands on a desk top with its plane face in contact with the desk top. The desk top is tilted until it makes an angle $\alpha$ with the horizontal, where $\tan\alpha = \frac{7}{26}$. The desk top is rough enough to prevent slipping and the model is about to topple.

**b** Find $r$ in terms of $h$. **[E]**

**Solution:**

**a**

| Shape | Mass | Mass ratio | Position – distance of centre of mass from $O$ |
|---|---|---|---|
| Cylinder | $\rho\pi r^2 h$ | 3 | $+\frac{h}{2}$ |
| Cone | $\frac{1}{3}\rho\pi(2r)^2 h$ | 4 | $-\frac{h}{4}$ |
| Tree | $\rho\pi r^2 h\left(1+\frac{4}{3}\right)$ | 7 | $\bar{x}$ |

The centre of mass lies on the axis of symmetry.

Draw a table showing masses and positions of centre of mass.

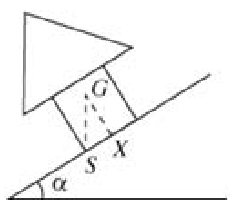
$$\circlearrowleft M(0): 3\times\frac{h}{2}-4\times\frac{h}{4}=7\bar{x}$$

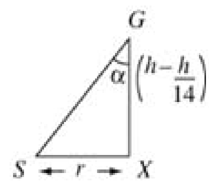
Take moments and let $OG=\bar{x}$.

$$\therefore \frac{3h}{2}-h=7\bar{x}$$

$$\therefore \bar{x}=\frac{h}{14}$$

**b**




Draw a diagram showing $G$ directly above $S$, the lowest point of the base of the cylinder.

Enlarge $\Delta GSX$, where $X$ is the centre of the circular base.

$$\tan\alpha=\frac{r}{h-\frac{h}{14}}=\frac{7}{26}$$

$$\therefore r=\frac{7}{26}\left(\frac{13h}{14}\right)$$

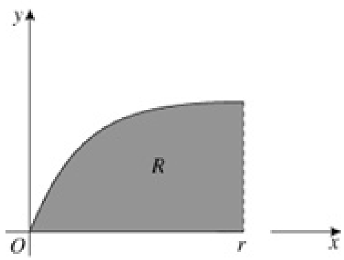
Use $\tan\alpha$ to find $r$ in terms of $h$.

$$r=\frac{1}{4}h$$

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 33**

**Question:**




The diagram shows the region $R$ bounded by the curve with equation $y^2 = rx$, where $r$ is a positive constant, the $x$-axis and the line $x = r$. A uniform solid of revolution $S$ is formed by rotating $R$ through one complete revolution about the $x$-axis.

**a** Show that the distance of the centre of mass of $S$ from $O$ is $\frac{2}{3}r$.

The solid is placed with its plane face on a plane which is inclined at an angle $\alpha$ to the horizontal. The plane is sufficiently rough to prevent $S$ from sliding. Given that $S$ does not topple,

**b** find, to the nearest degree, the maximum value of $\alpha$. **[E]**

**Solution:**

**a** The centre of mass lies on the axis of symmetry $OX$. Let the distance of the centre of mass of $S$ from $O$ be $\bar{x}$ and let it be at point $G$.

Use $\bar{x} = \dfrac{\pi \int y^2 x \, dx}{\pi \int y^2 \, dx}$ — Use the standard formula.

$= \dfrac{\int_0^r rx^2 \, dx}{\int_0^r rx \, dx}$ — Substitute $y^2 = rx$ into the formula.

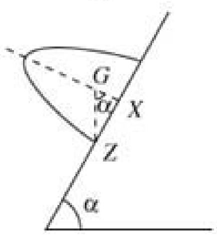
— Integrate numerator and denominator.

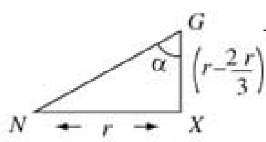
$= \dfrac{\left[\frac{1}{3}rx^3\right]_0^r}{\left[\frac{1}{2}rx^2\right]_0^r}$ — Substitute the limits $\phi$ and $r$.

$= \frac{1}{3}r^4 \div \frac{1}{2}r^3$

$\therefore \bar{x} = \frac{2}{3}r$

**b**




Draw a sketch showing $G$ vertically above the lowest point of the plane face.

Show an enlargement of $\Delta GXN$.
$GX = OX - OG$
$= r - \frac{2}{3}r$

$\tan\alpha = \dfrac{r}{r - \frac{2}{3}r}$

$= r \div \frac{1}{3}r$

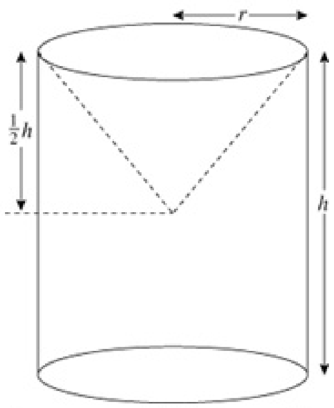
$= 3$

$\therefore \alpha = 72°$ (nearest degree)

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 34**

**Question:**




An ornament $S$ is formed by removing a solid right circular cone, of radius $r$ and height $\frac{1}{2}h$, from a solid uniform cylinder, of radius $r$ and height $h$, as shown in the diagram.

**a** Show that the distance of the centre of mass $S$ from its plane face is $\frac{17}{40}h$.

The ornament is suspended from a point on the circular rim of its open end. It hangs in equilibrium with its axis of symmetry inclined at an angle $\alpha$ to the horizontal. Given that $h = 4r$,

**b** find, in degrees to one decimal place, the value of $\alpha$. **[E]**

**Solution:**

**a**

| Shape | Mass | Mass ratios | Distance of centre of mass from base |
|---|---|---|---|
| Cylinder | $\pi\rho r^2 h$ | 1 | $\frac{h}{2}$ |
| Cone | $\frac{1}{3}\pi\rho r^2\left(\frac{h}{2}\right)$ | $\frac{1}{6}$ | $h-\frac{1}{4}\left(\frac{h}{2}\right)$ |
| Ornament | $\frac{5}{6}\pi\rho r^2 h$ | $\frac{5}{6}$ | $\bar{x}$ |

Draw a table showing masses and position of centre of mass on axis of symmetry.

Take moments about $O$, centre of plane base:

$$1\times\frac{h}{2}-\frac{1}{6}\times\frac{7h}{8}=\frac{5}{6}\bar{x}$$

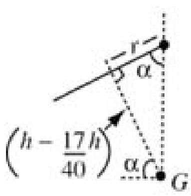
$$\therefore \frac{5}{6}\bar{x}=\frac{h}{2}-\frac{7h}{48}$$

$$=\frac{17h}{48}$$

$$\therefore \bar{x}=\frac{17h}{48}\div\frac{5}{6}=\frac{17h}{48}\times\frac{6}{5}=\frac{17h}{40}$$

Take moments and solve to find $\bar{x}$.

**b**




$$\tan\alpha=\frac{h-\frac{17h}{40}}{r}$$

Also $h=4r$

$$\therefore \tan\alpha=\frac{\frac{23}{40}\times 4r}{r}$$

$$=2.3$$

$$\therefore \alpha=66.5^\circ \text{ (1 d.p.)}$$

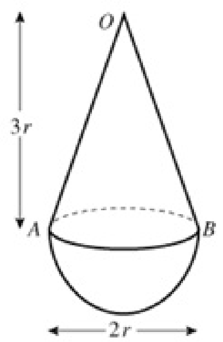
Draw a diagram showing the centre of mass $G$ below the point of suspension.

Use $\tan\alpha$ and inverse tan to find the angle $\alpha$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 35**

**Question:**




A child's toy consists of a uniform solid hemisphere, of mass $M$ and base radius $r$, joined to a uniform solid right circular cone of mass $m$, where $2m < M$. The cone has vertex $O$, base radius $r$ and height $3r$. Its plane face, with diameter $AB$, coincides with the plane face of the hemisphere, as shown in the diagram above.

**a** Show that the distance of the centre of mass of the toy from $AB$ is $\frac{3(M-2m)}{8(M+m)}r$.

The toy is placed with $OA$ on a horizontal surface. The toy is released from rest and does not remain in equilibrium.

**b** Show that $M > 26m$. **[E]**

**Solution:**

**a**

| Shape | Mass | Distance of centre of mass from $AB$ |
|---|---|---|
| Hemisphere | $M$ | $+\frac{3}{8}r$ |
| Cone | $m$ | $-\frac{1}{4}\times 3r$ |
| Toy | $m+M$ | $\bar{x}$ |

Draw a table showing mass and distance of centre of mass from $AB$.

$\circlearrowleft M(AB)$

$$(m+M)\bar{x} = +\frac{3}{8}Mr - \frac{3}{4}mr$$

$$= \frac{3r}{8} - (2m+M)$$

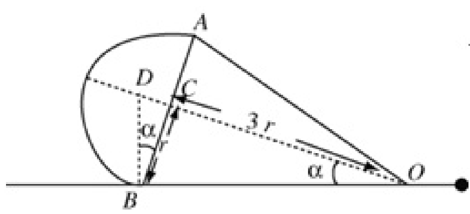
Take moments about $AB$.

$$\therefore \bar{x} = \frac{3(M-2m)}{8(M+m)}r$$

Make $\bar{x}$ the subject of the formula.

where the centre of mass is on the axis of symmetry at a distance $\bar{x}$ from $AB$ in the direction away from $O$.

**b**




Draw a diagram with point $D$ on the axis of symmetry above point $B$.

No equilibrium $\Rightarrow \bar{x} > CD$

$$\tan\alpha = \frac{r}{3r} = \frac{CD}{r}$$

This is the condition for toppling.

$$\therefore CD = \frac{1}{3}r$$

Express $CD$ in terms of $r$.

So $\bar{x} > CD \Rightarrow \frac{3(M-2m)}{8(M+m)}r > \frac{1}{3}r$

i.e. $9(M-2m) > 8(M+m)$

$\therefore M > 26$ m

Substitute and express $M$ in terms of $m$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

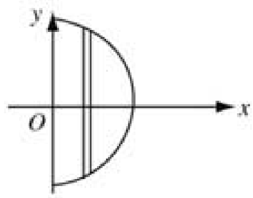
**Review Exercise 2**
**Exercise A, Question 36**

**Question:**

Use integration to show that the centre of mass of a uniform semi-circular lamina, of radius $a$, is a distance $\frac{4a}{3\pi}$ from the mid-point of its straight edge, $O$. A semi-circular lamina, of radius $b$ with $O$ as the mid-point of its straight edge, is removed from the first lamina. Show that the centre of mass of the resulting lamina is at a distance $\bar{x}$ from $O$, where $\bar{x} = \frac{4}{3\pi}\frac{(a^2+ab+b^2)}{(a+b)}$

Hence find the position of the centre of mass of a uniform semi-circular arc of radius $a$. **[E]**

**Solution:**




The centre of mass lies an the $x$-axis from symmetry.
An elemental strip of area is $2y\delta x$

$$\therefore\ \bar{x} = \frac{\rho\int 2xy\delta x}{\rho\int 2y\delta x}$$

Substitute $y = (a^2 - x^2)^{\frac{1}{2}}$ into the formula for $\bar{x}$.

The boundary of the semi-circle has equation $x^2 + y^2 = a^2 \ 0 \le x \le a$

$$\therefore\ \bar{x} = \frac{\rho\int_0^a 2x(a^2 - x^2)^{\frac{1}{2}}\,dx}{\rho \times \frac{\pi a^2}{2}}$$

The denominator is the area of semi-circle times $\rho$. i.e. $\rho \times \frac{\pi a^2}{2}$.

$$= \frac{2}{\pi a^2}\left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a$$

Integrate using the reverse of the chain rule.

$$= \frac{4}{3\pi a^2}\left[a^3\right] = \frac{4a}{3\pi}$$, as required.

| Shape | Mass | Ratio of mass | Distance of centre of mass from $O$ |
|---|---|---|---|
| Semi-circle radius $a$ | $\frac{1}{2}\pi\rho a^2$ | $a^2$ | $\frac{4a}{3\pi}$ |
| Semi-circle radius $b$ | $\frac{1}{2}\pi\rho b^2$ | $b^2$ | $\frac{4b}{3\pi}$ |
| Remainder | $\frac{1}{2}\pi\rho(a^2 - b^2)$ | $a^2 - b^2$ | $\bar{x}$ |

Draw a table and complete with mass ratios and distances of centres of mass from $O$.

$$\circlearrowleft M(O): a^2 \times \frac{4a}{3\pi} - b^2 \times \frac{4b}{3\pi} = (a^2 - b^2)\bar{x}$$

Take moments to find $\bar{x}$.

$$\therefore\ \bar{x} = \frac{4}{3\pi}\frac{\left[a^3 - b^3\right]}{(a^2 - b^2)} = \frac{4}{3\pi}\frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)}$$

$$= \frac{4}{3\pi}\frac{(a^2 + ab + b^2)}{(a+b)}$$

Factorise numerator and denominator to simplify the fraction.

As $b \to a$, the area becomes a circular arc and

$$\bar{x} \to \frac{4}{3\pi} \times \frac{3a^2}{2a} = \frac{2a}{\pi}$$

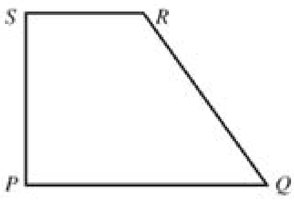
Let $b = a$ in the formula and obtain the limiting value.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 37**

**Question:**

A uniform triangular lamina $ABC$ has $\angle ABC = 90°$ and $AB = c$. Using integration show that the centre of mass of the lamina is at a distance $\frac{1}{3}c$ from $BC$.

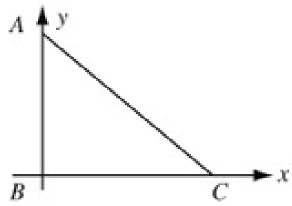



The diagram shows a uniform lamina in which $PQ = PS = 2a, SR = a$.
The centre of mass of the lamina is $G$.

**a** Show that the distance of $G$ from $PS$ is $\frac{7}{9}a$.

**b** Find the distance of $G$ from $PQ$. **[E]**

**Solution:**




Choose $B$ as the origin. The direction $BA$ is the $y$-axis and the direction $BC$ is the $x$-axis.

Let the equation of the line $AC$ be $y = c - mx$.

Then $\bar{x} = \dfrac{\frac{1}{2}\int y^2 \, dx}{\int y \, dx}$

i.e. $\bar{y} = \dfrac{\frac{1}{2}\int_0^{\frac{c}{m}} (c - mx)^2 \, dx}{\int_0^{\frac{c}{m}} c - mx \, dx}$

$y = c - mx$ so when $y = 0$ $x = \dfrac{c}{m}$, which gives the upper limit for the integral.

$$= \frac{\frac{1}{2}\left[\frac{-1}{3m}(c - mx)^3\right]_0^{\frac{c}{m}}}{\left[\frac{-1}{2m}(c - mx)^2\right]_0^{\frac{c}{m}}}$$

Integrate numerator and denominator.

$$= \frac{1}{6}\frac{c^3}{m} \div \frac{1}{2}\frac{c^2}{m}$$

Substitute limits.

$= \frac{1}{3}c$ as required.

| Shape | Mass | Position of centre of mass |
|---|---|---|
| Rectangle | $2a^2\rho$ | $\left(\frac{a}{2}, a\right)$ |
| Triangle | $a^2\rho$ | $\left(\frac{4a}{3}, \frac{2a}{3}\right)$ |
| Lamina | $3a^2\rho$ | $(\bar{x}, \bar{y})$ |

Complete a table with mass and positions of centres of mass.

$$\circlearrowleft 2a^2\rho\begin{pmatrix}\frac{a}{2}\\ a\end{pmatrix} + a^2\rho\begin{pmatrix}\frac{4a}{3}\\ \frac{2a}{3}\end{pmatrix} = 3a^2\rho\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\therefore \begin{pmatrix}a + \frac{4}{3}a\\ 2a + \frac{2}{3}a\end{pmatrix} = \begin{pmatrix}3\bar{x}\\ 3\bar{y}\end{pmatrix}$$

Use a vector equation an as in $M2$.

$$\therefore \bar{x} = \frac{7}{9}a, \ \bar{y} = \frac{8}{9}a$$

**a** Distance of $G$ from $PS$ is $\frac{7}{9}a$

**b** Distance of $G$ from $PQ$ is $\frac{8}{9}a$

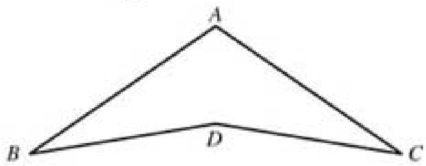
Give the answers clearly.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 38**

**Question:**

A uniform triangular lamina $XYZ$ has $XY = XZ$ and the perpendicular distance of $X$ from $YZ$ is $h$. Prove, by integration, that the centre of mass of the lamina is at a distance $\frac{2h}{3}$ from $X$.



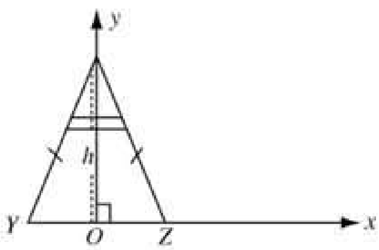

A uniform triangular lamina $ABC$ has $AB = AC = 5a, BC = 8a$ and $D$ is the centre of mass of the lamina. The triangle $BCD$ is removed from the lamina, leaving the plate $ABDC$ shown in the diagram.

**a** Show that the distance of the centre of mass of the plate from $A$ is $\frac{5a}{3}$.

The plate, which is of mass $M$, has a particle of mass $M$ attached at $B$. The loaded plate is suspended from $C$ and hangs in equilibrium.

**b** Prove that in this position $CB$ makes an angle of $\arctan\frac{1}{9}$ with the vertical. **[E]**

**Solution:**




Let $O$ be the mid-point of $YZ$. Let $OZ$ be the $x$-axis and $OX$ the $y$-axis

$$\bar{y} = \frac{\int y2x\,dy}{\int 2x\,dy} \text{ where}$$

The triangle is isosceles and the $y$-axis is the line of symmetry.

$$y = h - mx \Rightarrow x = \frac{h-y}{m}$$

The equation describes $XZ$ which gives $x$ in terms of $y$.

$$\therefore \bar{y} = \frac{2\int_0^h y\left(\frac{h-y}{m}\right)dy}{2\int_0^h \frac{h-y}{m}\,dy}$$

Integrate numerator and denominator.

$$= \frac{\left[\frac{hy^2}{2m} - \frac{y^3}{3m}\right]_0^h}{\left[\frac{hy}{m} - \frac{y^2}{2m}\right]_0^h}$$

Substitute limits.

$$= \frac{h^3}{6m} \div \frac{h^2}{2m}$$

$$= \frac{h}{3}$$

You obtained distance from $O$.

$\therefore$ The centre of mass of the lamina is at a distance $h - \frac{h}{3} = \frac{2h}{3}$ from $X$.

Subtract to give distance from $X$.

**a**

| Shape | Mass | Distance of centre of mass from $A$ |
|---|---|---|
| $\Delta ABC$ | $12\rho a^2$ | $2a$ |
| $\Delta BDC$ | $4\rho a^2$ | $2a+\frac{2a}{3}$ |
| Remainder | $8\rho a^2$ | $\bar{x}$ |

Draw a table completing masses and positions of centres of mass.

$$\circlearrowleft M(A): 12\rho a^2 \times 2a - 4\rho a^2\left(\frac{8a}{3}\right) = 8\rho a^2 \bar{x}$$

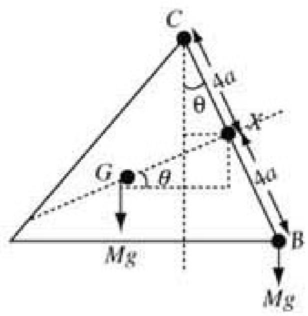
Take moments and make $\bar{x}$ the subject of the formula.

$$\therefore 24\rho a^2 - \frac{32\rho a^3}{3} = 8\rho a^2\,\bar{x}$$

$$\therefore 8\bar{x} = \frac{40a}{3} \Rightarrow \bar{x} = \frac{5a}{3}$$

Draw a diagram showing $B$, $C$ and $G$ and $X$ the mid-point of $BC$

**b**




The distance $GX = 3a - \frac{5a}{3}$

$= \frac{4a}{3}$

$$\circlearrowleft M(C): Mg \times 8a\sin\theta = Mg\left[\frac{4a}{3}\cos\theta - 4a\sin\theta\right]$$

$$\therefore 12Mga\sin\theta = \frac{4}{3}Mga\cos\theta$$

$CB$ makes angle $\theta$ with vertical. Also $GX$ makes an angle $\theta$ with the horizontal.

$$\therefore \frac{\sin\theta}{\cos\theta} = \frac{4}{3} \div 12$$

$$\text{i.e. } \tan\theta = \frac{1}{9}$$

Divide by $\cos\theta$ as $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

So $CB$ makes an angle $\arctan\left(\frac{1}{9}\right)$ with the vertical.

This is the required answer.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 39**

**Question:**

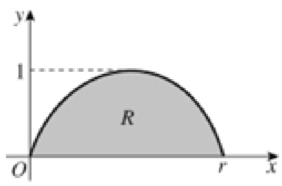



Figure 1

A uniform lamina occupies the region $R$ bounded by the $x$-axis and the curve $y = \sin x, 0 \le x \le \pi$, as shown in Figure 1.

**a** Show, by integration, that the $y$-coordinate of the centre of mass of the lamina is $\frac{\pi}{8}$.

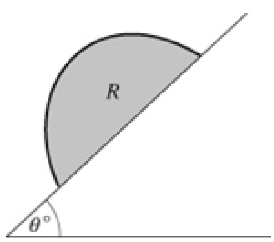



Figure 2

A uniform prism $S$ has cross section $R$. The prism is placed with its rectangular face on a table which is inclined at an angle $\theta$ to the horizontal. The cross section $R$ lies in a vertical plane as shown in Figure 2. The table is sufficiently rough to prevent $S$ sliding. Given that $S$ does not topple,

**b** find the largest possible value of $\theta$. **[E]**

**Solution:**

**a**

$$\bar{y} = \frac{\frac{1}{2}\int y^2 \, dx}{\int y \, dx}$$

$$= \frac{\frac{1}{2}\int_0^{\pi} \sin^2 x \, dx}{\int_0^{\pi} \sin x \, dx}$$

$$= \frac{\frac{1}{4}\int_0^{\pi} 1 - \cos 2x \, dx}{\left[-\cos x\right]_0^{\pi}}$$

$$= \frac{1}{4}\frac{\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}}{1+1}$$

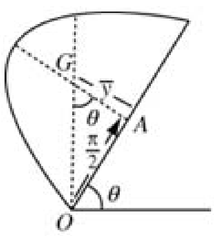
$$= \frac{1}{8}[\pi] = \frac{\pi}{8}$$

Substitute $y = \sin x$ into the standard formula

Use double angle formula to integrate $\sin^2 x$.

Integrate and substitute limits.

**b**




When $S$ is on the point of toppling, $G$ is above $O$.
Let $A$ be the point mid-way along the base of $S$.

$$\tan\theta = \frac{\frac{\pi}{2}}{\bar{y}} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}}$$

$$= 4$$

$\therefore \theta = 76.0°$ (3 s.f.) or $76°$ (nearest degree)

Use triangle $GOA$.

Give your answer to 3 s.f. or to the nearest degree.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 40**

**Question:**

A closed container $C$ consists of a thin uniform hollow hemispherical bowl of radius $a$, together with a lid. The lid is a thin uniform circular disc, also of radius $a$. The centre $O$ of the disc coincides with the centre of the hemispherical bowl. The bowl and its lid are made of the same material.

**a** Show that the centre of mass of $C$ is at a distance $\frac{1}{3}a$ from $O$.

The container $C$ has mass $M$. A particle of mass $\frac{1}{2}M$ is attached to the container at a point $P$ on the circumference of the lid. The container is then placed with a point of its curved surface in contact with a horizontal plane. The container rests in equilibrium with $P$, $O$ and the point of contact in the same vertical plane.

**b** Find, to the nearest degree, the angle made by the line $PO$ with the horizontal. **[E]**

**Solution:**

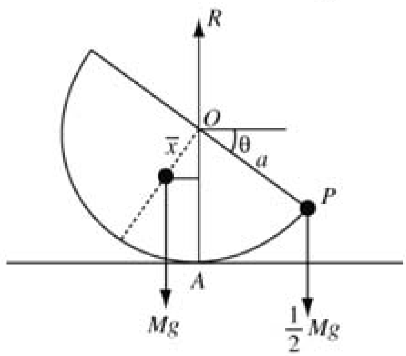
| Shape | Mass | Distance of centre of mass from $O$ |
|---|---|---|
| Circular disc | $\pi a^2 \rho$ | 0 |
| Hemispherical bowl | $2\pi a^2 \rho$ | $\frac{1}{2}a$ |
| Closed container | $3\pi a^2 \rho$ | $\overline{x}$ |

Draw a table with masses or mass ratios and distances of centres of mass from $O$.

$\circlearrowleft M(O): 0 + 2\pi a^2 \rho \times \frac{a}{2} = 3\pi a^2 \rho \overline{x}$

$\therefore\ \overline{x} = \frac{a}{3}$

Take moments and solve to find $\overline{x}$, the distance of the centre of mass of $C$ from $O$.




Draw a diagram showing $O$ above the point of contact with the weight of $P$ acting to one side and the weight of $C$ balancing on the other side.

$\circlearrowleft M(A): Mg \times \frac{a}{3} \sin\theta = \frac{1}{2} Mg \times a\cos\theta$

$\therefore\ \tan\theta = \frac{3}{2}$

$\theta \approx 56°$

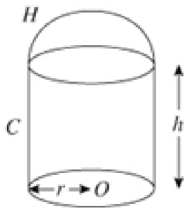
Use $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Give your answer to the nearest degree, as requested.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 41**

**Question:**




A body consists of a uniform solid circular cylinder $C$, together with a uniform solid hemisphere $H$ which is attached to $C$. The plane face of $H$ coincides with the upper plane face of $C$, as shown in the diagram. The cylinder $C$ has base radius $r$, height $h$ and mass $3M$. The mass of $H$ is $2M$. The point $O$ is the centre of the base of $C$.

**a** Show that the distance of the centre of mass of the body from $O$ is $\dfrac{14h+3r}{20}$

The body is placed with its plane face on a rough plane which is inclined at an angle $\alpha$ to the horizontal, where $\tan\alpha = \dfrac{4}{3}$. The plane is sufficiently rough to prevent slipping. Given that the body is on the point of toppling,

**b** find $h$ in terms of $r$.

**Solution:**

**a**

| Shape | Mass | Distance of centre of mass from $O$ |
|---|---|---|
| $H$ | $2M$ | $h+\frac{3}{8}r$ |
| $C$ | $3M$ | $\frac{h}{2}$ |
| Total body | $5M$ | $\bar{x}$ |

Draw a table showing masses and positions of centres of mass.

$$\circlearrowleft M(O): 2M\left(h+\frac{3}{8}r\right)+3M\times\frac{h}{2}=5M\bar{x}$$

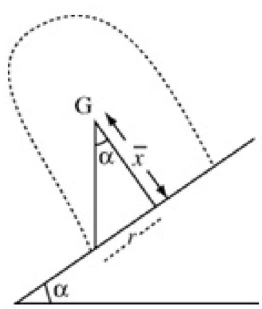
$$\therefore 5\bar{x}=2h+\frac{3}{4}r+\frac{3h}{2}$$

$$=\frac{7h}{2}+\frac{3r}{4}$$

$$\therefore \bar{x}=\frac{14h+3r}{20}$$

Take moments and make $\bar{x}$ the subject of the formula.

**b**




Draw a diagram showing the centre of mass $G$ above the lowest point of the plane circular base.

$$\tan\alpha=\frac{r}{\bar{x}}=\frac{20r}{14h+3r}$$

$$\text{As }\tan\alpha=\frac{4}{3}$$

$$\therefore \frac{20r}{14h+3r}=\frac{4}{3}$$

$$\therefore 60r=56h+12r$$

$$\therefore 48r=56h$$

$$\therefore h=\frac{48}{56}r=\frac{6}{7}r$$

Use trigonometry on the triangle shown in the figure to find $\alpha$.

Make $h$ the subject of the formula.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 42**

**Question:**

A bowl consists of a uniform solid metal hemisphere, of radius $a$ and centre $O$, from which is removed the solid hemisphere of radius $\frac{1}{2}a$ with the same centre $O$.

**a** Show that the distance of the centre of mass of the bowl from $O$ is $\frac{45}{112}a$.

The bowl is fixed with its plane face uppermost and horizontal. It is now filled with liquid. The mass of the bowl is $M$ and the mass of the liquid is $kM$, where $k$ is a constant. Given that the distance of the centre of mass of the bowl and liquid together from $O$ is $\frac{17}{48}a$,

**b** find the value of $k$. **[E]**

**Solution:**

**a**

| Shape | Mass | Mass ratios | Distance of centre of mass from $O$ |
|---|---|---|---|
| Large hemisphere | $\frac{2}{3}\pi\rho a^3$ | 8 | $\frac{3}{8}a$ |
| Small hemisphere | $\frac{2}{3}\pi\rho\left(\frac{a}{2}\right)^3$ | 1 | $\frac{3}{16}a$ |
| Remainder | $\frac{2}{3}\pi\rho\frac{7a^3}{8}$ | 7 | $\bar{x}$ |

Complete a table showing the mass ratios and positions of the centres of mass.

$$\circlearrowleft M(O): 8\times\frac{3}{8}a - 1\times\frac{3}{16}a = 7\bar{x}$$
$$\therefore \frac{45}{16}a = 7\bar{x}$$
$$\therefore \bar{x} = \frac{45a}{112}$$

Take moments and make $\bar{x}$ the subject of the formula.

**b**

| Shape | Mass ratios | Distance of centre of mass from $O$ |
|---|---|---|
| Bowl | $M$ | $\frac{45}{112}a$ |
| Liquid | $kM$ | $\frac{3}{16}a$ |
| Bowl + liquid | $(k+1)M$ | $\frac{17a}{48}$ |

Complete a second table.

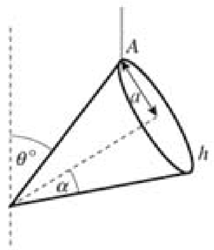
$$\circlearrowleft M(O): M\times\frac{45a}{112} + kM\times\frac{3}{16}a = (k+1)M\times\frac{17a}{48}$$
$$\therefore M\left(\frac{45a}{112}-\frac{17a}{48}\right) = kM\left(\frac{17a}{48}-\frac{3a}{16}\right)$$
$$\therefore k = \frac{2}{7}$$

Take moments and make $k$ the subject of the formula.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

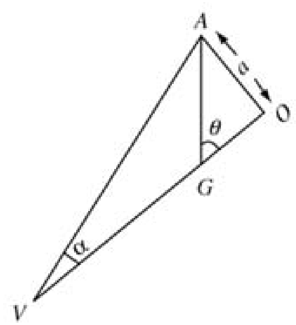
**Review Exercise 2**
**Exercise A, Question 43**

**Question:**




A uniform solid right circular cone has base radius $a$ and semi-vertical angle $\alpha$, where $\tan\alpha = \frac{1}{3}$. The cone is freely suspended by a string attached at a point $A$ on the rim of its base, and hangs in equilibrium with its axis of symmetry making an angle of $\theta°$ with the upward vertical, as shown in the diagram. Find, to one decimal place, the value of $\theta$. **[E]**

**Solution:**




Let $V$ be the vertex of the cone and $O$ be the centre of its base. Let $G$ be the position of its centre of mass. Draw a diagram showing $G$ vertically below $A$.

From $\Delta VAO$, $\tan\alpha = \frac{OA}{OV} = \frac{a}{h}$

where $h$ is the height of the cone.

$\therefore \frac{1}{3} = \frac{a}{h}$

Using $\tan\alpha = \frac{1}{3}$

$\therefore h = 3a$

$\therefore OG = \frac{1}{4} \times 3a$

$= \frac{3a}{4}$

Use the known result for the centre of mass of a solid cone.

Then from $\Delta GAO$

$\tan\theta = \frac{a}{\frac{3a}{4}} = \frac{4}{3}$

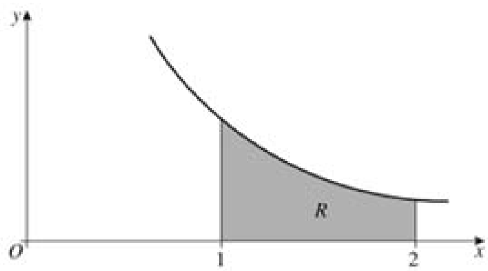
$\therefore \theta = 53.1°$ (1 d.p.)

Give your answer to 1 decimal place as requested.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

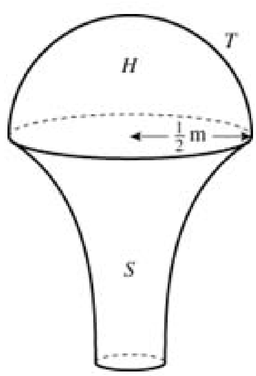
**Review Exercise 2**
**Exercise A, Question 44**

**Question:**




The shaded region $R$ is bounded by the curve with equation $y = \frac{1}{2x^2}$, the $x$-axis and the lines $x = 1$ and $x = 2$, as shown above. The unit of length on each axis is 1 m. A uniform solid $S$ has the shape made by rotating $R$ through $360°$ about the $x$-axis.

**a** Show that the centre of mass of $S$ is $\frac{2}{7}$ m from its larger plane face.




A sporting trophy $T$ is a uniform solid hemisphere $H$ joined to the solid $S$. The hemisphere has radius $\frac{1}{2}$ m and its plane face coincides with the larger plane face of $S$, as shown above. Both $H$ and $S$ are made of the same material.

**b** Find the distance of the centre of mass of $T$ from its plane face. **[E]**

**Solution:**

**a** Use the formula

$$\bar{x} = \frac{\int \pi x y^2 \, dx}{\int \pi y^2 \, dx}$$

Substitute $y = \frac{1}{2x^2}$ into the standard formula.

$$= \frac{\pi \int_1^2 x \times \frac{1}{4} x^{-4} \, dx}{\pi \int_1^2 \frac{1}{4} x^{-4} \, dx}$$

Integrate and use the limits $x = 1$ and $x = 2$.

$$= \frac{\frac{1}{4}\int_1^2 x^{-3} \, dx}{\frac{1}{4}\int_1^2 x^{-4} \, dx}$$

$$= \frac{\left[\frac{-1}{2} x^{-2}\right]_1^2}{\left[\frac{-1}{3} x^{-3}\right]_1^2}$$

$$= \frac{\left[\frac{-1}{8} + \frac{1}{2}\right]}{\left[\frac{-1}{24} + \frac{1}{3}\right]}$$

$$= \frac{3}{8} \div \frac{7}{24}$$

$$= \frac{9}{7}$$

The centre of mass lies on the axis of symmetry at a distance $\frac{9}{7}$ from $O$.

Subtract 1 to give the required distance.

$\therefore$ The centre of mass is $\left(\frac{9}{7} - 1\right) = \frac{2}{7}$ m from the larger plane face.

**b**

| Shape | Mass | Distance of centre of mass from common face |
|---|---|---|
| Solid $S$ | $\frac{7}{96}\pi\rho$ | $\frac{2}{7}$ |
| Hemisphere $H$ | $\frac{2}{3}\pi\rho\times\left(\frac{1}{2}\right)^3$ | $\frac{-3}{8}\times\frac{1}{2}$ |
| Trophy $T$ | $\pi\rho\left(\frac{7}{96}+\frac{1}{12}\right)$ | $\bar{x}$ |

Draw a table of masses and positions of centre of mass.

The mass of solid S, i.e. $\frac{7}{96}\pi\rho$ is obtained from the denominator in part **a**.

$\circlearrowleft M$ (common face):

$$\frac{7}{96}\pi\rho\times\frac{2}{7}-\frac{1}{12}\pi\rho\times\frac{3}{16}=\pi\rho\times\frac{5}{32}\bar{x}$$

$$\text{i.e. } \frac{1}{48}\pi\rho-\frac{1}{64}\pi\rho=\pi\rho\times\frac{5}{32}\bar{x}$$

$$\therefore \frac{1}{192}=\frac{5}{32}\bar{x}$$

$$\therefore \bar{x}=\frac{1}{192}\div\frac{5}{32}=\frac{1}{30}$$

This is the distance of the centre of mass from the common face.

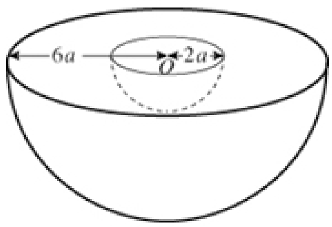
$\therefore$ Distance of centre of mass from the plane face is $1-\frac{1}{30}=\frac{29}{30}$ m or 0.967 m.

As the height of $S$ is 1 m subtract $\bar{x}$ from 1 m to give the required answer.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

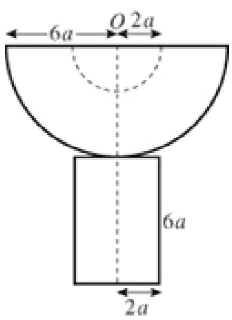
**Review Exercise 2**
**Exercise A, Question 45**

**Question:**




A uniform solid hemisphere, of radius $6a$ and centre $O$, has a solid hemisphere of radius $2a$, and centre $O$, removed to form a bowl $B$ as shown.

**a** Show that the centre of mass of $B$ is $\frac{30}{13}a$ from $O$.




The bowl $B$ is fixed to a plane face of a uniform solid cylinder made from the same material as $B$. The cylinder has radius $2a$ and height $6a$ and the combined solid $S$ has an axis of symmetry which passes through $O$, as shown above.

**b** Show that the centre of mass of $S$ is $\frac{201}{61}a$ from $O$.

The plane surface of the cylindrical base of $S$ is placed on a rough plane inclined at $12°$ to the horizontal. The plane is sufficiently rough to prevent slipping.

**c** Determine whether or not $S$ will topple. **[E]**

**Solution:**

**a**

| Shape | Mass | Mass ratio | Distance of centre of mass from $O$ |
|---|---|---|---|
| Large hemisphere | $\frac{2}{3}\pi\rho(6a)^3$ | 27 | $\frac{3}{8}\times 6a$ |
| Small hemisphere | $\frac{2}{3}\pi\rho(2a)^3$ | 1 | $\frac{3}{8}\times 2a$ |
| Remainder | $\frac{2}{3}\pi\rho(6^3-2^3)a^3$ | 26 | $\overline{x}$ |

Complete a table of mass and positions of centres of mass.

$$\circlearrowleft M(O): 26\overline{x} = 27\times\frac{3}{8}\times 6a - 1\times\frac{3}{8}\times 2a$$

$$\text{i.e. } 26\overline{x} = \frac{243a}{4} - \frac{3a}{4}$$

$$= 60a$$

$$\therefore \overline{x} = \frac{30a}{13}$$

Take moments and make $\overline{x}$ the subject of the formula.

**b**

| Shape | Mass | Mass ratio | Distance of centre of mass from $O$ |
|---|---|---|---|
| Bowl $B$ | $\frac{416}{3}\pi\rho a^3$ | 52 | $\frac{30a}{13}$ |
| Cylinder | $24\pi\rho a^3$ | 9 | $6a+3a$ |
| Combined solid | $\frac{488}{3}\pi\rho a^3$ | 61 | $\overline{y}$ |

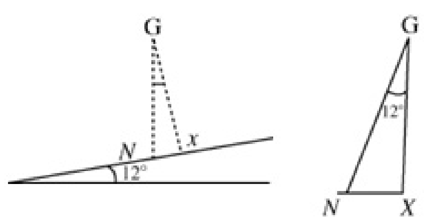
Complete a second table of mass and positions of centres of mass.

$$\circlearrowleft M(O): 52\times\frac{30a}{13} + 9\times 9a = 61\overline{y}$$

$$\therefore 120a + 81a = 61\overline{y}$$

$$\therefore \overline{y} = \frac{201}{61}a$$

**c**




Draw a clear diagram showing position of centre of mass $G$.

Let $N$ be the point on the plane vertically below $G$, and let $X$ be the centre of the cylindrical base of $S$.

$$GX = 12a - \frac{201}{61}a = \frac{531}{61}a$$

$$\therefore NX = GX \tan 12^{0}_{a} = 1.85a \text{ (3 s.f.)}$$

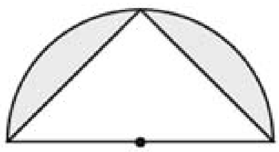
As $NX < 2a$, $S$ will not topple.

$S$ will topple when point $N$ is outside the base circle.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

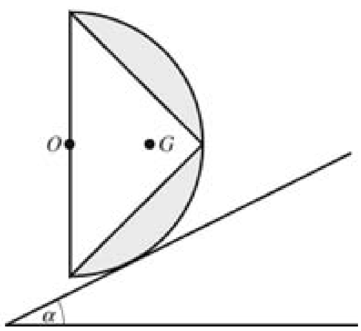
**Review Exercise 2**
**Exercise A, Question 46**

**Question:**




The diagram shows a cross section of a solid formed by the removal of a right circular cone, of base radius $a$ and height $a$, from a uniform solid hemisphere of base radius $a$. The plane bases of the cone and the hemisphere are coincident, both having centre $O$.

Show that $G$, the centre of mass of the solid, is at a distance $\frac{a}{2}$ from $O$.




The second diagram shows a cross section of the solid resting in equilibrium with a point of its curved surface in contact with a rough inclined plane of inclination $\alpha$. Given that $O$ and $G$ are in the same vertical plane through a line of greatest slope of the inclined plane, and that $OG$ is horizontal, show that $\alpha = \frac{\pi}{6}$. Given that $\alpha = \frac{\pi}{6}$, find the smallest possible value of the coefficient of friction between the solid and the plane. **[E]**

**Solution:**

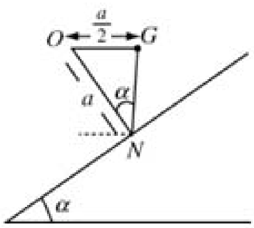
| Shape | Mass | Mass ratio | Distance from $O$ of centre of mass |
|---|---|---|---|
| Hemisphere | $\frac{2}{3}\pi\rho a^3$ | 2 | $\frac{3}{8}a$ |
| Cone | $\frac{1}{3}\pi\rho a^3$ | 1 | $\frac{a}{4}$ |
| Remainder | $\frac{1}{3}\pi\rho a^3$ | 1 | $\bar{x}$ |

Draw a table.

$$\circlearrowleft M(O): 2\times\frac{3}{8}a - 1\times\frac{a}{4} = 1\bar{x}$$

$$\therefore\ \bar{x} = \frac{a}{2}$$

Take moments.



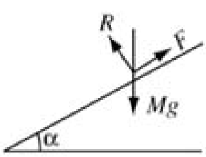

Draw a figure with the centre of mass $G$ above the point of contact $N$.

From $\Delta OGN$ $\sin\alpha = \frac{\frac{a}{2}}{a} = \frac{1}{2}$

$$\therefore\ \alpha = \frac{\pi}{6}$$

Use trigonometry to find $\alpha$.

For limiting equilibrium, when the solid is about to slip $F = \mu R$




Draw another figure showing the normal reaction force $R$ and the friction force $F$.

R(↗) $F = mg\sin\alpha$

R(↖) $R = mg\cos\alpha$

$$F \le \mu R \Rightarrow mg\sin\alpha \le \mu mg\cos\alpha$$

$$\Rightarrow \mu \ge \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \mu \ge \tan\alpha$$

As $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

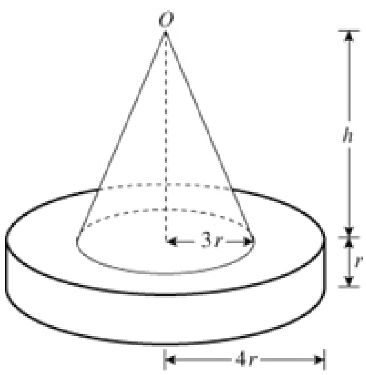
$\alpha = \frac{\pi}{6} \Rightarrow \mu \ge \frac{1}{\sqrt{3}}$ so $\frac{1}{\sqrt{3}}$ is the smallest value of $\mu$

Use $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 47**

**Question:**




An experimental plastic traffic bollard $B$ is made by joining a uniform solid cylinder to a uniform solid right circular cone of the same density. They are joined to form a symmetrical solid, in such a way that the centre of the plane face of the cone coincides with the centre of one of the plane faces of the cylinder, as shown in the diagram. The cylinder has radius $4r$ and height $r$. The cone has vertex $O$, base radius $3r$ and height $h$.

**a** Show that the distance from $O$ of the centre of mass of $B$ is $\frac{32r^2+64rh+9h^2}{4(16r+3h)}$

The bollard is placed on a rough plane which is inclined at an angle $\alpha$ to the horizontal. The circular base of $B$ is in contact with the inclined plane. Given that $h=4r$ and that $B$ is on the point of toppling,

**b** find $\alpha$, to the nearest degree. **[E]**

**Solution:**

**a**

| Shape | Mass | Ratio of mass | Distance from $O$ of centre of mass |
|---|---|---|---|
| Cone | $\frac{1}{3}\pi\rho(3r)^2h$ | $3h$ | $\frac{3}{4}h$ |
| Cylinder | $\pi\rho(4r)^2r$ | $16r$ | $h+\frac{r}{2}$ |
| Bollard | $\pi\rho(16r+3h)r^2$ | $16r+3h$ | $\bar{x}$ |

Complete the table showing mass and positions of centre of mass.

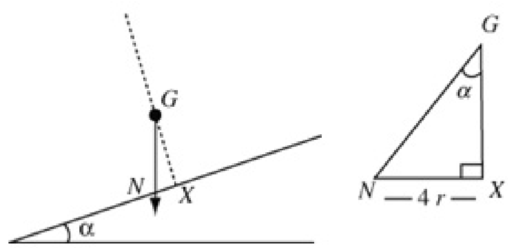
$$\circlearrowleft M(O):\ 3h\times\frac{3h}{4}+16r\left(h+\frac{r}{2}\right)=(16r+3h)\bar{x}$$

Take moments about $O$.

$$\therefore \frac{9h^2}{4}+16rh+8r^2=(16r+3h)\bar{x}$$

$$\therefore \bar{x}=\frac{32r^2+64rh+9h^2}{4(16r+3h)}$$

**b**




Draw a diagram showing the centre of mass $G$ above $N$ a point on the plane which is the lowest point on the base of the bollard.

Let $X$ be the centre point of the base

$GX=OX-\bar{x}$

Also $h=4r, \therefore OX=5r$ and

$$\bar{x}=\frac{32r^2+256r^2+144r^2}{4(28r)}$$

$$=\frac{432r}{112}=\frac{27r}{7}$$

$$\therefore GX=5r-\frac{27r}{7}=\frac{8r}{7}$$

Substitute $h=4r$ into the expression for $\bar{x}$ and evaluate $h+r-\bar{x}$.

From $\Delta GNX$ : $\tan\alpha=\frac{4r}{\frac{8r}{7}}=\frac{28}{8}$

$=3.5$

$\therefore \alpha=74°$ (nearest degree)

Use trigonometry to find angle $\alpha$.

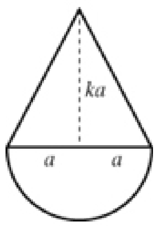
# Solutionbank M3

## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 48**

**Question:**

**a** Show, by integration, that the centre of mass of a uniform solid hemisphere, of radius $R$, is at a distance $\frac{3}{8}R$ from its plane face.




The diagram shows a uniform solid top made from a right circular cone of base radius $a$ and height $ka$ and a hemisphere of radius $a$. The circular plane faces of the cone and hemisphere are coincident.

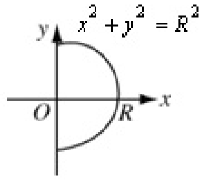
**b** Show that the distance of the centre of mass of the top from the vertex $V$ of the cone is $\frac{(3k^2+8k+3)a}{4(k+2)}$

The manufacturer requires the top to have its centre of mass situated at the centre of the coincident plane faces.

**c** Find the value of $k$ for this requirement. **[E]**

**Solution:**

**a**




Use $\bar{x} = \dfrac{\rho\int \pi x y^2\,dx}{\frac{2}{3}\pi R^3 \rho}$

A hemisphere is generated when a semicircle is notated through 180° about the $x$-axis.

i.e. $\bar{x} = \dfrac{\int_0^R \pi x(R^2 - x^2)\,dx}{\frac{2}{3}\pi R^3}$

Substitute $y^2 = R^2 - x^2$ into the standard equation.

$= \left[\pi R^2 \dfrac{x^2}{2} - \dfrac{\pi x^4}{4}\right]_0^R \div \dfrac{2}{3}\pi R^3$

Integrate and substitute the limits.

$= \dfrac{\pi R^4}{4} \times \dfrac{3}{2\pi R^3}$

$= \dfrac{3}{8}R$

**b**

| Shape | Mass | Mass ratio | Position of centre of mass |
|---|---|---|---|
| Cone | $\frac{1}{3}\pi\rho a^2 ka$ | $k$ | $\frac{3}{4}ka$ |
| Hemisphere | $\frac{2}{3}\pi\rho a^3$ | 2 | $ka + \frac{3}{8}a$ |
| Top | $\frac{1}{3}\pi\rho a^3(k+2)$ | $k+2$ | $\bar{x}$ |

Draw a table with component masses and positions of centres of mass, measured from $V$.

$\circlearrowleft M(V): k\left(\dfrac{3}{4}ka\right) + 2\left(ka + \dfrac{3a}{8}\right) = (k+2)\bar{x}$

Take moments about $V$.

$\therefore \dfrac{3}{4}k^2 a + 2ka + \dfrac{3a}{4} = (k+2)\bar{x}$

i.e. $\bar{x} = \dfrac{(3k^2 + 8k + 3)a}{4(k+2)}$

**c** i.e. $\bar{x} = ka$

$\therefore \dfrac{3k^2 + 8k + 3}{4(k+2)} = k$

i.e. $3k^2 + 8k + 3 = 4k^2 + 8k$

$\therefore k^2 = 3$ so $k = \sqrt{3}$

Put $\bar{x} = ka$ and solve resulting equation to find $k$.

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 49**

**Question:**

**a** A uniform solid hemisphere $H$ has base radius $a$ and the centre of its plane circular face is $C$.

The plane face of a second hemisphere $K$, of radius $\frac{a}{2}$, and made of the same material as $H$, is stuck to the plane face of $H$, so that the centres of the two plane faces coincide at $C$, to form a uniform composite body $S$. Given that the mass of $K$ is $M$, show that the mass of $S$ is $9M$, and find, in terms of $a$, the distance of the centre of mass of the body $S$ from $C$.

**b** A particle $P$, of mass $M$, is attached to a point on the edge of the circular face of $H$ of the body $S$. The body $S$ with $P$ attached is placed with a point of the curved surface of the part $H$ in contact with a horizontal plane and rests in equilibrium. Find the tangent of the acute angle made by the line $PC$ with the horizontal. **[E]**

**Solution:**

**a**

| Shape | Mass | Distance of $C$ of mass from $C$ |
|---|---|---|
| $H$ | $8M$ | $\frac{3a}{8}$ |
| $K$ | $M$ | $\frac{-3a}{16}$ |
| $S$ | $M+8M$ | $\bar{x}$ |

The hemisphere $K$ has mass $M$. $H$ is a similar figure with double the radius so it has 8 times the volume and mass $8M$. $S$ has mass $M+8M=9M$

Draw a table listing masses and positions of centres of mass.

$$\circlearrowleft M(C): 8M\times\frac{3a}{8} - M\times\frac{3a}{16} = 9M\bar{x}$$

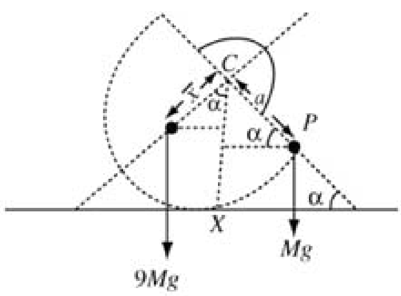
$$\therefore\ 3a-\frac{3}{16}a = 9\bar{x}$$

$$\therefore\ \bar{x} = \frac{45a}{16}\div 9$$

$$= \frac{5a}{16}$$

Take moments and make $\bar{x}$ the subject of the formula.

**b**




Draw a diagram showing the composite body with particle $P$ attached resting in equilibrium with $C$ above the point of contact with the plane, $X$.

$$\circlearrowleft M(X): Mg\times a\cos\alpha = 9Mg\bar{x}\sin\alpha$$

$$\therefore\ \tan\alpha = \frac{a}{9\bar{x}}$$

$$= \frac{16}{45}$$

Take moments about $X$.

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

# Solutionbank M3
## Edexcel AS and A Level Modular Mathematics

**Review Exercise 2**
**Exercise A, Question 50**

**Question:**

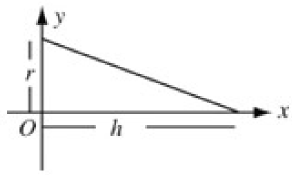
**a** Prove, by integration, that the position of the centre of mass of a uniform solid right circular cone is one quarter of the way up the axis from the base.

**b** From a uniform solid right circular cone of height $H$ is removed a cone with the same base and height $h$, the two axes coinciding. Show that the centre of mass of the remaining solid $S$ is a distance $\frac{1}{4}(3H - h)$ from the vertex of the original cone.

**c** The solid $S$ is suspended by two vertical strings, one attached to the vertex and the other attached to a point on the bounding circular base. Given that $S$ is in equilibrium, with its axis of symmetry horizontal, find, in terms of $H$ and $h$, the ratio of the magnitude of the tension in the string attached to the vertex to that in the other string. **[E]**

**Solution:**

**a**




Draw a diagram and find the equation of the generator of the cone.

The line shown is rotated around the $x$-axis.
The triangular region generates a solid cone.

The equation of the line is $y = r - \frac{r}{h}x$

Use $\bar{x} = \frac{\int \pi x y^2 \mathrm{d}x}{\int \pi y^2 \mathrm{d}x}$

$$= \frac{\pi \int_0^h x\left(r^2 - \frac{2r^2}{h}x + \frac{r^2x^2}{h^2}\right)\mathrm{d}x}{\frac{1}{3}\pi r^2 h}$$

Substitute $y = \left(r - \frac{rx}{h}\right)$ into the standard equation.

$$= \frac{3\pi}{\pi r^2 h}\int_0^h xr^2 - \frac{2r^2}{h}x^2 + \frac{r^2}{h^2}x^3 \mathrm{d}x$$

Expand.

$$= \frac{3\pi}{\pi r^2 h}\left[\frac{x^2r^2}{2} - \frac{2r^2x^3}{3h} + \frac{r^2x^4}{4h^2}\right]_0^h$$

Integrate

$$= \frac{3\pi}{\pi r^2 h}\left[\frac{h^2r^2}{2} - \frac{2h^2r^2}{3h} + \frac{h^4r^2}{4h^2}\right]$$

Substitute limits.

$$= \frac{3}{r^2 h} \times \frac{h^2r^2}{12}$$

$$= \frac{h}{4}$$

**b**

| Shape | Mass | Mass ratio | Distance from base of centre of mass |
|---|---|---|---|
| Large cone | $\frac{1}{3}\pi\rho r^2 h$ | $H$ | $\frac{H}{4}$ |
| Small cone | $\frac{1}{3}\pi\rho r^2 h$ | $h$ | $\frac{h}{4}$ |
| Remainder | $\frac{1}{3}\pi\rho r^2 (H-h)$ | $H-h$ | $\bar{x}$ |

Complete the table.

$$H \times \frac{H}{4} - h \times \frac{h}{4} = (H - h)\bar{x}$$

Take moments.

$$\therefore \bar{x} = \frac{1}{4}\frac{(H^2 - h^2)}{(H - h)} = \frac{1}{4}(H + h)$$

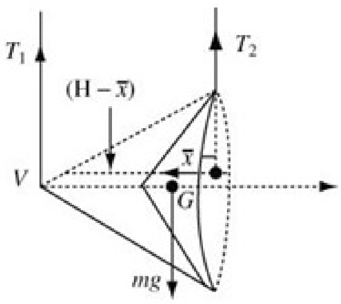
Factorise, using difference of two squares and simplify.

$\therefore$ Distance from the vertex is $H - \frac{1}{4}(H + h)$

Calculate $H - \bar{x}$.

$$= \frac{3}{4}H - \frac{1}{4}h$$

$$= \frac{1}{4}(3H - h)$$




Let the tension in the string attached to the vertex be $T_1$ and tension in the other string be $T_2$.

$$\circlearrowleft M(G): T_2\bar{x} = T_1(H - \bar{x})$$

Take moments about $G$, the position of the centre of mass.

$$\therefore \frac{T_1}{T_2} = \frac{\bar{x}}{H - \bar{x}}$$

$$= \frac{\frac{1}{4}(H + h)}{\frac{1}{4}(3H - h)}$$

This is distance of $G$ from vertex.

This is distance of $G$ from base.

$$= \frac{H + h}{3H - h}$$