Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Further kinematics
Exercise A, Question 1

Question:

A particle P is moving in a straight line. Initially P is moving through a point O with speed 4 m s^{-1}. At time t seconds after passing through O the acceleration of P is $3e^{-0.25t} \text{ m s}^{-2}$ in the direction OP. Find the velocity of the particle at time t seconds.

Solution:

$$v = \int a \, dt = \int 3e^{-0.25t} \, dt$$
$$= -12e^{-0.25t} + A$$

When $t = 0, v = 4$

$$4 = -12 + A \Rightarrow A = 16$$
$$v = 16 - 12e^{-0.25t}$$

The velocity of the particle at time t seconds is $(16 - 12e^{-0.25t}) \text{m s}^{-1}$.

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Further kinematics
Exercise A, Question 2

Question:

A particle P is moving along the x-axis in the direction of x increasing. At time t seconds, the velocity of P is $(t\sin t)\text{m s}^{-1}$. When $t = 0$, P is at the origin. Show that when $t = \frac{\pi}{2}$, P is 1 metre from O.

Solution:

$x = \int v \, dt = \int t \sin t \, dt$

Using integration by parts

$x = -t\cos t + \int \cos t \, dt$

$= -t\cos t + \sin t + A$

When $t = 0, x = 0$

$0 = 0 + 0 + A \Rightarrow A = 0$

$x = -t\cos t + \sin t$

When $t = \frac{\pi}{2}$

$x = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$

Hence P is one metre from O, as required.

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Further kinematics
Exercise A, Question 3

Question:

At time t seconds the velocity, v m s^{-1}, of a particle moving in a straight line is given by $v = \frac{4}{3+2t^0}$ $t \geq 0$.

When $t = 0$, the particle is at a point A. When $t = 3$, the particle is at the point B.

Find the distance between A and B.

Solution:

$s = \int v \, dt = \int \frac{4}{3+2t} \, dt = 2\ln(3+2t) + C$ where s is the displacement from point A.

When $t = 0, s = 0$

$$0 = 2\ln 3 + C \Rightarrow C = -2\ln 3$$

$$s = 2\ln(3+2t) - 2\ln 3 = 2\ln\left(\frac{3+2t}{3}\right)$$

When $t = 3$

$$s = 2\ln\left(\frac{3+6}{3}\right) = 2\ln 3$$

$$AB = 2\ln 3 \text{ m}$$

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Further kinematics
Exercise A, Question 4

Question:

A particle P is moving along the x-axis in the positive direction. At time t seconds the acceleration of P is $4e^{\frac{1}{2}t}$ m s^{-1} in the positive direction. When $t = 0$, P is at rest. Find the distance P moves in the interval $0 \le t \le 2$. Give your answer to 3 significant figures.

Solution:

$v = \int a \, dt = \int 4e^{\frac{1}{2}t} \, dt = 8e^{\frac{1}{2}t} + A$

When $t = 0, v = 0$

$0 = 8 + A \Rightarrow A = -8$

$v = 8e^{\frac{1}{2}t} - 8$

The distance moved in the interval $0 \le t \le 2$ is given by

$s = \int v \, dt = \int_0^2 \left(8e^{\frac{1}{2}t} - 8 \right) dt$

$= \left[16\, e^{\frac{1}{2}t} - 8t \right]_0^2 = (16e^1 - 16) - 16$

$= 16e - 32 \approx 11.5$

The distance moved is 11.5 m (3 s.f.).

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Further kinematics
Exercise A, Question 5

Question:

A particle P is moving along the x-axis. At time t seconds the displacement of P from O is x m and the velocity of P is $(4\cos 3t \text{ m s}^{-1})$, both measured in the direction Ox. When $t = 0$ the particle P is at the origin O. Find

a the magnitude of the acceleration when $t = \frac{\pi}{12}$,

b x in terms of t,

c the smallest positive value of t for which P is at O.

Solution:

a $a = \frac{dv}{dt} = -12\sin 3t$

When $t = \frac{\pi}{12}$

$$a = -12\sin\frac{\pi}{4} = -12 \times \frac{1}{\sqrt{2}} = -6\sqrt{2}$$

The magnitude of the acceleration when $t = \frac{\pi}{12}$ is $6\sqrt{2}$ m s^{-2}.

b $x = \int v \, dt = \int 4\cos 3t \, dt = \frac{4}{3}\sin 3t + A$

$t = 0, x = 0$

When $0 = \frac{4}{3} \times 0 + A \Rightarrow A = 0$

$$x = \frac{4}{3}\sin 3t$$

c When P is at O, $x = 0$

$x = \frac{4}{3}\sin 3t = 0 \Rightarrow \sin 3t = 0$

The smallest positive value of t is given by

$3t = \pi \Rightarrow t = \frac{\pi}{3}$

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Further kinematics
Exercise A, Question 6

Question:

A particle P is moving along a straight line. Initially P is at rest. At time t seconds P has velocity v m s^{-1} and acceleration a m s^{-2} where

$$a = \frac{6t}{(2+t^2)^0} \, t \geq 0.$$

Find v in terms of t.

Solution:

$$v = \int a \, \mathrm{d}t = \int \frac{6t}{(2+t^2)^2} \, \mathrm{d}t$$

Let $u = 2 + t^2$, then $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$

$$v = \int \frac{6t}{(2+t^2)^2} \, \mathrm{d}t = \int \frac{3}{(2+t^2)^2} \times 2t \, \mathrm{d}t$$

$$= \int \frac{3}{u^2} \, \mathrm{d}u = \int 3u^{-2} \, \mathrm{d}u$$

$$= \frac{3u^{-1}}{-1} + A = A - \frac{3}{u}$$

$$= A - \frac{3}{2+t^2}$$

When $t = 0, v = 0$

$$0 = A - \frac{3}{2} \Rightarrow A = \frac{3}{2}$$

$$v = \frac{3}{2} - \frac{3}{2+t^2}$$

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Further kinematics
Exercise A, Question 7

Question:

A particle P is moving along the x-axis. At time t seconds the velocity of P is v m s^{-1} in the direction of x increasing, where

$$v = \begin{cases} 4, & 0 \le t \le 3 \\ 5 - \frac{3}{t}, & 3 < t \le 6 \end{cases}$$

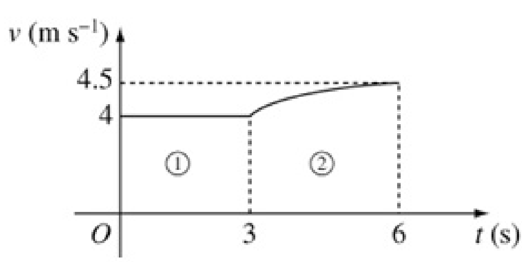
When $t = 0$, P is at the origin O.

a Sketch a velocity–time graph to illustrate the motion of P in the interval $0 \le t \le 6$.

b Find the distance of P from O when $t = 6$.

Solution:

a



b The distance moved in the first three seconds is represented by the area labelled ①.

Let this area be A_1. $A_1 = 3 \times 4 = 12$

The distance travelled in the next three seconds is represented by the area labelled ②.

Let this area be A_2.

$$A_2 = \int_3^6 \left(5 - \frac{3}{t}\right) dt$$

$$= \left[5t - 3\ln t\right]_3^6 = (30 - 3\ln 6) - (15 - 3\ln 3)$$

$$= 15 - 3\ln 2$$

The distance of P from O when $t = 6$ is $(12 + 15 - 3\ln 2)$ m $= (27 - 3\ln 2)$ m .

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Further kinematics
Exercise A, Question 8

Question:

A particle P is moving in a straight line with acceleration $\left(\sin\frac{1}{2}t\right)\text{m s}^{-2}$ at time t seconds, $t \geq 0$. The particle is initially at rest at a point O. Find

a the speed of P when $t = 2\pi$,

b the distance of P from O when $t = \frac{\pi}{2}$.

Solution:

a $v = \int a \, dt = \int \sin\frac{1}{2}t \, dt = -2\cos\frac{1}{2}t + A$

When $t = 0, v = 0$

$$0 = -2 + A \Rightarrow A = 2$$

$$v = 2 - 2\cos\frac{1}{2}t$$

When $t = 2\pi$

$$v = 2 - 2\cos\pi = 2 - (2 \times -1) = 4$$

The speed of P when $t = 2\pi$ is 4 m s^{-1}.

b $x = \int v \, dt = \int\left(2 - 2\cos\frac{1}{2}t\right)dt = 2t - 4\sin\frac{1}{2}t + B$

When $t = 0, x = 0$

$$0 = 0 - 0 + B \Rightarrow B = 0$$

$$x = 2t - 4\sin\frac{1}{2}t$$

When $t = \frac{\pi}{2}$

$$x = 2 \times \frac{\pi}{2} - 4\sin\frac{\pi}{4} = \pi - 4 \times \frac{1}{\sqrt{2}} = \pi - 2\sqrt{2}$$

The distance of P from O when $t = \frac{\pi}{2}$ is $(\pi - 2\sqrt{2})$ m.

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Further kinematics
Exercise A, Question 9

Question:

A particle P is moving along the x-axis. At time t seconds P has velocity v m s^{-1} in the direction x increasing and an acceleration of magnitude $4e^{0.2t}$ m s^{-2} in the direction x decreasing. When $t = 0$, P is moving through the origin with velocity 20 m s^{-1} in the direction x increasing. Find

a v in terms of t,

b the maximum value of x attained by P during its motion.

Solution:

a $v = \int a \, dt = \int -4e^{0.2t} \, dt = -20e^{0.2t} + A$

When $t = 0, v = 20$

$20 = -20 + A \Rightarrow A = 40$

$v = 40 - 20e^{0.2t}$

b $x = \int v \, dt = \int (40 - 20e^{0.2t}) \, dt = 40t - 100e^{0.2t} + B$

When $t = 0, x = 0$

$0 = 0 - 100 + B \Rightarrow B = 100$

$x = 40t - 100e^{0.2t} + 100$

The maximum value of x occurs when

$\frac{dx}{dt} = v = 40 - 20e^{0.2t} = 0$

$e^{0.2t} = 2$

$0.2t = \ln 2$

$t = 5\ln 2$

The maximum value of x is given by

$x = 40 \times 5\ln 2 - 100 \times 2 + 100 = 200\ln 2 - 100$

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Further kinematics
Exercise A, Question 10

Question:

A car is travelling along a straight road. As it passes a sign S, the driver applies the brakes. The car is modelled as a particle. At time t seconds the car is x m from S and its velocity, v m s^{-1}, is modelled by the equation $v = \frac{3200}{c + dt}$ where c and d are constants.

Given that when $t = 0$, the speed of the car is 40 m s^{-1} and its deceleration is 0.5 m s^{-2}, find

a the value of c and the value of d,

b x in terms of t.

Solution:

a $v = \frac{3200}{c + dt}$

When $t = 0, v = 40$

$$40 = \frac{3200}{c} \Rightarrow c = 80$$

$$v = \frac{3200}{80 + dt} = 3200(80 + dt)^{-1}$$

$$a = \frac{dv}{dt} = -3200d(80 + dt)^{-2} = -\frac{3200d}{(80 + dt)^2}$$

When $t = 0, a = -0.5$

$$-\frac{3200d}{80^2} = -0.5 \Rightarrow d = \frac{0.5 \times 80^2}{3200} = 1$$

$$c = 80, d = 1$$

b $x = \int v \, dt = \int \frac{3200}{80 + t} \, dt = 3200 \ln(80 + t) + A$

When $t = 0, x = 0$

$$0 = 3200 \ln 80 + A \Rightarrow A = -3200 \ln 80$$

$$x = 3200 \ln(80 + t) - 3200 \ln 80 = 3200 \ln\left(\frac{80 + t}{80}\right)$$

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Further kinematics
Exercise A, Question 11

Question:

A particle P is moving along a straight line. When $t = 0$, P is passing through a point A. At time t seconds after passing through A the velocity, v m s^{-1}, of P is given by
$v = e^{2t} - 11e^t + 15t$.
Find

a the values of t for which the acceleration is zero,

b the distance of P from A when $t = \ln 3$.

Solution:

a $a = \frac{dv}{dt} = 2e^{2t} - 11e^t + 15 = 0$

$(2e^t - 5)(2e^t - 3) = 0$

$e^t = 2.5, 3$

$t = \ln 2.5, \ln 3$

b $x = \int v \, dt = \int (e^{2t} - 11e^t + 15t) \, dt = \frac{e^{2t}}{2} - 11e^t + \frac{15t^2}{2} + A$

When $t = 0, x = 0$

$0 = \frac{1}{2} - 11 + 0 + A \Rightarrow A = \frac{21}{2}$

$x = \frac{e^{2t}}{2} - 11e^t + \frac{15t^2}{2} + \frac{21}{2}$

When $t = \ln 3$

$$x = \frac{e^{2\ln 3}}{2} - 11e^{\ln 3} + \frac{15(\ln 3)^2}{2} + \frac{21}{2}$$

$$= \frac{9}{2} - 33 + \frac{15(\ln 3)^2}{2} + \frac{21}{2} = \frac{15(\ln 3)^2}{2} - 18 \approx -8.95$$

As distance is a positive quantity, the required distance is $\left(18 - \frac{15(\ln 3)^2}{2}\right)$ m.

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Further kinematics
Exercise A, Question 12

Question:

A particle P moves along a straight line. At time t seconds (where $t \geq 0$) the velocity of P is $[2t + \ln(t+2)]$ m s^{-1}. Find

a the value of t for which the acceleration has magnitude 2.2 m s^{-2}.

b the distance moved by P in the interval $1 \leq t \leq 4$.

Solution:

a $a = \frac{dv}{dt} = 2 + \frac{1}{t+2} = 2.2$

$\frac{1}{t+2} = 0.2 \Rightarrow t + 2 = 5 \Rightarrow t = 3$

b $x = \int v \, dt = \int_1^4 (2t + \ln(t+2)) \, dt$

Using integration by parts

$$\int \ln(t+2) \, dt = \int 1.\ln(t+2) \, dt$$

$$= t\ln(t+2) - \int \frac{t}{t+2} \, dt = t\ln(t+2) - \int \left(1 - \frac{2}{t+2}\right) dt$$

$$= t\ln(t+2) - t + 2\ln(t+2) = (t+2)\ln(t+2) - t$$

Hence $x = \left[t^2 + (t+2)\ln(t+2) - t\right]_1^4$

$$= (16 + 6\ln 6 - 4) - (1 + 3\ln 3 - 1)$$

$$= 12 + 6\ln 6 - 3\ln 3 = 12 + 3\ln 6^2 - 3\ln 3$$

$$= 12 + 3\ln\left(\frac{36}{3}\right) = 12 + 3\ln 12$$

The distance moved by P in the interval $1 \leq t \leq 4$ is $(12 + 3\ln 12)$ m.

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Further kinematics
Exercise B, Question 1

Question:

A particle P moves along the x-axis. At time $t=0$, P passes through the origin O with velocity 5 m s^{-1} in the direction of x increasing. At time t seconds, the velocity of P is $v\text{ m s}^{-1}$ and $OP = x\text{ m}$. The acceleration of P is $\left(2+\frac{1}{2}x\right)\text{m s}^{-2}$, measured in the positive x direction. Find v^2 in terms of x.

Solution:

$$a = 2+\frac{1}{2}x$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2+\frac{1}{2}x$$
$$\frac{1}{2}v^2 = \int\left(2+\frac{1}{2}x\right)dx$$
$$= 2x+\frac{x^2}{4}+A$$
At $x=0, v=5$
$$\frac{1}{2}\times 25 = 0+0+A \Rightarrow A=\frac{25}{2}$$
$$\frac{1}{2}v^2 = 2x+\frac{x^2}{4}+\frac{25}{2}$$
$$v^2 = \frac{x^2}{2}+4x+25$$

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Further kinematics
Exercise B, Question 2

Question:

A particle P moves along a straight line. When its displacement from a fixed point O on the line is x m and its velocity is v m s^{-1}, the deceleration of P is $4x$ m s^{-2}. At $x = 2$, $v = 8$. Find v in terms of x.

Solution:

$$a = -4x$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$$
$$\frac{1}{2}v^2 = \int(-4x)\,dx$$
$$= -2x^2 + A$$
At $x = 2, v = 8$
$$\frac{1}{2} \times 64 = -8 + A \Rightarrow A = 40$$
$$\frac{1}{2}v^2 = -2x^2 + 40$$
$$v^2 = 80 - 4x^2$$
$$v = \pm\sqrt{(80 - 4x^2)}$$

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Further kinematics
Exercise B, Question 3

Question:

A particle P is moving along the x-axis in the direction of x increasing. At $OP = x$ m$(x > 0)$, the velocity of P is v m s^{-1} and its acceleration is of magnitude $\frac{4}{x^2}$ m s^{-2} in the direction of x increasing. Given that at $x = 2, v = 6$ find the value of x for which P is instantaneously at rest.

Solution:

$$a = \frac{4}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4}{x^2}$$

$$\frac{1}{2}v^2 = \int (4x^{-2})\,dx$$

$$= \frac{4x^{-1}}{-1} + A = A - \frac{4}{x}$$

At $x = 2, v = 6$

$$\frac{1}{2} \times 36 = A - 2 \Rightarrow A = 20$$

$$\frac{1}{2}v^2 = 20 - \frac{4}{x}$$

When $v = 0$

$$0 = 20 - \frac{4}{x} \Rightarrow x = \frac{4}{20} = \frac{1}{5}$$

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Further kinematics
Exercise B, Question 4

Question:

A particle P moves along a straight line. When its displacement from a fixed point O on the line is x m and its velocity is v m s^{-1}, the acceleration of P is of magnitude $25x$ m s^{-2} and is directed towards O. At $x = 0, v = 40$. In its motion P is instantaneously at rest at two points A and B. Find the distance between A and B.

Solution:

$a = -25x$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -25x$

$\frac{1}{2}v^2 = \int(-25x)\,\mathrm{d}x$

$= -\frac{25}{2}x^2 + A$

At $x = 0, v = 40$

$\frac{1}{2}\times 1600 = -0 + A \Rightarrow A = 800$

$\frac{1}{2}v^2 = -\frac{25}{2}x^2 + 800$

When $v = 0$

$0 = -\frac{25}{2}x^2 + 800 \Rightarrow x^2 = 64 \Rightarrow x = \pm 8$

$AB = 16$ m

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Further kinematics
Exercise B, Question 5

Question:

A particle P is moving along the x-axis. At $OP = x$ m, the velocity of P is v m s^{-1} and its acceleration is of magnitude kx^2 m s^{-2}, where k is a positive constant, in the direction of x decreasing. At $x = 0$, $v = 16$. The particle is instantaneously at rest at $x = 20$. Find

a the value of k,

b the velocity of P when $x = 10$.

Solution:

a $a = -kx^2$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -kx^2$$

$$\frac{1}{2}v^2 = \int(-kx^2)dx$$

$$= -\frac{kx^3}{3} + A$$

At $x = 0, v = 16$

$$\frac{1}{2} \times 256 = -0 + A \Rightarrow A = 128$$

$$\frac{1}{2}v^2 = -\frac{kx^3}{3} + 128$$

When $v = 0, x = 20$

$$0 = -\frac{8000k}{3} + 128$$

$$k = \frac{3 \times 128}{8000} = \frac{6}{125}$$

b $\frac{1}{2}v^2 = -\frac{6}{125} \times \frac{x^2}{3} + 128$

$$v^2 = 256 - \frac{4}{125}x^3$$

$$x = 10$$

When $v^2 = 256 - \frac{4}{125} \times 1000 = 224$

$$v = \pm\sqrt{224} = \pm 4\sqrt{14}$$

The velocity of P when $x = 10$ is $\pm 4\sqrt{14}$ m s^{-1} as the particle will pass through this position in both directions.

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Further kinematics
Exercise B, Question 6

Question:

A particle P is moving along the x-axis in the direction of x increasing. At $OP = x$ m, the velocity of P is v m s^{-1} and its acceleration is of magnitude $8x^3$ m s^{-2} in the direction PO. At $x = 2$, $v = 32$. Find the value of x for which $v = 8$.

Solution:

$$a = -8x^3$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -8x^3$$
$$\frac{1}{2}v^2 = \int(-8x^3)dx$$
$$= -2x^4 + A$$
At $x = 2, v = 32$
$$\frac{1}{2} \times 1024 = A - 32 \Rightarrow A = 544$$
$$\frac{1}{2}v^2 = 544 - 2x^4$$
$$v^2 = 1088 - 4x^4$$
When $v = 8$
$$64 = 1088 - 4x^4 \Rightarrow x^4 = 256$$
$$x = 256^{\frac{1}{4}} = 4$$

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Further kinematics
Exercise B, Question 7

Question:

A particle P is moving along the x-axis. When the displacement of P from the origin O is x m, the velocity of P is v m s^{-1} and its acceleration is $6\sin\frac{x}{3}$ m s^{-2}. At $x = 0$, $v = 4$. Find

a v^2 in terms of x,

b the greatest possible speed of P.

Solution:

a $a = 6\sin\frac{x}{3}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6\sin\frac{x}{3}$$

$$\frac{1}{2}v^2 = \int\left(6\sin\frac{x}{3}\right)dx$$

$$= -18\cos\frac{x}{3} + A$$

At $x = 0, v = 4$

$$\frac{1}{2} \times 16 = -18 + A \Rightarrow A = 26$$

$$\frac{1}{2}v^2 = -18\cos\frac{x}{3} + 26$$

$$v^2 = 52 - 36\cos\frac{x}{3}$$

b The greatest value of v^2 occurs when $\cos\frac{x}{3} = -1$.

The greatest value of v^2 is given by

$v^2 = 52 + 36 = 88$

$v = \pm\sqrt{88} = \pm 2\sqrt{22}$

The greatest possible speed of P is $2\sqrt{22}$ m s^{-1} (≈ 9.38 m s^{-1}).

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Further kinematics
Exercise B, Question 8

Question:

A particle P is moving along the x-axis. At $x = 0$, the velocity of P is 2 m s^{-1} in the direction of x increasing. At $OP = x\text{ m}$, the velocity of P is $v\text{ m s}^{-1}$ and its acceleration is $(2+3e^{-x})\text{ m s}^{-2}$. Find the velocity of P at $x = 3$. Give your answer to 3 significant figures.

Solution:

$a = 2 + 3e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 + 3e^{-x}$

$\frac{1}{2}v^2 = \int (2+3e^{-x})\,dx$

$= 2x - 3e^{-x} + A$

At $x = 0, v = 2$

$\frac{1}{2} \times 4 = 0 - 3 + A \Rightarrow A = 5$

$\frac{1}{2}v^2 = 2x - 3e^{-x} + 5$

$v^2 = 4x - 6e^{-x} + 10$

At $x = 3$

$v^2 = 12 - 6e^{-3} + 10 = 21.701\ldots$

$v = \sqrt{(21.701\ldots)} = 4.658\ldots$

The velocity of P at $x = 3$ is 4.66 m s^{-1} (3 s.f.), in the direction of x increasing.

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Further kinematics
Exercise B, Question 9

Question:

A particle P moves away from the origin O along the positive x-axis. The acceleration of P is of magnitude $\frac{4}{2x+1}$ m s^{-2}, where $OP = x$ m, directed towards O. Given that the speed of P at O is 4 m s^{-1}, find

a the speed of P at $x = 10$,

b the value of x at which P is instantaneously at rest.

Give your answers to 3 significant figures.

Solution:

a $a = -\frac{4}{2x+1}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{4}{2x+1}$$

$$\frac{1}{2}v^2 = \int\left(-\frac{4}{2x+1}\right)dx$$

$$= -2\ln(2x+1) + A$$

At $x = 0, v = 4$

$$\frac{1}{2}\times 16 = -0 + A \Rightarrow A = 8$$

$$\frac{1}{2}v^2 = -2\ln(2x+1) + 8$$

$$v^2 = 16 - 4\ln(2x+1)$$

At $x = 10$

$$v^2 = 16 - 4\ln 21 = 3.821910\ldots$$

$$v = 1.954\ldots$$

The speed of P at $x = 10$ is 1.95 m s^{-1} (3 s.f.).

b When $v = 0$

$$0 = 16 - 4\ln(2x+1) \Rightarrow \ln(2x+1) = 4$$

$$2x + 1 = e^4 \Rightarrow x = \frac{e^4 - 1}{2} = 26.799\ldots = 26.8 \text{ (3 s.f.)}.$$

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Further kinematics
Exercise B, Question 10

Question:

A particle P is moving along the positive x-axis. At $OP = x$ m, the velocity of P is v m s^{-1} and its acceleration is $\left(x - \frac{4}{x^3}\right)$m s^{-2}. The particle starts from the position where $x = 1$ with velocity 3 m s^{-1} in the direction of x increasing. Find

a v in terms of x,
b the least speed of P during its motion.

Solution:

a $a = x - \frac{4}{x^3}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - \frac{4}{x^3}$$

$$\frac{1}{2}v^2 = \int (x - 4x^{-3})dx$$

$$= \frac{x^2}{2} - \frac{4x^{-2}}{-2} + A = \frac{x^2}{2} + \frac{2}{x^2} + A$$

At $x = 1, v = 3$

$$\frac{1}{2} \times 9 = \frac{1}{2} + 2 + A \Rightarrow A = 2$$

$$\frac{1}{2}v^2 = \frac{x^2}{2} + \frac{2}{x^2} + 2$$

$$v^2 = x^2 + 4 + \frac{4}{x^2} = \left(x + \frac{2}{x}\right)^2$$

$$v = x + \frac{2}{x}$$

b The minimum value of v occurs when $\frac{dv}{dt} = a = 0$

$$x - \frac{4}{x^3} = 0 \Rightarrow x^4 = 4 \Rightarrow x = \sqrt{2} \text{ (as } P \text{ moves on the positive } x\text{-axis, } x > 0)$$

At $x = \sqrt{2}$

$$v = \sqrt{2} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$$

The least speed of P during its motion is $2\sqrt{2}$ m s^{-1}.

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Further kinematics
Exercise B, Question 11

Question:

A particle P is moving along the x-axis. Initially P is at the origin O moving with velocity 15 m s^{-1} in the direction of x increasing. When the displacement of P from O is x m, its acceleration is of magnitude $\left(10+\frac{1}{4}x\right)\text{m s}^{-2}$ directed towards O. Find the distance P moves before first coming to instantaneous rest.

Solution:

$$a=-\left(10+\frac{1}{4}x\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)=-10-\frac{1}{4}x$$

$$\frac{1}{2}v^2=\int\left(-10-\frac{1}{4}x\right)\mathrm{d}x$$

$$=-10x-\frac{x^2}{8}+A$$

At $x=0, v=15$

$$\frac{1}{2}\times 225=-0-0+A\Rightarrow A=\frac{225}{2}$$

$$\frac{1}{2}v^2=-10x-\frac{x^2}{8}+\frac{225}{2}$$

$$v^2=225-20x-\frac{x^2}{4}=-\frac{x^2+80x-900}{4}=-\frac{(x+90)(x-10)}{4}$$

$$v=0\Rightarrow x=10,-90$$

As P is initially moving in the direction of x increasing, it reaches $x=10$ before $x=-90$. The distance P moves before first coming to instantaneous rest is 10 m.

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Further kinematics
Exercise B, Question 12

Question:

A particle P is moving along the x-axis. At time t seconds, P is x m from O, has velocity v m s^{-1} and acceleration of magnitude $6x^{\frac{1}{3}}$ m s^{-2} in the direction of x increasing. When $t = 0$, $x = 8$ and $v = 12$. Find

a v in terms of x,

b x in terms of t.

Solution:

a $a = 6x^{\frac{1}{3}}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^{\frac{1}{3}}$$

$$\frac{1}{2}v^2 = \int 6x^{\frac{1}{3}}\,dx = \frac{6x^{\frac{4}{3}}}{\frac{4}{3}} + A = \frac{9}{2}x^{\frac{4}{3}} + A$$

$$v^2 = 9x^{\frac{4}{3}} + B, \text{ where } B = 2A$$

At $x = 8, v = 12$

$$144 = 9 \times 16 + B \Rightarrow B = 0$$

$$v^2 = 9x^{\frac{4}{3}}$$

$$v = 3x^{\frac{2}{3}}$$

b $v = \frac{dx}{dt} = 3x^{\frac{2}{3}}$

Separating the variables and integrating

$$\int x^{-\frac{2}{3}}\,dx = \int 3\,dt$$

$$3x^{\frac{1}{3}} = 3t + C$$

When $t = 0, x = 8$

$$3 \times 2 = 0 + C \Rightarrow C = 6$$

$$3x^{\frac{1}{3}} = 3t + 6$$

$$x^{\frac{1}{3}} = t + 2$$

$$x = (t + 2)^3$$

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Further kinematics
Exercise C, Question 1

Question:

A particle P moves along a straight line. When the displacement of P from a fixed point on the line is x m, its velocity is v m s^{-1} and its acceleration is of magnitude $\frac{6}{x^2}$ m s^{-2} in the direction of x increasing. At $x = 3, v = 4$.
Find v in terms of x.

Solution:

$$a = \frac{6}{x^2} = 6x^{-2}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^{-2}$$
$$\frac{1}{2}v^2 = \int 6x^{-2}\ dx = \frac{6x^{-1}}{-1} + A$$
$$\frac{1}{2}v^2 = A - \frac{6}{x}$$
At $x = 3, v = 4$
$$\frac{1}{2} \times 16 = A - 2 \Rightarrow A = 10$$
$$\frac{1}{2}v^2 = 10 - \frac{6}{x}$$
$$v^2 = 20 - \frac{12}{x}$$
$$v = \sqrt{\left(20 - \frac{12}{x}\right)}$$

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Further kinematics
Exercise C, Question 2

Question:

A particle P is moving along the x-axis. At time t seconds, the displacement of P from the origin O is x m and the velocity of P is $4e^{0.5t}$ m s^{-1} in the direction Ox. When $t = 0$, P is at O. Find

a x in terms of t,

b the acceleration of P when $t = \ln 9$.

Solution:

a $v = \frac{dx}{dt} = 4e^{0.5t}$

$x = \int 4e^{0.5t}\,dt = 8e^{0.5t} + A$

When $t = 0, x = 0$

$0 = 8 + A \Rightarrow A = -8$

$x = 8e^{0.5t} - 8 = 8(e^{0.5t} - 1)$

b $a = \frac{dv}{dt} = 2e^{0.5t}$

When $t = \ln 9$

$a = 2e^{0.5\ln 9} = 2e^{\ln 3} = 2 \times 3 = 6$

The acceleration of P when $t = \ln 9$ is 6 m s^{-2}.

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Further kinematics
Exercise C, Question 3

Question:

A particle is moving along the x-axis. At time $t = 0$, P is passing through the origin O with velocity 8 m s^{-1} in the direction of x increasing. When P is x m from O, its acceleration is $\left(3 + \frac{1}{4}x\right)\text{m s}^{-2}$ in the direction of x decreasing.

Find the positive value of x for which P is instantaneously at rest.

Solution:

$$a = -\left(3 + \frac{1}{4}x\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -3 - \frac{1}{4}x$$

$$\frac{1}{2}v^2 = \int\left(-3 - \frac{1}{4}x\right)\mathrm{d}x = -3x - \frac{1}{8}x^2 + A$$

At $x = 0, v = 8$

$$32 = -0 - 0 + A \Rightarrow A = 32$$

$$\frac{1}{2}v^2 = 32 - 3x - \frac{1}{8}x^2$$

When $v = 0$

$$0 = 32 - 3x - \frac{1}{8}x^2$$

$$x^2 + 24x - 256 = 0$$

$$(x + 32)(x - 8) = 0$$

As $x > 0$

$$x = 8$$

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Further kinematics
Exercise C, Question 4

Question:

A particle P is moving on the x-axis. When P is a distance x metres from the origin O, its acceleration is of magnitude $\frac{15}{4x^2}$ m s^{-2} in the direction OP. Initially P is at the point where $x = 5$ and is moving toward O with speed 6 m s^{-1}.
Find the value of x where P first comes to rest.

Solution:

$$a = \frac{15}{4x^2} = \frac{15}{4}x^{-2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{15}{4}x^{-2}$$

$$\frac{1}{2}v^2 = \int \frac{15}{4}x^{-2}\ \mathrm{d}x = -\frac{15}{4}x^{-1} + A$$

$$\frac{1}{2}v^2 = A - \frac{15}{4x}$$

At $x = 5, v = -6$

$$18 = A - \frac{15}{20} \Rightarrow A = 18\frac{3}{4} = \frac{75}{4}$$

$$\frac{1}{2}v^2 = \frac{75}{4} - \frac{15}{4x} = \frac{15}{4}\left(5 - \frac{1}{x}\right)$$

When $v = 0$

$$5 - \frac{1}{x} = 0 \Rightarrow x = \frac{1}{5}$$

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Further kinematics
Exercise C, Question 5

Question:

A particle P moves along the x-axis in the direction x increasing. At time t seconds, the velocity of P is $v\text{ m s}^{-1}$ and its acceleration is $20te^{-t^2}\text{ m s}^{-2}$. When $t=0$ the speed of P is 8 m s^{-1}. Find

a v in terms of t,

b the limiting velocity of P.

Solution:

a $a=\dfrac{dv}{dt}=20t\,e^{-t^2}$

$v=\int 20t\,e^{-t^2}\,dt=-10e^{-t^2}+A$

When $t=0, v=8$

$8=-10+A \Rightarrow A=18$

$v=18-10e^{-t^2}$

b As $t\to\infty, e^{-t^2}\to 0$ and $v\to 18$

The limiting velocity of P is 18 m s^{-1}.

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Further kinematics
Exercise C, Question 6

Question:

A particle P moves along a straight line. Initially P is at rest at a point O on the line. A time t seconds, where $t \geq 0$, the acceleration of P is $\frac{18}{(2t+3)^3}$ m s^{-2} directed away from O.
Find the value of t for which the speed of P is 0.48 m s^{-1}.

Solution:

$$a = \frac{dv}{dt} = \frac{18}{(2t+3)^3} = 18(2t+3)^{-3}$$
$$v = \int 18(2t+3)^{-3}\, dt = \frac{18}{-2\times 2}(2t+3)^{-2} + A$$
$$= A - \frac{9}{2(2t+3)^2}$$
When $t = 0, v = 0$
$$0 = A - \frac{9}{2\times 3^2} \Rightarrow A = \frac{1}{2}$$
$$v = \frac{1}{2} - \frac{9}{2(2t+3)^2}$$
When $v = 0.48$
$$0.48 = \frac{1}{2} - \frac{9}{2(2t+3)^2} \Rightarrow \frac{9}{2(2t+3)^2} = 0.02$$
$$(2t+3)^2 = \frac{9}{2\times 0.02} = 225$$
$$t \geq 0$$
As $2t+3 = \sqrt{225} = 15$
$$t = \frac{15-3}{2} = 6$$

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Further kinematics
Exercise C, Question 7

Question:

A particle P is moving along the x-axis. At time t seconds, the velocity of P is v m s^{-1} and the acceleration of P is $(3-x)$m s^{-2} in the direction x increasing. Initially P is at the origin O and is moving with speed 4 m s^{-1} in the direction x increasing. Find

a v^2 in terms of x,
b the maximum value of v.

Solution:

a $a = 3 - x$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - x$$

$$\frac{1}{2}v^2 = \int (3-x)\,dx = 3x - \frac{x^2}{2} + A$$

$$v^2 = B + 6x - x^2, \text{ where } B = 2A$$

At $x = 0, v = 4$

$$16 = B + 0 - 0 \Rightarrow B = 16$$

$$v^2 = 16 + 6x - x^2$$

b $v^2 = 16 + 6x - x^2 = 25 - 9 + 6x - x^2$

$$= 25 - (x-3)^2$$

As $(x-3)^2 \geq 0, v^2 \leq 25$

The greatest value of v is 5.

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Further kinematics
Exercise C, Question 8

Question:

A particle P is moving along the x-axis. At time $t = 0$, P passes through the origin O. After t seconds the speed of P is v m s^{-1}, $OP = x$ metres and the acceleration of P is $\frac{x^2(5-x)}{2}$ m s^{-2} in the direction x increasing. At $x = 10$, P is instantaneously at rest.

Find

a an expression for v^2 in terms of x,

b the speed of P when $t = 0$.

Solution:

a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{x^2(5-x)}{2} = \frac{5x^2}{2} - \frac{x^3}{2}$

$\frac{1}{2}v^2 = \int\left(\frac{5x^2}{2} - \frac{x^3}{2}\right)dx = \frac{5x^3}{6} - \frac{x^4}{8} + A$

$v^2 = \frac{5x^3}{3} - \frac{x^4}{4} + B$, where $B = 2A$

At $x = 10, v = 0$

$0 = \frac{5000}{3} - \frac{10\,000}{4} + B \Rightarrow B = \frac{2500}{3}$

$v^2 = \frac{5x^3}{3} - \frac{x^4}{4} + \frac{2500}{3}$

b When $t = 0, x = 0$

$v^2 = \frac{2500}{3} \Rightarrow v = (\pm)\frac{50}{\sqrt{3}} = (\pm)\frac{50\sqrt{3}}{3}$

The speed of P when $t = 0$ is $\frac{50\sqrt{3}}{3}$ m s^{-1}.

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Further kinematics
Exercise C, Question 9

Question:

A particle P moves away from the origin along the positive x-axis. At time t seconds, the acceleration of P is $\frac{20}{5x+2}$ m s^{-2}, where $OP = x$ m, directed away from O. Given that the speed of P is 3 m s^{-1} at $x = 0$, find, giving your answers to 3 significant figures,

a the speed of P at $x = 12$,

b the value of x when the speed of P is 5 m s^{-1}.

Solution:

a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{20}{5x+2}$

$\frac{1}{2}v^2 = \int \frac{20}{5x+2}\,dx = 4\ln(5x+2) + A$

$v^2 = 8\ln(5x+2) + B$, where $B = 2A$

At $x = 0, v = 3$

$9 = 8\ln 2 + B \Rightarrow B = 9 - 8\ln 2$

$v^2 = 8\ln(5x+2) - 8\ln 2 + 9 = 8\ln\left(\frac{5x+2}{2}\right) + 9$

At $x = 12$

$v^2 = 8\ln 31 + 9 = 36.471\ldots$

$v = \sqrt{36.471\ldots} = 6.039\ldots$

The speed of P at $x = 12$ is 6.04 m s^{-1} (3 s.f.).

b When $v = 5$

$25 = 8\ln\left(\frac{5x+2}{2}\right) + 9$

$\ln\left(\frac{5x+2}{2}\right) = \frac{25-9}{8} = 2$

$\frac{5x+2}{2} = e^2$

$x = \frac{2e^2 - 2}{5} = 2.56$ (3 s.f.)

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Further kinematics
Exercise C, Question 10

Question:

A car moves along a horizontal straight road. At time t seconds the acceleration of the car is $\frac{100}{(2t+5)^2}$ m s^{-2} in the direction of motion of the car. When $t=0$, the car is at rest. Find

a an expression for v in terms of t,

b the distance moved by the car in the first 10 seconds of its motion.

Solution:

a $a = \frac{dv}{dt} = \frac{100}{(2t+5)^2} = 100(2t+5)^{-2}$

$v = \int 100(2t+5)^{-2}\, dt = \frac{100}{2x-1}(2t+5)^{-1} + A$

$= A - \frac{50}{2t+5}$

When $t=0, v=0$

$0 = A - \frac{50}{5} \Rightarrow A = 10$

$v = 10 - \frac{50}{2t+5}$

b $v = \frac{dx}{dt} = 10 - \frac{50}{2t+5}$

$x = \int \left(10 - \frac{50}{2t+5}\right) dt = 10t - 25\ln(2t+5) + B$

When $t=0, x=0$

$0 = -25\ln 5 + B \Rightarrow B = 25\ln 5$

$x = 10t - 25\ln(2t+5) + 25\ln 5$

When $t = 10$

$x = 100 - 25\ln 25 + 25\ln 5 = 100 - 25\ln\frac{25}{5} = 100 - 25\ln 5$

The distance moved by the car in the first 10 seconds of its motion is $(100 - 25\ln 5)$ m (≈ 59.8 m).

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Further kinematics
Exercise C, Question 11

Question:

A particle P is moving in a straight line with acceleration $\cos^2 t$ m s^{-2} at time t seconds. The particle is initially at rest at a point O.

a Find the speed of P when $t = \pi$.

b Show that the distance of P from O when $t = \frac{\pi}{4}$ is $\frac{1}{64}(\pi^2 + 8)$m .

Solution:

a $a = \frac{dv}{dt} = \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$

$v = \int\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)dt = \frac{1}{2}t + \frac{1}{4}\sin 2t + A$

When $t = 0, v = 0$

$0 = 0 + 0 + A \Rightarrow A = 0$

$v = \frac{1}{2}t + \frac{1}{4}\sin 2t$

When $t = \pi$

$v = \frac{\pi}{2} + \frac{1}{4}\sin 2\pi = \frac{\pi}{2} + 0 = \frac{\pi}{2}$

The speed of P when $t = \pi$ is $\frac{\pi}{2}$ m s^{-1}.

b The distance of P from O when $t = \frac{\pi}{4}$ is given by

$x = \int_0^{\frac{\pi}{4}}\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right)dt = \left[\frac{1}{4}t^2 - \frac{1}{8}\cos 2t\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\pi^2}{64} - \frac{1}{8}\cos\frac{\pi}{2}\right) - \left(0 - \frac{1}{8}\right)$

$= \frac{\pi^2}{64} + \frac{1}{8} = \frac{1}{64}(\pi^2 + 8)$

The distance of P from O when $t = \frac{\pi}{4}$ is $\frac{1}{64}(\pi^2 + 8)$m , as required.

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Further kinematics
Exercise C, Question 12

Question:

A particle P is moving along the x-axis. At time t seconds, the velocity of P is v m s^{-1} in the direction of x increasing, where

$$v = \begin{cases} \frac{1}{2}t^2, & 0 \le t \le 4 \\ 8e^{4-t}, & t > 4 \end{cases}$$

When $t = 0$, P is at the origin O. Find

a the acceleration of P when $t = 2.5$,

b the acceleration of P when $t = 5$,

c the distance of P from O when $t = 6$.

Solution:

a When $t = 2.5$, $v = \frac{1}{2}t^2$

$a = \frac{dv}{dt} = t$

When $t = 2.5, a = 2.5$

The acceleration of P when $t = 2.5$ is 2.5 m s^{-2} in the direction of x increasing.

b When $t = 5$, $v = 8e^{4-t}$

$a = \frac{dv}{dt} = -8e^{4-t}$

When $t = 5, a = -8e^{4-5} = -8e^{-1}$

The acceleration of P when $t = 5$ is $8e^{-1} \text{ m s}^{-2}$ in the direction of x decreasing.

c The distance of P from O when $t = 6$ is given by

$$x = \int_0^4 \frac{1}{2}t^2 \, dt + \int_4^6 8e^{4-t} \, dt$$

$$= \left[\frac{t^3}{6}\right]_0^4 + \left[-8e^{4-t}\right]_4^6 = \frac{64}{6} - 8e^{-2} + 8$$

$$= \frac{56}{3} - 8e^{-2}$$

The distance of P from O when $t = 6$ is $\left(\frac{56}{3} - 8e^{-2}\right)\text{m} \approx 17.6 \text{ m}$ (3 s.f.).

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Further kinematics
Exercise C, Question 13

Question:

A particle P is moving along the x-axis. When $t = 0$, P is passing through O with velocity 3 m s^{-1} in the direction of x increasing. When $0 \le x \le 4$ the acceleration is of magnitude $\left(4+\frac{1}{2}x\right)\text{ms}^{-2}$ in the direction of x increasing. At $x = 4$, the acceleration of P changes. For $x > 4$, the magnitude of the acceleration remains $\left(4+\frac{1}{2}x\right)\text{m s}^{-2}$ but it is now in the direction of x decreasing.

a Find the speed of P at $x = 4$.

b Find the positive value of x for which P is instantaneously at rest. Give your answer to 2 significant figures.

Solution:

a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 4 + \frac{1}{2}x$

$\frac{1}{2}v^2 = 4x + \frac{x^2}{4} + A$

$v^2 = 8x + \frac{x^2}{2} + B$, where $B = 2A$

At $x = 0, v = 3$

$9 = 0 + 0 + B \Rightarrow B = 9$

$v^2 = 8x + \frac{x^2}{2} + 9$

At $x = 4$

$v^2 = 32 + 8 + 9 = 49 \Rightarrow v = 7$

The speed of P at $x = 4$ is $7\ \mathrm{m\ s^{-1}}$.

b $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -4 - \frac{1}{2}x$

$\frac{1}{2}v^2 = C - 4x - \frac{x^2}{4}$

$v^2 = D - 8x - \frac{x^2}{2}$, where $D = 2C$

At $x = 4, v = 7$

$49 = D - 32 - 8 \Rightarrow D = 89$

$v^2 = 89 - 8x - \frac{x^2}{2}$

When $v = 0$

$x^2 + 16x = 178 \Rightarrow x^2 + 16x + 64 = 242$

$(x+8)^2 = 242 \Rightarrow x = 11\sqrt{2} - 8$, as $x > 0$

$x = 7.6$ (2 s.f.)

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Further kinematics
Exercise C, Question 14

Question:

A particle P is moving along the x-axis. At time t seconds, P has velocity v m s^{-1} in the direction x increasing and an acceleration of magnitude $\frac{2t+3}{t+1}$ m s^{-2} in the direction x increasing. When $t = 0$, P is at rest at the origin O. Find

a v in terms of t,

b the distance of P from O when $t = 2$.

Solution:

a $a = \frac{dv}{dt} = \frac{2t+3}{t+1} = 2 + \frac{1}{t+1}$

$v = 2t + \ln(t+1) + A$

When $t = 0, v = 0$

$0 = 0 + A \Rightarrow A = 0$

$v = 2t + \ln(t+1)$

b The distance of P from O when $t = 2$ is given by

$$x = \int_0^2 (2t + \ln(t+1))dt$$

Using integration by parts

$$\int \ln(t+1)dt = \int 1\ln(t+1)dt = t\ln(t+1) - \int \frac{t}{t+1}dt$$

$$= t\ln(t+1) - \int\left(1 - \frac{1}{t+1}\right)dt = t\ln(t+1) - t + \ln(t+1)$$

$$= (t+1)\ln(t+1) - t \ (+C)$$

Hence $x = \left[t^2 - t + (t+1)\ln(t+1)\right]_0^2 = 2 + 3\ln 3$

The distance of P from O when $t = 2$ is $(2 + 3\ln 3)$ m.

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Further kinematics
Exercise C, Question 15

Question:

A particle P is moving along the x-axis. At time t seconds P is x m from O, has velocity v m s^{-1} and acceleration of magnitude $(4x+6)$ m s^{-2} in the direction of x increasing. When $t=0$, P is passing through O with velocity 3 m s^{-1} in the direction of x increasing. Find

a v in terms of x,

b x in terms of t.

Solution:

a $a=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)=4x+6$

$\frac{1}{2}v^2=2x^2+6x+A$

$v^2=4x^2+12x+B$, where $B=2A$

At $x=0, v=3$

$9=0+0+B \Rightarrow B=9$

$v^2=4x^2+12x+9=(2x+3)^2$

As v is increasing as x increases

$v=2x+3$

b $v=\frac{\mathrm{d}x}{\mathrm{d}t}=2x+3$

Separating the variables and integrating

$\int \frac{1}{2x+3}\,\mathrm{d}x=\int 1\,\mathrm{d}t$

$\frac{1}{2}\ln(2x+3)=t+C$

$\ln(2x+3)=2t+2C$

$2x+3=\mathrm{e}^{2t+2C}=D\mathrm{e}^{2t}$, where $D=\mathrm{e}^{2C}$

When $t=0, x=0$

$3=D\mathrm{e}^0 \Rightarrow D=3$

$2x+3=3\mathrm{e}^{2t}$

$x=\frac{3}{2}(\mathrm{e}^{2t}-1)$

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Elastic strings and springs
Exercise A, Question 1

Question:

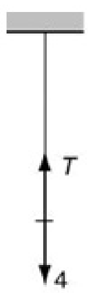
One end of a light elastic string is attached to a fixed point. A force of 4 N is applied to the other end of the string so as to stretch it. The natural length of the string is 3 m and the modulus of elasticity is λ N. Find the total length of the string when

a $\lambda = 30$,

b $\lambda = 12$,

c $\lambda = 16$.

Solution:



$(\uparrow) T = 4$

$T = \frac{\lambda x}{3}$ ← Hooke's Law

So, $\frac{\lambda x}{3} = 4$

$\Rightarrow x = \frac{12}{\lambda}$

$\therefore$ Total length of string, $L = 3 + \frac{12}{\lambda}$

a $\lambda = 30$: $L = 3 + \frac{12}{30}$
$= 3.4$ m

b $\lambda = 12$: $L = 3 + \frac{12}{12}$
$= 4$ m

c $\lambda = 16$: $L = 3 + \frac{12}{16}$
$= 3.75$ m

Solutionbank M3

Edexcel AS and A Level Modular Mathematics

Elastic strings and springs

Exercise A, Question 2

Question:

The length of an elastic spring is reduced to 0.8 m when a force of 20 N compresses it. Given that the modulus of elasticity of the spring is 25 N, find its natural length.

Solution:

by Hooke's Law,

$$20 = \frac{25(l - 0.8)}{l}$$

$$4L = 5l - 4$$

$$4 = l$$

Natural length is 4 m.

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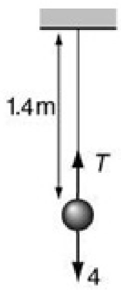
Elastic strings and springs
Exercise A, Question 3

Question:

An elastic spring of modulus of elasticity 20 N has one end fixed. When a particle of mass 1 kg is attached to the other end and hangs at rest, the total length of the spring is 1.4 m. The particle of mass 1 kg is removed and replaced by a particle of mass 0.8 kg. Find the new length of the spring.

Solution:



Let natural length be l

$$(\uparrow) T = g = 9.8$$

$$T = \frac{20(1.4 - l)}{l}$$

$$9.8 = 20\frac{(1.4 - l)}{l}$$

$$9.8l = 28 - 20l$$

$$29.8l = 28 \Rightarrow l = \frac{28}{29.8} = \frac{140}{149}$$

$$0.8g = \frac{20x}{\left(\frac{140}{149}\right)}$$

$$0.8g = \frac{20x \times 149}{140_7}$$

$$\frac{5.6g}{149} = x$$

$$x \approx 0.3683\ldots$$

Total length of string is $0.3683 + \frac{140}{149}$

$= 1.31\text{ m (3 s.f.)}$

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Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise A, Question 4

Question:

A light elastic spring, of natural length a and modulus of elasticity λ, has one end fixed. A scale pan of mass M is attached to its other end and hangs in equilibrium. A mass m is gently placed in the scale pan. Find the distance of the new equilibrium position below the old one.

Solution:

$$Mg = \frac{\lambda x_1}{a} \Rightarrow x_1 = \frac{Mga}{\lambda}$$

$$(M+m)g = \frac{\lambda x_2}{a} \Rightarrow x_2 = \frac{(M+m)ga}{\lambda}$$

$$\therefore x_2 - x_1 = \frac{ga}{\lambda}(M+m-M) = \frac{mga}{\lambda}$$

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Elastic strings and springs
Exercise A, Question 5

Question:

An elastic string has length a_1 when supporting a mass m_1 and length a_2 when supporting a mass m_2. Find the natural length and modulus of elasticity of the string.

Solution:

$$m_1 g = \frac{\lambda(a_1 - l)}{l} \quad ①$$

$$m_2 g = \frac{\lambda(a_2 - l)}{l} \quad ②$$

Dividing,

$$\frac{m_1}{m_2} = \frac{a_1 - l}{a_2 - l}$$

$$m_1(a_2 - l) = m_2(a_1 - l)$$

$$m_1 a_2 - m_2 a_1 = l(m_1 - m_2)$$

$$l = \frac{m_1 a_2 - m_2 a_1}{m_1 - m_2}$$

$$m_1 g - m_2 g = \frac{\lambda a_1}{l} - \lambda - \left(\frac{\lambda a_2}{l} - \lambda\right)$$

$$lg(m_1 - m_2) = \lambda(a_1 - a_2)$$

$$\lambda = gl\frac{(m_1 - m_2)}{(a_1 - a_2)}$$

$$= g\frac{(m_1 - m_2)}{(a_1 - a_2)}\frac{(m_1 a_2 - m_2 a_1)}{(m_1 - m_2)}$$

$$= g\frac{(m_1 a_2 - m_2 a_1)}{(a_1 - a_2)}$$

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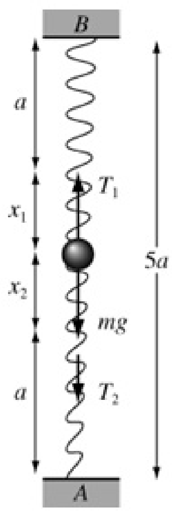
Elastic strings and springs
Exercise A, Question 6

Question:

A light elastic spring has natural length $2a$ and modulus of elasticity $2mg$. A particle of mass m is attached to the mid-point of the spring. One end of the spring, A, is attached to the floor of a room of height $5a$ and the other end is attached to the ceiling of the room at a point B vertically above A. Find the distance of the particle below the ceiling when it is in equilibrium.

Solution:



$$2a + x_1 + x_2 = 5a$$

$$x_1 + x_2 = 3a \quad ①$$

$$(\uparrow) T_1 = mg + T_2$$

$$\frac{2mgx_1}{a} = mg + \frac{2mgx_2}{a}$$

$$2x_1 = a + 2x_2$$

$$x_1 - x_2 = \frac{1}{2}a \quad ②$$

$$① + ②: \quad 2x_1 = \frac{7a}{2} \Rightarrow x_1 = \frac{7a}{4}$$

$$\therefore \text{ Distance below ceiling is } a + \frac{7a}{4} = \frac{11a}{4}$$

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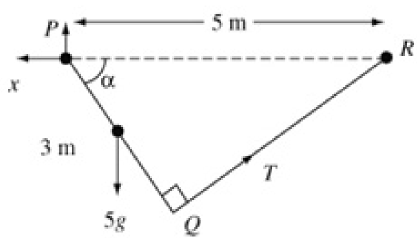
Elastic strings and springs
Exercise A, Question 7

Question:

A uniform rod PQ, of mass 5 kg and length 3 m, has one end, P, smoothly hinged to a fixed point. The other end, Q is attached to one end of a light elastic string of modulus of elasticity 30 N. The other end of the string is attached to a fixed point R which is on the same horizontal level as P with $RP = 5\,\text{m}$. The system is in equilibrium and $\angle PQR = 90°$. Find

a the tension in the string,
b the natural length of the string.

Solution:



$PQR = 90° \Rightarrow QR = 4\,\text{m}$

$\cos\alpha = \frac{3}{5};\ \sin\alpha = \frac{4}{5}$

a $m(P),\ 5g \times \frac{3}{2}\cos\alpha = 3T$

$5g \times \frac{3}{2} \times \frac{3}{5} = 3T$

$T = \frac{3g}{2} = 14.7\,\text{N}$

Tension is 14.7 N.

b $14.7 = \frac{30(4-l)}{l}$

$14.7l = 120 - 30l$

$44.7l = 120$

$l = 2.68\ldots$

Natural length is 2.7 m (2 s.f.)

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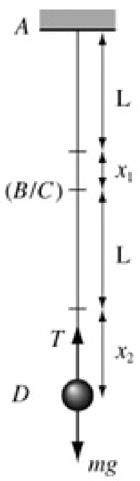
Elastic strings and springs
Exercise A, Question 8

Question:

A light elastic string AB has natural length l and modulus of elasticity $2mg$. Another light elastic string CD has natural length l and modulus of elasticity $4mg$. The strings are joined at their ends B and C and the end A is attached to a fixed point. A particle of mass m is hung from the end D and is at rest in equilibrium. Find the length AD.

Solution:



$(\uparrow) T = mg$ (throughout the length)

so, $mg = \frac{2mgx_1}{l} \Rightarrow x_1 = \frac{1}{2}l$

and $mg = \frac{4mgx_2}{l} \Rightarrow x_2 = \frac{1}{4}l$

$\therefore AD = 2l + x_1 + x_2$

$= \frac{11l}{4}$

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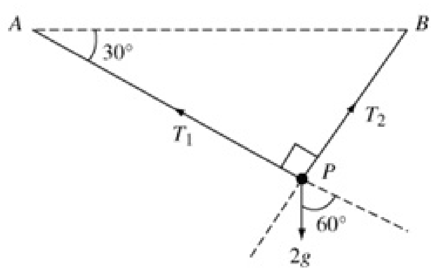
Elastic strings and springs
Exercise A, Question 9

Question:

An elastic string PA has natural length 0.5 m and modulus of elasticity 9.8 N. The string PB is inextensible. The end A of the elastic string and the end B of the inextensible string are attached to two fixed points which are on the same horizontal level. The end P of each string is attached to a 2 kg particle. The particle hangs in equilibrium below AB, with PA making an angle of $30°$ with AB and PA perpendicular to PB. Find

a the length of PA,
b the length of PB,
c the tension of PB.

Solution:



a (↖ along PA),

$T_1 = 2g\cos 60° = g = 9.8$ N

so, $\frac{9.8x_1}{0.5} = 9.8$

$x_1 = 0.5$

$\therefore AP = 0.5 + 0.5$

$= 1$ m

b $\frac{PB}{1} = \tan 30° = \frac{1}{\sqrt{3}}$ m

≈ 0.577 m

$= 0.58$ m (2 s.f.)

c (↗ along PB),

$T_2 = 2g\cos 30°$

$= 2g\frac{\sqrt{3}}{2}$

$= g\sqrt{3}$ N

≈ 17 N(2 s.f.)

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Elastic strings and springs
Exercise A, Question 10

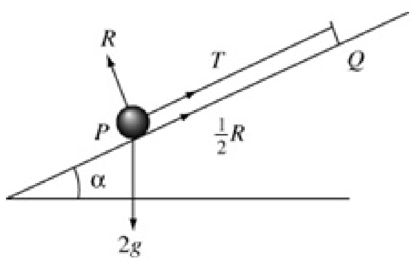
Question:

A particle of mass 2 kg is attached to one end P of a light elastic string PQ of modulus of elasticity 20 N and natural length 0.8 m. The end Q of the string is attached to a point on a rough plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the particle and the plane is $\frac{1}{2}$. The particle rests in limiting equilibrium, on the point of sliding down the plane, with PQ along a line of greatest slope. Find

a the tension in the string,
b the length of the string.

Solution:

a



$$(\nwarrow) R = 2g\cos\alpha = \frac{8g}{5}$$

$$\therefore F = \frac{1}{2}\times\frac{8g}{5} = \frac{4g}{5}$$

$$(\nearrow) T + \frac{4g}{5} = 2g\sin\alpha = \frac{6g}{5}$$

$$T = \frac{2g}{5}$$
$$= 3.92\text{ N}$$
$$= 3.9\text{ N}(2\text{ s.f.})$$

b $3.92 = \frac{20x}{0.8}$

$x = 0.1568\text{ m}$

$\therefore$ Length of string is $0.9568 = 0.96$ m (2 s.f.)

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Elastic strings and springs
Exercise B, Question 1

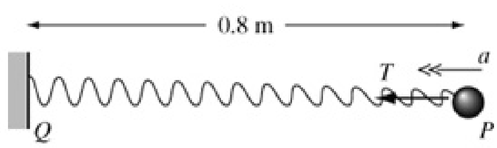
Question:

A particle of mass 4 kg is attached to one end P of a light elastic spring PQ, of natural length 0.5 m and modulus of elasticity 40 N. The spring rests on a smooth horizontal plane with the end O fixed. The particle is held at rest and then released. Find the initial acceleration of the particle

a if $PQ = 0.8$ m initially,

b if $PQ = 0.4$ m initially.

Solution:



a $(\leftarrow) T = 4a$

$$T = \frac{40 \times 0.3}{0.5}$$
$$= 24 \text{ N}$$
$$\therefore 24 = 4a$$
$$6 = a$$

initial acceleration is 6 m s^{-2}

b

$(\rightarrow) S = 4a$

$$S = \frac{40 \times 0.1}{0.5}$$
$$= 8 \text{ N}$$
$$\therefore 8 = 4a$$
$$2 = a$$

initial acceleration is 2 m s^{-2}

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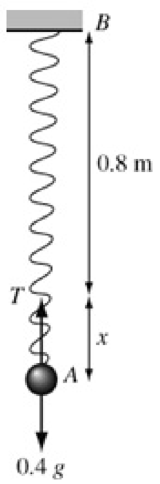
Elastic strings and springs
Exercise B, Question 2

Question:

A particle of mass 0.4 kg is fixed to one end A of a light elastic spring AB, of natural length 0.8 m and modulus of elasticity 20 N. The other end B of the spring is attached to a fixed point. The particle hangs in equilibrium. It is then pulled vertically downwards through a distance 0.2 m and released from rest. Find the initial acceleration of the particle.

Solution:



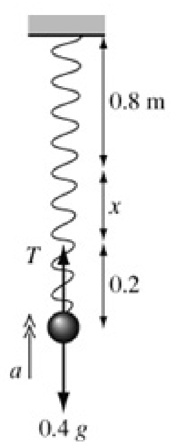
In equilibrium position

$$(\uparrow) T = 0.4g$$

$$T = \frac{20x}{0.8} = 25x$$

$$25x = 0.4g$$

$$x = \frac{2g}{125}$$



After further extension,

$$(\uparrow) T - 0.4g = 0.4a$$

$$T = \frac{20(x+0.2)}{0.8} = 25x + 5$$

So, $25x + 5 - 0.4g = 0.4a$

$$a = \frac{5}{0.4} = 12.5$$

initial acceleration is 12.5 m s^{-2}

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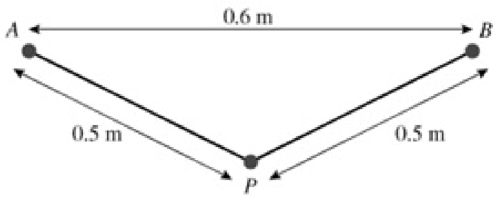
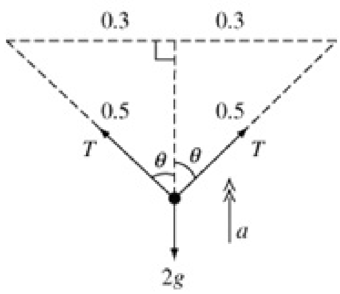
Elastic strings and springs
Exercise B, Question 3

Question:



A particle P of mass 2 kg is attached to the mid-point of a light elastic string, of natural length 0.4 m and modulus of elasticity 20 N. The ends of the elastic string are attached to two fixed points A and B which are on the same horizontal level, with $AB = 0.6$ m. The particle is held in the position shown, with $AP = BP = 0.5$ m, and released from rest. Find the initial acceleration of the particle and state its direction.

Solution:



$$(\uparrow) 2T\cos\theta - 2g = 2a$$

$$\frac{4T}{5} - g = a$$

$$T = \frac{20 \times 0.6}{0.4} = 30$$

$$\frac{4}{5} \times 30 - 9.8 = a$$

$$14.2 = a$$

initial acceleration is 14.2 m s^{-2} upwards

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Edexcel AS and A Level Modular Mathematics

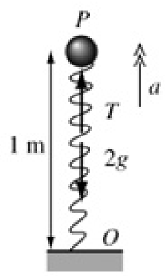
Elastic strings and springs

Exercise B, Question 4

Question:

A particle of mass 2 kg is attached to one end P of a light elastic spring. The other end Q of the spring is attached to a fixed point O. The spring has natural length 1.5 m and modulus of elasticity 40 N. The particle is held at a point which is 1 m vertically above O and released from rest. Find the initial acceleration of the particle, stating its magnitude and direction.

Solution:



$(\uparrow) T - 2g = 2a$

$$T = \frac{40 \times 0.5}{1.5}$$

$$= \frac{40}{3}$$

So, $\frac{40}{3} - 19.6 = 2a$

$$a = -3.13$$

magnitude of initial acceleration is 3.13 m s^{-2} and direction is downwards

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Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise C, Question 1

Question:

An elastic spring has natural length 0.6 m and modulus of elasticity 8 N. Find the work done when the spring is stretched from its natural length to a length of 1 m.

Solution:

$$\text{work done} = \frac{\lambda x^2}{2l} = \frac{8 \times 0.4^2}{2 \times 0.6}$$
$$= 1.06 \text{ J}$$

Solutionbank M3

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Elastic strings and springs
Exercise C, Question 2

Question:

An elastic spring, of natural length 0.8 m and modulus of elasticity of 4 N, is compressed to a length of 0.6 m. Find the elastic potential energy stored in the spring.

Solution:

$$\text{work done} = \frac{\lambda x^2}{2l} = \frac{4 \times 0.2^2}{2 \times 0.8}$$
$$= 0.1\text{ J}$$

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Elastic strings and springs
Exercise C, Question 3

Question:

An elastic string has natural length 1.2 m and modulus of elasticity 10 N. Find the work done when the string is stretched from a length 1.5 m to a length 1.8 m.

Solution:

$$\begin{aligned}\text{work done } &= \frac{10\times 0.6^2}{2\times 1.2} - \frac{10\times 0.3^2}{2\times 1.2}\\ &= \frac{10}{2.4}(0.6^2 - 0.3^2)\\ &= \frac{10}{2.4}\times 0.9\times 0.3\\ &= 1.125\text{ J}\end{aligned}$$

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Elastic strings and springs
Exercise C, Question 4

Question:

An elastic spring has natural length 0.7 m and modulus of elasticity 20 N. Find the work done when the spring is stretched from a length

a 0.7 m to 0.9 m
b 0.8 m to 1.0 m
c 1.2 m to 1.4 m.

Note that your answer to **a**, **b** and **c** are all different.

Solution:

a $\frac{20}{2\times 0.7}(0.2^2 - 0^2) = 0.571 \text{ J (3 s.f.)}$

b $\frac{20}{2\times 0.7}(0.3^2 - 0.1^2)$

$= \frac{20}{1.4}\times 0.4\times 0.2 = 1.14 \text{ J (3 s.f.)}$

c $\frac{20}{2\times 0.7}(0.7^2 - 0.5^2)$

$= \frac{20}{1.4}\times 1.2\times 0.2 = 3.43 \text{ J (3 s.f.)}$

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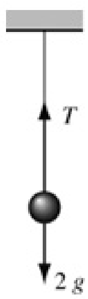
Elastic strings and springs
Exercise C, Question 5

Question:

A light elastic spring has natural length 1.2 m and modulus of elasticity 10 N. One end of the spring is attached to a fixed point. A particle of mass 2 kg is attached to the other end and hangs in equilibrium. Find the energy stored in the spring.

Solution:



$$(\uparrow) T = 2g$$

$$\frac{10e}{1.2} = 2g$$

$$e = \frac{2.4g}{10} = 0.24g$$

$$\text{energy stored} = \frac{10 \times (0.24g)^2}{2 \times 1.2}$$

$$= 23.0\text{ J} = 23\text{ J (2 s.f.)}$$

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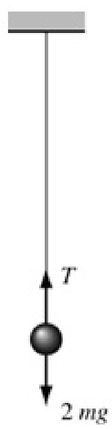
Elastic strings and springs
Exercise C, Question 6

Question:

An elastic string has natural length a. One end is fixed. A particle of mass $2m$ is attached to the free end and hangs in equilibrium, with the length of the string $3a$. Find the elastic potential energy stored in the string.

Solution:



$$(\uparrow) T = 2mg$$

$$\frac{\lambda \times 2a}{a} = 2mg$$

$$\Rightarrow \lambda = mg$$

$$\text{E.E.} = \frac{\lambda x^2}{2l} = \frac{mg(2a)^2}{2a} = 2mga$$

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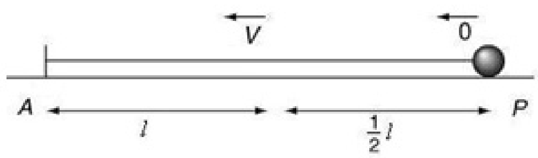
Elastic strings and springs
Exercise D, Question 1

Question:

An elastic string, of natural length l and modulus of elasticity mg, has one end fixed to a point A on a smooth horizontal table. The other end is attached to a particle P of mass m. The particle is held at a point on the table with $AP = \frac{3l}{2}$ and is released. Find the speed of the particle when the string reaches its natural length.

Solution:



Conservation of energy
K.E. gain = E.E. loss

$$\frac{1}{2}mV^2 = \frac{mg\left(\frac{1}{2}l\right)^2}{2l}$$

$$V^2 = \frac{1}{4}gl$$

$$V = \frac{1}{2}\sqrt{gl}$$

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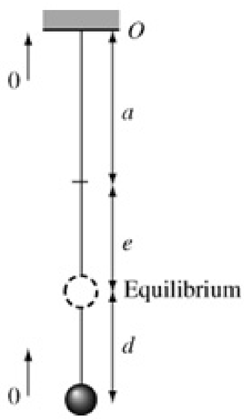
Elastic strings and springs
Exercise D, Question 2

Question:

A particle of mass m is suspended from a fixed point O by a light elastic string, of natural length a and modulus of elasticity $4mg$. The particle is pulled vertically downwards a distance d from its equilibrium position and released from rest. If the particle just reaches O, find d.

Solution:



At equilibrium, $T = mg$

$$\frac{4mge}{a} = mg \Rightarrow e = \frac{1}{4}a$$

Conservation of energy

$$\text{P.E. gain} = \text{E.E. loss}$$

$$mg\left(a + \frac{1}{4}a + d\right) = \frac{4mg\left(\frac{1}{4}a + d\right)^2}{2a}$$

$$\frac{5a^2}{4} + ad = 2\left(\frac{a^2}{16} + \frac{ad}{2} + d^2\right)$$

$$\frac{5a^2}{4} = \frac{a^2}{8} + 2d^2$$

$$\frac{9a^2}{16} = d^2$$

$$\left(\text{ignore solution} - \frac{3a}{4}\right) \quad \frac{3a}{4} = d$$

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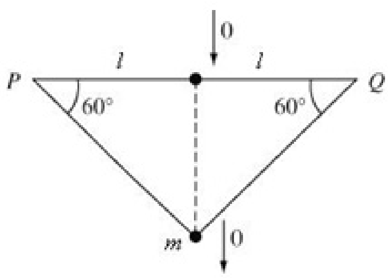
Elastic strings and springs
Exercise D, Question 3

Question:

A light elastic spring of natural length $2l$ has its ends attached to two points P and Q which are at the same horizontal level. The length PQ is $2l$. A particle of mass m is fastened to the midpoint of the spring and is held at the mid-point of PQ. The particle is released from rest and first comes to instantaneous rest when both parts of the string make an angle of $60°$ with the line PQ. Find the modulus of elasticity of the spring.

Solution:



Conservation of energy
P.E. loss = E.E. gain

$$mgl\tan 60° = \frac{2\times\lambda\left(\frac{l}{\cos 60°}-l\right)^2}{2l}$$

$$mgl\sqrt{3} = \lambda l$$

modulus is $mg\sqrt{3}$

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Elastic strings and springs
Exercise D, Question 4

Question:

A light elastic string, of natural length 1 m and modulus of elasticity 21.6 N has one end attached to a fixed point O. A particle of mass 2 kg is attached to the other end. The particle is held at a point which is 3 m vertically below O and released from rest. Find

a the speed of the particle when the string first becomes slack,

b the distance from O when the particle first comes to rest.

Solution:

a Conservation of energy

K.E. gain + P.E. gain = E.E. loss

$$\frac{1}{2}\times 2\times V^2 + 2g\times 2 = \frac{21.6\times 2^2}{2\times 1}$$

$$V^2 = 43.2 - 39.2$$

$$= 4$$

$$V = 2\text{ m s}^{-1}$$

b

Conservation of energy

K.E. loss = P.E. gain

$$\frac{1}{2}mV^2 = mgd$$

$$2 = gd$$

$$0.20\text{ m (2 s.f.)} = \frac{2}{g} = d$$

distance from O is 0.80 m (2 s.f.)

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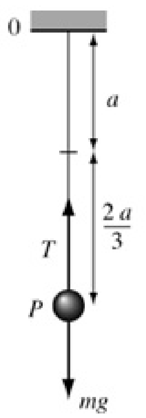
Elastic strings and springs
Exercise D, Question 5

Question:

A particle P is attached to one end of a light elastic string of natural length a. The other end of the string is attached to a fixed point O. When P hangs at rest in equilibrium, the distance OP is $\frac{5a}{3}$. The particle is now projected vertically downwards from O with speed U and first comes to instantaneous rest at a distance $\frac{10a}{3}$ below O. Find U in terms of a and g.

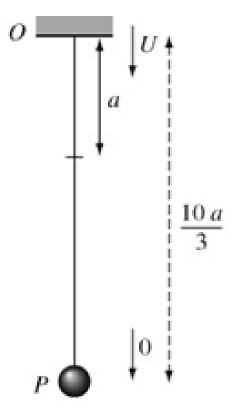
Solution:



$$(\uparrow)T = mg$$

$$\frac{\lambda}{a}\times\frac{2a}{3} = mg$$

$$\lambda = \frac{3mg}{2}$$



$$\text{K.E. loss} + \text{P.E. loss} = \text{E.E. gain}$$

$$\frac{1}{2}mU^2 + mg\frac{10a}{3} = \frac{3mg}{2}\frac{\left(\frac{7a}{3}\right)^2}{2a}$$

$$\frac{U^2}{2} + \frac{10ag}{3} = \frac{3g}{4a}\frac{49a^2}{9}$$

$$\frac{U^2}{2} = \frac{49ag}{12} - \frac{10ag}{3}$$

$$U^2 = \frac{9ag\times 2}{12}$$

$$U = \sqrt{\frac{3ag}{2}}$$

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Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise D, Question 6

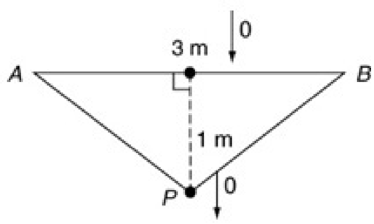
Question:

A particle P of mass 1 kg is attached to the mid-point of a light elastic string, of natural length 3 m and modulus λ N. The ends of the string are attached to two points A and B on the same horizontal level with $AB = 3\text{m}$. The particle is held at the mid-point of AB and released from rest. The particle falls vertically and comes to instantaneous rest at a point which is 1 m below the mid-point of AB.

Find **a** the value of λ,

b the speed of P when it is 0.5 m below the initial position.

Solution:



$$AP = \sqrt{1.5^2 + 1^2} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$$

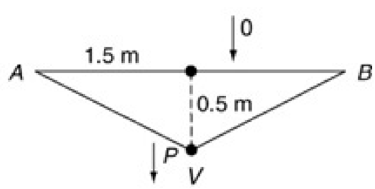
a P.E. loss = E.E. gain

$$g \times 1 = \frac{2\lambda\left(\frac{\sqrt{13}}{2} - \frac{3}{2}\right)^2}{2 \times 1.5}$$

$$\lambda = \frac{2 \times 3g}{(\sqrt{13} - 3)^2} = 80.176 \times 2$$

$$= 160 \text{ N } (2 \text{ s.f.})$$

b



$$AP = \sqrt{1.5^2 + 0.5^2} = \frac{\sqrt{10}}{2}$$

K.E. gain + E.E. gain = P.E. loss

$$\frac{1}{2}V^2 + \frac{2\lambda\left(\frac{\sqrt{10}}{2} - \frac{3}{2}\right)^2}{2 \times 1.5} = 0.5g$$

$$V^2 = g - \frac{(\sqrt{10} - 3)^2}{3} \times \lambda$$

$$V = 2.896 = 2.9 \text{ m s}^{-1} (2 \text{ s.f.})$$

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Elastic strings and springs
Exercise D, Question 7

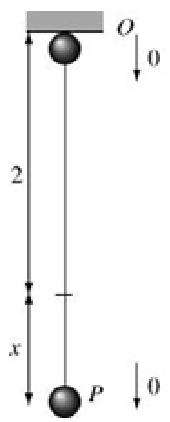
Question:

A light elastic string of natural length 2 m and modulus of elasticity 117.6 N has one end attached to a fixed point O. A particle P of mass 3 kg is attached to the other end. The particle is held at O and released from rest.

a Find the distance fallen by P before it first comes to rest.

b Find the greatest speed of P during the fall.

Solution:



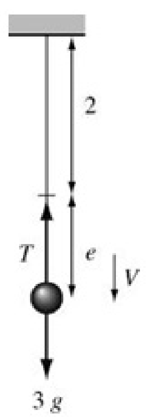
a

$$\text{P.E. loss} = \text{E.E. gain}$$
$$3g(2+x) = \frac{117.6}{4}x^2$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$
$$x = 2(x-1)$$

Ignore negative root.

falls 4 m

b Greatest speed at equilibrium position



$$(\uparrow) T = 3g$$
$$\frac{117.6 \times e}{2} = 3g$$
$$e = 0.5\text{ m}$$
$$\text{E.E. gain} + \text{K.E. gain} = \text{P.E. loss}$$
$$\frac{117.6(0.5)^2}{2\times 2} + \frac{1}{2}\times 3V^2 = 3g \times 2.5$$
$$7.35 + 1.5V^2 = 73.5$$
$$V = 6.64$$
$$= 6.6\text{ m s}^{-1}\text{ (2 s.f.)}$$

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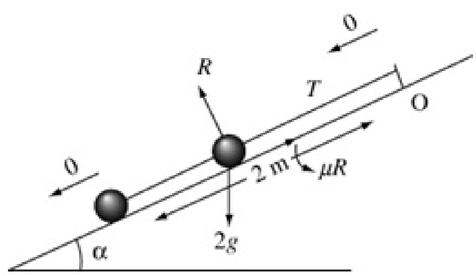
Elastic strings and springs
Exercise D, Question 8

Question:

A particle P of mass 2 kg is attached to one end of a light elastic string of natural length 1 m and modulus of elasticity 40 N. The other end of the string is fixed to a point O on a rough plane which is inclined at an angle α, where $\tan\alpha = \frac{3}{4}$. The particle is held at O and released from rest. Given that P comes to rest after moving 2 m down the plane, find the coefficient of friction between the particle and the plane.

Solution:



$$(\nwarrow) R = 2g\cos\alpha = \frac{8g}{5}$$

$$\text{Work done against friction} = \text{P.E. loss} - \text{E.E. gain}$$

$$\mu\frac{8g}{5}\times 2 = 2g\times 2\sin\alpha - \frac{40\times 1^2}{2\times 1}$$

$$\mu\frac{16g}{5} = \frac{12g}{5} - 20$$

$$\mu = \frac{12g-100}{16g}$$

$$= 0.11\,(2\text{ s.f.})$$

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Elastic strings and springs
Exercise E, Question 1

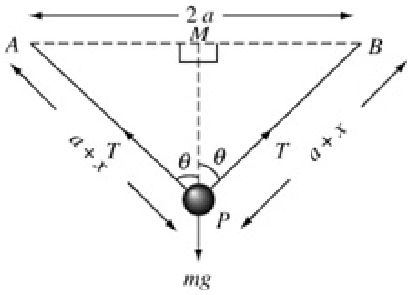
Question:

A particle of mass m is supported by two light elastic strings, each of natural length a and modulus of elasticity $\frac{15mg}{16}$. The other ends of the strings are attached to two fixed points A and B where A and B are in the same horizontal line with $AB = 2a$. When the particle hangs at rest in equilibrium below AB, each string makes an angle θ with the vertical.

a Verify that $\cos\theta = \frac{4}{5}$.

b How much work must be down to raise the particle to the mid-point of AB?

Solution:



$(\uparrow) 2T\cos\theta = mg$ ①

by Hooke's Law

$T = \dfrac{15mgx}{16a}$ ②

$\sin\theta = \dfrac{a}{a+x}$ ③

a If $\cos\theta = \dfrac{4}{5}, T = \dfrac{5mg}{8}$ from ①

$\text{so, } \dfrac{5mg}{8} = \dfrac{15mgx}{16a}$ from ②

$\dfrac{2a}{3} = x$

$\text{than } \dfrac{3}{5} = \dfrac{a}{a+\dfrac{2a}{3}}$ from ③

which is true.

b work done on particle = overall gain in energy

= P.E. gain − E.E. loss

$PM = (a+x)\cos\theta$

$= \left(a + \dfrac{2a}{3}\right)\dfrac{4}{5}$

$= \dfrac{4a}{3}$

$\therefore \text{P.E. gain } = mg\dfrac{4a}{3}$

E.E. loss = initial E.E. − final E.E.

$= \dfrac{15mg}{16\times 2a}\left(2x\left(\dfrac{2a}{3}\right)^2 - 0^2\right)$

$= \dfrac{15mg4a^2\times 2}{16\times 2a\times 9}$

$= \dfrac{5mga}{12}$

$\text{So, work done} = \dfrac{4mga}{3} - \dfrac{5mga}{12}$

$= \dfrac{mga}{12}(16-5)$

$= \dfrac{11mga}{12}$

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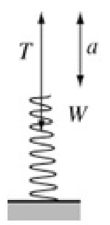
Elastic strings and springs
Exercise E, Question 2

Question:

A light elastic spring is such that a weight of magnitude W resting on the spring produces a compression a. The weight W is allowed to fall onto the spring from a height of $\frac{3a}{2}$ above it. Find the maximum compression of the spring in the subsequent motion.

Solution:

Let l be the natural length of the spring.
Let λ be the modulus of the spring.

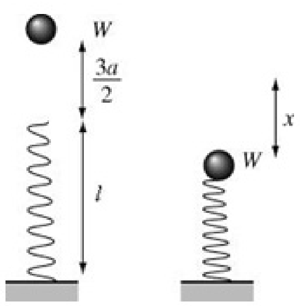


$(\uparrow) T = W$

by Hooke's Law,

$$T = \frac{\lambda a}{l}$$

$$\therefore W = \frac{\lambda a}{l} \text{ i.e. } \frac{W}{a} = \frac{\lambda}{l}$$



Using conservation of energy,
P.E. loss of W = E.E. gain of spring

$$W\left(\frac{3a}{2} + x\right) = \frac{\lambda x^2}{2l}$$

$$\text{so, } W\left(\frac{3a}{2} + x\right) = \frac{Wx^2}{2a}$$

Substitute for $\frac{\lambda}{l}$ from above.

$$3a^2 + 2ax = x^2$$

$$0 = x^2 - 2ax - 3a^2$$

$$0 = (x - 3a)(x + a)$$

$$\therefore x = 3a \text{ or } -a$$

$\therefore$ maximum compression is $3a$

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Elastic strings and springs
Exercise E, Question 3

Question:

A light elastic string of natural length 0.5 m is stretched between two points P and Q on a smooth horizontal table. The distance PQ is 0.75 m and the tension in the string is 15 N.

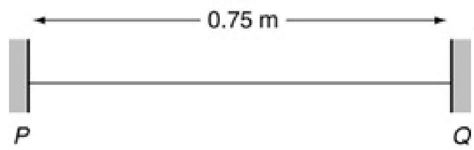
a Find the modulus of elasticity of the string.

A particle of mass 0.5 kg is attached to the mid-point of the string. The particle is pulled 0.1 m towards Q and released from rest.

b Find the speed of the particle as it passes through the mid-point of PQ.

Solution:

a

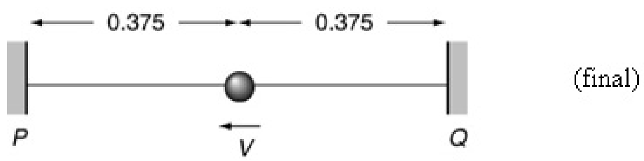


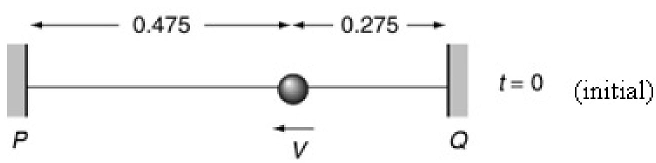
$x = 0.75 - 0.5 = 0.25$

by Hooke's Law, $15 = \frac{\lambda \times 0.25}{0.5}$

$\Rightarrow \lambda = 30$ N

b





Using conservation of energy

K.E. gain = E.E. loss

$$\begin{aligned}
\text{E.E. loss} &= \text{initial E.E.} - \text{final E.E.} \\
&= \frac{30}{2 \times 0.25}\{0.225^2 + 0.025^2 - 2 \times 0.125^2\} \\
&= 60(0.05125 - 0.03125) \\
&= 1.2 \text{ J}
\end{aligned}$$

$$\frac{1}{2} \times \frac{1}{2} \times v^2 = 1.2$$

$$\text{So, } v^2 = 4.8$$

$$v = 2.19 \text{ m s}^{-1} \text{ (3 s.f.)}$$

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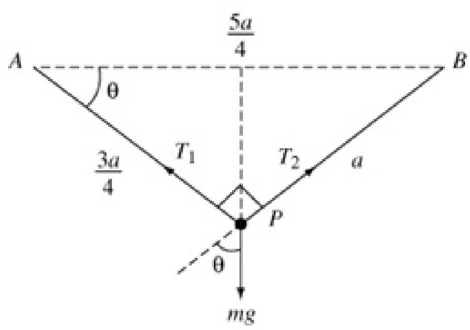
Elastic strings and springs
Exercise E, Question 4

Question:

A particle P of mass m is attached to two strings AP and BP. The points A and B are on the same horizontal level and $AB = \frac{5a}{4}$.
The string AP is inextensible and $AP = \frac{3a}{4}$.
The string BP is elastic and $BP = a$.
The modulus of elasticity of BP is λ. Show that the natural length of BP is $\frac{5\lambda a}{3mg + 5\lambda}$.

Solution:



ΔABP is 3, 4, 5 so $A\hat{P}B = 90°$.
($\nearrow$, along PB) $T_2 = mg\cos\theta = \frac{3mg}{5}$
by Hooke's Law, $T_2 = \frac{\lambda(-l + a)}{l}$

$$\text{So,}\quad \lambda\frac{(-l+a)}{l} = \frac{3mg}{5}$$
$$5\lambda(-l + a) = 3mgl$$
$$5\lambda l + 3mgl = 5\lambda a$$
$$l(5\lambda + 3mg) = 5\lambda a$$
$$l = \frac{5\lambda a}{(5\lambda + 3mg)}$$

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Elastic strings and springs
Exercise E, Question 5

Question:

A light elastic string, of natural length a and modulus of elasticity $5mg$, has one end attached to the base of a vertical wall. The other end of the string is attached to a small ball. The ball is held at a distance $\frac{3a}{2}$ from the wall, on a rough horizontal plane, and released from rest. The coefficient of friction between the ball and the plane is $\frac{1}{5}$.

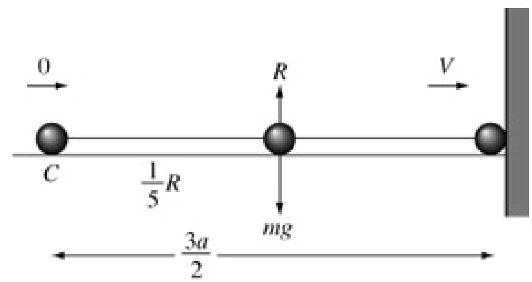
a Find, in terms of a and g, the speed V of the ball as it hits the wall.

The ball rebounds from the wall with speed $\frac{2V}{5}$.

b Find the distance from the wall at which the ball comes to rest.

Solution:

a

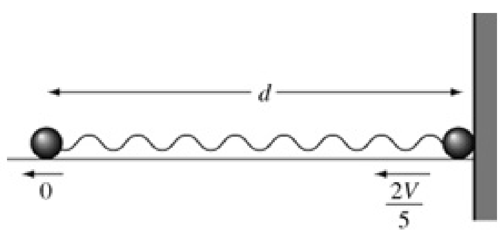


$(\uparrow) R = mg$

$\therefore \text{Friction} = \frac{1}{5}mg$

$$\text{work done against friction} = \text{overall loss in energy}$$
$$= \text{E.E. loss} - \text{K.E. gain}$$
$$\frac{1}{5}mg\frac{3a}{2} = \frac{5mg\left(\frac{a}{2}\right)^2}{2a} - \frac{1}{2}mV^2$$
$$\frac{3ag}{5} = \frac{5ag}{4} - V^2$$
$$V^2 = \frac{5ag}{4} - \frac{3ag}{5} = \frac{ag(25-12)}{20}$$
$$V = \sqrt{\frac{13ag}{20}}$$

b



Friction will be same.
Assume string is still slack when ball comes to rest.

$$\text{Work done against friction} = \text{K.E. loss}$$
$$\frac{1}{5}mg\,d = \frac{1}{2}m\left(\frac{2V}{5}\right)^2 = \frac{1}{2}m\frac{4V^2}{25}$$
$$\frac{1}{5}gd = \frac{1}{2}\times\frac{4}{25}\times\frac{13ag}{20}$$
$$d = \frac{13a}{50}$$

As d is less than a, the assumption that the string is still slack is valid.

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Elastic strings and springs
Exercise E, Question 6

Question:

a Using integration, show that the work done in stretching a light elastic string of natural length l and modulus of elasticity λ, from length l to length $(l+x)$ is $\frac{\lambda x^2}{2l}$.

b The same string is stretched from a length $(l+a)$ to a length $(l+b)$ where $b > a$. Show that the work done is the product of the mean tension and the distance moved.

Solution:

a work done $= \int_0^x T \, ds = \int_0^x \frac{\lambda s}{l} \, ds$

$$= \frac{\lambda}{2l}\left[s^2\right]_0^x$$

$$= \frac{\lambda x^2}{2l}$$

b work done = E.E. gain of string

$$= \frac{\lambda}{2l}(b^2 - a^2)$$

$$= \frac{\lambda}{2l}(b+a)(b-a)$$

$$= \frac{1}{2}\left(\frac{\lambda b}{l} + \frac{\lambda a}{l}\right)(b-a)$$

$$= \frac{1}{2}\left(T_b + T_a\right)(b-a)$$

= mean of tensions × distance moved

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Elastic strings and springs
Exercise E, Question 7

Question:

A light elastic string has natural length l and modulus $2mg$. One end of the string is attached to a particle P of mass m. The other end is attached to a fixed point C on a rough horizontal plane. Initially P is at rest at a point D on the plane where $CD = \frac{4l}{3}$.

a Given that P is in limiting equilibrium, find the coefficient of friction between P and the plane.

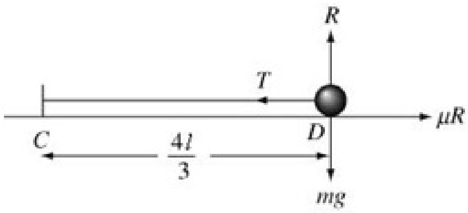
The particle P is now moved away from C to a point E on the plane where $CE = 2l$.

b Find the speed of P when the string returns to its natural length.

c Find the total distance moved by P before it comes to rest.

Solution:

a



$(\uparrow) R = mg \quad (\rightarrow) \mu R = T$

$\mu mg = T$

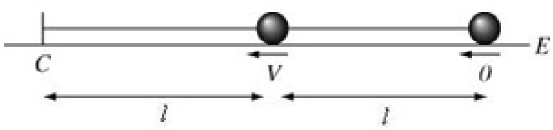
by Hooke's Law,

$$T = \frac{2mg}{l}\frac{l}{3} = \frac{2mg}{3}$$

$$\therefore \mu mg = \frac{2mg}{3}$$

$$\mu = \frac{2}{3}$$

b



work done against friction = overall loss in energy

= E.E. loss − K.E. gain

$$\frac{2}{3}mg\,l = \frac{2mgl^2}{2l} - \frac{1}{2}mV^2$$

$$\frac{1}{2}V^2 = gl - \frac{2}{3}gl - \frac{1}{3}gl$$

$$V^2 = \frac{2}{3}gl$$

$$V = \sqrt{\frac{2gl}{3}}$$

c String is now slack

work done against friction = K.E. loss

$$\frac{2}{3}mg\,d = \frac{1}{2}m \times \frac{2}{3}gl$$

$$d = \frac{1}{2}l$$

$\therefore$ total distance travelled is $\frac{3l}{2}$

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Elastic strings and springs
Exercise E, Question 8

Question:

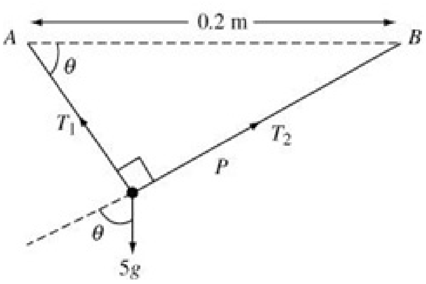
A light elastic string of natural length 0.2 m has its ends attached to two fixed points A and B which are on the same horizontal level with $AB = 0.2$ m . A particle of mass 5 kg is attached to the string at the point P where $AP = 0.15$ m. The system is released and P hangs in equilibrium below AB with $A\hat{P}B = 90°$.

a If $B\hat{A}P = \theta$, show that the ratio of the extension of AP and BP is $\frac{4\cos\theta - 3}{4\sin\theta - 1}$.

b Hence show that $\cos\theta(4\cos\theta - 3) = 3\sin\theta(4\sin\theta - 1)$.

Solution:

a



extension of $AP = 0.2\cos\theta - 0.15$

extension of $BP = 0.2\sin\theta - 0.05$

$$\therefore \text{ ratio is } \frac{0.2\cos\theta - 0.15}{0.2\sin\theta - 0.05} \times \frac{20}{20} = \frac{4\cos\theta - 3}{4\sin\theta - 1}$$

b (↗) along PB: $T_2 = 5g\cos\theta$

(↙) along PA: $T_1 = 5g\sin\theta$

$$\text{so, } \frac{T_2}{T_1} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{\lambda x_2}{0.05} \times \frac{0.15}{\lambda x_1} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{3x_2}{x_1} = \frac{\cos\theta}{\sin\theta}$$

$$\text{i.e. } \frac{x_1}{x_2} = \frac{3\sin\theta}{\cos\theta}$$

$$\text{i.e. } \frac{4\cos\theta - 3}{4\sin\theta - 1} = \frac{3\sin\theta}{\cos\theta}$$

$$3\sin\theta(4\sin\theta - 1) = \cos\theta(4\cos\theta - 3)$$

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Elastic strings and springs
Exercise E, Question 9

Question:

A particle of mass 3 kg is attached to one end of a light elastic string, of natural length 1 m and modulus of elasticity 14.7 N. The other end of the string is attached to a fixed point. The particle is held in equilibrium by a horizontal force of magnitude 9.8 N with the string inclined to the vertical at an angle θ.

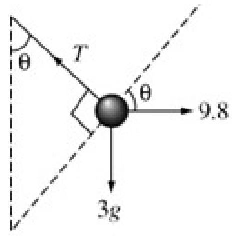
a Find the value of θ.

b Find the extension of the string.

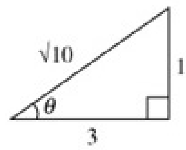
c If the horizontal force is removed, find the magnitude of the least force that will keep the string inclined at the same angle.

Solution:

a



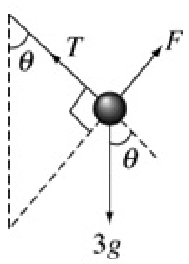
(↗ perpendicular to string)



$$9.8\cos\theta = 3g\sin\theta$$
$$\frac{1}{3} = \tan\theta$$
$$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.4°$$

b $(\rightarrow)\ T\sin\theta = 9.8$
$$T = 9.8\sqrt{10}$$
$$\frac{14.7\times x}{1} = 9.8\sqrt{10}$$
$$x = \frac{2\sqrt{10}}{3}\text{ m} \approx 2.1\text{ m (2 s.f.)}$$

c



least force will be perpendicular to string

$(\nearrow) F = 3g\sin\theta$
$$= \frac{3g}{\sqrt{10}}\text{ N}$$
$$= \frac{3g\sqrt{10}}{10}\text{ N}$$
$$= 9.3\text{ N (2 s.f.)}$$

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Elastic strings and springs
Exercise E, Question 10

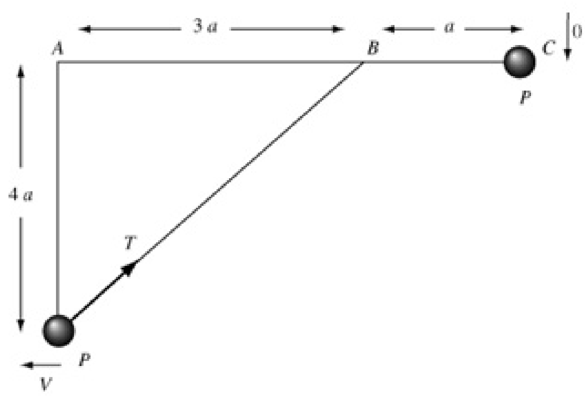
Question:

Two points A and B are on the same horizontal level with $AB = 3a$. A particle P of mass m is joined to A by a light inextensible string of length $4a$ and is joined to B by a light elastic string, of natural length a and modulus of elasticity $\frac{mg}{4}$. The particle P is held at the point C, on AB produced, such that $BC = a$ and both strings are taut. The particle P is released from rest.

a Show that when AP is vertical the speed of P is $2\sqrt{ga}$.

b Find the tension in the elastic string in this position.

Solution:



a by conservation of energy,

K.E. gain + E.E. gain = P.E. loss

$$\frac{1}{2}mv^2 + \frac{mg}{4}\frac{x^2}{2a} = mg4a$$

$BP = 5a$ (3, 4, 5 Δ)

So, $x = 4a$

$$\therefore \frac{1}{2}mv^2 + \frac{mg}{4}\cdot\frac{16a^2}{2a} = mg\,4a$$

$$v^2 + 4ga = 8ga$$

$$v^2 = 4ga$$

$$v = 2\sqrt{ga}$$

b $x = 4a : T = \frac{mg}{4} \times \frac{4a}{a}$

$= mg$

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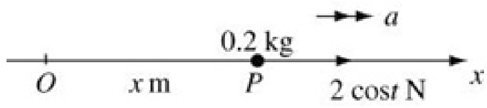
Further dynamics
Exercise A, Question 1

Question:

A particle P of mass 0.2 kg is moving on the x-axis. At time t seconds P is x metres from the origin O. The force acting on P has magnitude $2\cos t$ N and acts in the direction OP. When $t = 0$, P is at rest at O. Calculate

a the speed of P when $t = 2$,
b the speed of P when $t = 3$,
c the time when P first comes to instantaneous rest,
d the distance OP when $t = 2$,
e the distance OP when P first comes to instantaneous rest.

Solution:



a $F = ma$

$2\cos t = 0.2a$

$0.2\frac{dv}{dt} = 2\cos t$

Force is a function of time so use $a = \frac{dv}{dt}$.

$v = \frac{2}{0.2}\int \cos t \, dt$

Integrate to obtain an expression for v.

$v = 10\sin t + c$

Don't forget the constant.

$t = 0 \quad v = 0$

$0 = 0 + c \therefore c = 0$

$v = 10\sin t$

$t = 2 \quad v = 10\sin 2 = 9.092\ldots$

When $t = 2$ the speed of P is 9.09 m s^{-1} (3 s.f.)

b $t = 3 \quad v = 10\sin 3 = 1.411\ldots$

When $t = 3$ the speed of P is 1.41 m s^{-1} (3 s.f.)

c $v = 0 \quad 0 = 10\sin t$

P comes to rest when $v = 0$.

$\sin t = 0$

$t = 0, \pi, \ldots$

P first comes to rest when $t = \pi$.

Exact answers are best.

d $v = 10\sin t$

$\frac{dx}{dt} = 10\sin t$

$x = 10\int \sin t \, dt$

$x = -10\cos t + K$

Integrate to obtain an expression for x.

$t = 0, x = 0 \quad 0 = -10 + K \therefore K = 10$

$x = -10\cos t + 10$

$t = 2 \quad x = -10\cos t + 10 = 14.16\ldots$

When $t = 2$ $OP = 14.2$ m (3 s.f.)

e $t = \pi \quad x = -10\cos\pi + 10$

$= 10 + 10 = 20$

When P comes to rest $OP = 20$ m.

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Further dynamics

Exercise A, Question 2

Question:

A van of mass 1200 kg moves along a horizontal straight road. At time t seconds, the resultant force acting on the car has magnitude $\frac{60\,000}{(t+5)^2}$ N and acts in the direction of motion of the van. When $t = 0$, the van is at rest. The speed of the van approaches a limiting value V m s^{-1}. Find

a the value of V,

b the distance moved by the van in the first 4 seconds of its motion.

Solution:

a $F = ma$

$$\frac{60\,000}{(t+5)^2} = 1200a$$

$$a = \frac{50}{(t+5)^2}$$

$$\frac{dv}{dt} = \frac{50}{(t+5)^2}$$

Force is a function of time so use $a = \frac{dv}{dt}$.

$$v = \int \frac{50}{(t+5)^2}\,dt$$

Integrate to obtain an expression for v.

$$v = -\frac{50}{(t+5)} + c$$

$$t = 0, v = 0 \quad \therefore\ 0 = -\frac{50}{5} + c$$

$$c = 10$$

$$v = -\frac{50}{t+5} + 10$$

As $t \to \infty$ $-\frac{50}{t+5} \to 0$

$\therefore\ V = 10$

b $v = -\frac{50}{(t+5)} + 10$

$$\frac{dx}{dt} = -\frac{50}{t+5} + 10$$

$$x = -50\ln(t+5) + 10t + K$$

$$t = 0,\ x = 0 \quad 0 = -50\ln 5 + K$$

$$K = 50\ln 5$$

$$\therefore\ x = -50\ln(t+5) + 10t + 50\ln 5$$

$$t = 4 \quad x = -50\ln 9 + 40 + 50\ln 5$$

$$x = 40 + 50\ln\frac{5}{9}$$

$$x = 10.61\ldots$$

The van moves 10.6 m in the first 4 seconds (3 s.f.)

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Further dynamics
Exercise A, Question 3

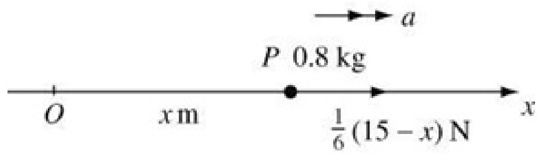
Question:

A particle P of mass 0.8 kg is moving along the x-axis. At time $t=0$, P passes through the origin O, moving in the positive x direction. At time t seconds, $OP=x$ metres and the velocity of P is $v\text{ m s}^{-1}$. The resultant force acting on P has magnitude $\frac{1}{6}(15-x)$N, and acts in the positive x direction. The maximum speed of P is 12 m s^{-1}.

a Explain why the maximum speed of P occurs when $x=15$.

b Find the speed of P when $t=0$.

Solution:



a Maximum speed ⇒ acceleration zero

⇒ force is zero

$\therefore \frac{1}{6}(15-x)=0 \quad \therefore x=15$

b $F=ma$

$\frac{1}{6}(15-x)=0.8a$

$a=\frac{1}{4.8}(15-x)$

$v\frac{dv}{dx}=\frac{1}{4.8}(15-x)$ ← Force is a function of x so use $a=v\frac{dv}{dx}$.

$\int v\,dv=\frac{1}{4.8}\int(15-x)dx$ ← Separate the variables.

$\frac{1}{2}v^2=\frac{1}{4.8}\left(15x-\frac{1}{2}x^2\right)+c$

$x=15, v=12$ ← **a** tells you the initial conditions.

$\frac{1}{2}\times 12^2=\frac{1}{4.8}\left(15\times 15-\frac{1}{2}\times 15^2\right)+c$

$c=\frac{1}{2}\times 12^2-\frac{1}{4.8}\times\frac{1}{2}\times 15^2$

$c=48.5625$

$\frac{1}{2}v^2=\frac{1}{4.8}\left(15x-\frac{1}{2}x^2\right)+48.5625$

$t=0, x=0 \quad v^2=2\times 48.5625$ ← P is at 0 when $t=0$.

$v=9.855$

When $t=0$ P's speed is 9.86 m s^{-1} (3 s.f.)

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Further dynamics
Exercise A, Question 4

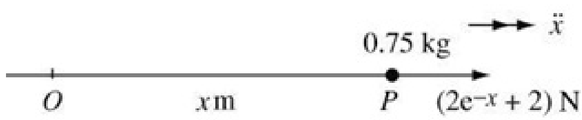
Question:

A particle P of mass 0.75 kg is moving in a straight line. At time t seconds after it passes through a fixed point on the line, O, the distance OP is x metres and the force acting on P has magnitude $(2e^{-x}+2)$ N and acts in the direction OP. Given that P passes through O with speed 5 m s^{-1}, calculate the speed of P when

a $x=3$,

b $x=7$.

Solution:



$$F = ma$$

$$(2e^{-x}+2) = 0.75\ddot{x}$$

$$0.75v\frac{dv}{dx} = 2e^{-x}+2$$

Force is a function of x so use $\ddot{x} = v\frac{dv}{dx}$.

$$0.75\int v\,dv = \int(2e^{-x}+2)\,dx$$

Separate the variables.

$$0.75\times\frac{1}{2}v^2 = -2e^{-x}+2x+c$$

$$x=0, v=5 \therefore 0.75\times\frac{1}{2}\times5^2 = -2+c$$

$$c = 0.75\times\frac{1}{2}\times5^2+2 = 11.375$$

$$\therefore 0.375v^2 = -2e^{-x}+2x+11.375$$

a $x=3$

$$v^2 = \frac{1}{0.375}(-2e^{-3}+6+11.375)$$

$$v = 6.787\ldots$$

When $x=3$ P's speed is 6.79 m s^{-1} (3 s.f.)

b $x=7$ $\quad v^2 = \frac{1}{0.375}(-2e^{-7}+14+11.375)$

$$v = 8.225\ldots$$

When $x=7$ P's speed is 8.23 m s^{-1} (3 s.f.)

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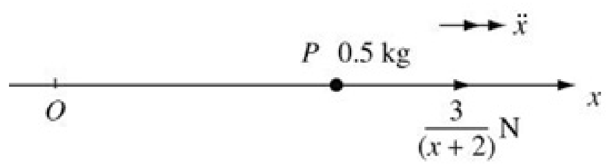
Further dynamics
Exercise A, Question 5

Question:

A particle P of mass 0.5 kg moves away from the origin O along the positive x-axis. When $OP = x$ metres the force acting on P has magnitude $\frac{3}{x+2}$ N and is directed away from O. When $x = 0$ the speed of P is 1.5 m s^{-1}. Find the value of x when the speed of P is 2 m s^{-1}.

Solution:



$$F = ma$$

$$\frac{3}{x+2} = 0.5\ddot{x}$$

Force is a function of x so use $\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$.

$$0.5v\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3}{x+2}$$

$$0.5\int v\,\mathrm{d}v = 3\int \frac{1}{x+2}\,\mathrm{d}x$$

$$0.5 \times \frac{1}{2}v^2 = 3\ln(x+2) + c$$

$$x = 0, v = 1.5$$

$$0.5 \times \frac{1}{2} \times 1.5^2 = 3\ln 2 + c$$

$$c = \frac{1.5^2}{4} - 3\ln 2$$

For the best final answer keep the exact value as long as possible.

$$\therefore \frac{1}{4}v^2 = 3\ln(x+2) + \frac{1.5^2}{4} - 3\ln 2$$

$$v = 2 \quad \frac{1}{4} \times 2^2 = 3\ln(x+2) + \frac{1.5^2}{4} - 3\ln 2$$

$$3\ln(x+2) = 1 - \frac{1.5^2}{4} + 3\ln 2$$

$$\ln(x+2) = 0.8389\ldots$$

$$x = e^{0.8389\ldots} - 2 = 0.3140\ldots$$

When P's speed is 2 m s^{-1}, $x = 0.314$ (3 s.f.)

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Further dynamics
Exercise A, Question 6

Question:

Calculate the magnitude of the impulse of a force of magnitude F N acting from time t_1 seconds to time t_2 seconds where

a $F = 3t^2 - \frac{1}{2}t \quad t_1 = 0, t_2 = 4$,

b $F = 2t + \frac{1}{3t-2} \quad t_1 = 1, t_2 = 2$,

c $F = 2\cos 4t \quad t_1 = 0, t_2 = \frac{\pi}{4}$,

d $F = 3 + e^{-0.5t} \quad t_1 = 0, t_2 = 4$.

Solution:

a Impulse $= \int_0^4 \left(3t^2 - \frac{1}{2}t\right)\mathrm{d}t$

$= \left[t^3 - \frac{1}{4}t^2\right]_0^4$

$= 64 - 4 - 0 = 60$

Impulse $= \int_{t_1}^{t_2} F \, \mathrm{d}t$ or see end for an alternative method.

The magnitude of the impulse is 60 Ns.

b Impulse $= \int_1^2 \left(2t + \frac{1}{3t-2}\right)\mathrm{d}t$

$= \left[t^2 + \frac{1}{3}\ln(3t-2)\right]_1^2$

$= 4 + \frac{1}{3}\ln(6-2) - \left(1 + \frac{1}{3}\ln 1\right)$

$= 3 + \frac{1}{3}\ln 4$

$= 3.462\ldots$

The magnitude of the impulse is 3.46 Ns (3 s.f.)

c Impulse $= \int_0^{\frac{\pi}{4}} 2\cos 4t \, \mathrm{d}t = \left[\frac{2}{4}\sin 4t\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}[\sin\pi - \sin 0]$

$= 1$

The magnitude of the impulse is 1 Ns

d Impulse $= \int_0^4 (3 + \mathrm{e}^{-0.5t})\mathrm{d}t$

$= \left[3t - 2\mathrm{e}^{-0.5t}\right]_0^4$

$= 12 - 2\mathrm{e}^{-2} - (0 - 2\times 1)$

$= 14 - 2\mathrm{e}^{-2}$

$= 13.72\ldots$

The magnitude of the impulse is 13.7 Ns (3 s.f.)

Alternative method for **a**

$F = ma$

$3t^2 - \frac{1}{2}t = m\frac{\mathrm{d}v}{\mathrm{d}t}$

$t^3 - \frac{t^2}{4} + c = mv$

$t = 0 \quad mv_1 = c$

$t = 4 \quad 64 - 4 + c = mv_2$

impulse $= mv_2 - mv_1$

$= 60 + c - c = 60$

The magnitude of the impulse is 60 Ns.

Use $F = ma$ to find mv at each of the required times.

Impulse = change in momentum.

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Further dynamics
Exercise A, Question 7

Question:

Calculate the work done by a force of magnitude F N directed along the x-axis which moves a particle from $x = x_1$ metres to $x = x_2$ metres where

a $F = 2x^{\frac{1}{2}} + \frac{1}{2}x^2 \quad x_1 = 1,\ x_2 = 4,$

b $F = 2\sin x + 3 \quad x_1 = 0,\ x_2 = \frac{\pi}{2},$

c $F = 3x^2 + e^{-2x} \quad x_1 = 1,\ x_2 = 3,$

d $F = \frac{3}{x} + \frac{2}{x-1} \quad x_1 = 2,\ x_2 = 4.$

Solution:

a work done $= \int_1^4 \left(2x^{\frac{1}{2}} + \frac{1}{2}x^2\right)dx$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} + \frac{1}{6}x^3\right]_1^4$$

$$= \frac{4}{3}\times 8 + \frac{1}{6}\times 64 - \left(\frac{4}{3} + \frac{1}{6}\right)$$

$$= 19\frac{5}{6} \quad (\text{or } 19.83\ldots)$$

The work done is $19\frac{5}{6}$ J (or 19.8 J (3 s.f.))

> work done $= \int_{x_1}^{x_2} F \, ds$ or see for an alternative method.

b work done $= \int_0^{\frac{\pi}{2}} (2\sin x + 3)dx$

$$= [-2\cos x + 3x]_0^{\frac{\pi}{2}}$$

$$= -2\cos\frac{\pi}{2} + \frac{3\pi}{2} - (-2\cos 0 + 0)$$

$$= \frac{3\pi}{2} + 2$$

The work done is $\left(\frac{3\pi}{2} + 2\right)$ J or 6.71 J (3 s.f.)

> Give the exact answer unless accuracy is specified.

c Work done $= \int_1^3 (3x^2 + e^{-2x})dx$

$$= \left[x^3 - \frac{1}{2}e^{-2x}\right]_1^3$$

$$= 27 - \frac{1}{2}e^{-6} - \left(1 - \frac{1}{2}e^{-2}\right)$$

$$= 26.06\ldots$$

The work done is 26.1 J (3 s.f.)

d Work done $= \int_2^4 \left(\frac{3}{x} + \frac{2}{x-1}\right) dx$

$$= \left[3\ln x + 2\ln(x-1)\right]_2^4$$
$$= 3\ln 4 + 2\ln 3 - (3\ln 2 + 2\ln 1)$$
$$= \ln 64 + \ln 9 - \ln 8 - 0$$
$$= \ln\left(\frac{64 \times 9}{8}\right) = \ln 72$$

The work done is ln 72 J or 4.28 J (3 s.f.)

Alternative method for a

$$F = ma$$

Use $F = ma$ with $a = v\frac{dv}{dx}$ and integrate to obtain $\frac{1}{2}mv^2$ for each value of x.

$$2x^{\frac{1}{2}} + \frac{1}{2}x^2 = mv\frac{dv}{dx}$$
$$m\int v\,dv = \int\left(2x^{\frac{1}{2}} + \frac{1}{2}x^2\right)dx$$
$$\frac{1}{2}mv^2 = \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{6}x^3 + c$$
$$x = 1 \quad \frac{1}{2}mv_1^2 = \frac{4}{3} + \frac{1}{6} + c = \frac{3}{2} + c$$
$$x = 4 \quad \frac{1}{2}mv_2^2 = \frac{4}{3} \times 8 + \frac{1}{6} \times 64 + c$$
$$\text{work done} = \frac{32}{3} + \frac{64}{6} - \frac{3}{2} = 19\frac{5}{6}$$

work done = change in K.E.

The work done is $19\frac{5}{6}$ J.

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Further dynamics

Exercise A, Question 8

Question:

A particle P of mass 1.5 kg is moving in a straight line. The particle is initially at rest at a point O on the line. At time t seconds (where $t \geq 0$) the force acting on P has magnitude $(3t+8)$N and acts in the direction OP. When $t = T$, P has speed 75 m s^{-1}.

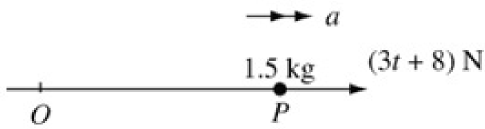
Calculate

a the magnitude of the impulse exerted by the force between the times $t = 1$ and $t = 4$,

b the speed of P when $t = 3$,

c the value of T.

Solution:



a Impulse $= \int_1^4 (3t+8)\,dt$

$= \left[\frac{3t^2}{2} + 8t\right]_1^4$

$= (24+32) - \left(\frac{3}{2}+8\right)$

$= 46.5$

The impulse has magnitude 46.5 Ns.

Impulse $= \int_{t_1}^{t_2} F\,dt$ or find velocities at $t=1$ and $t=4$ and use impulse = change in momentum.

b $F = ma$

$3t+8 = 1.5a$

$\frac{3}{2}\frac{dv}{dt} = 3t+8$

$\frac{3}{2}v = \int (3t+8)\,dt$

$\frac{3}{2}v = \frac{3t^2}{2} + 8t + c$

$t=0\ v=0 \Rightarrow c=0$

$t=3\ \frac{3}{2}v = \frac{27}{2} + 24$

$v = \frac{2}{3} \times \frac{75}{2}$

$v = 25$

When $t=3$ the speed of P is 25 m s^{-1}.

c $\frac{3}{2}v = \frac{3t^2}{2} + 8t$

$v = 75, t = T$

$\frac{3}{2} \times 75 = \frac{3T^2}{2} + 8T$

$3T^2 + 16T - 225 = 0$

$T = \frac{-16 \pm \sqrt{(16^2 + 4 \times 3 \times 225)}}{6}$

$= 6.394\ldots \text{ or } -11.72\ldots$

$T > 0 \quad \therefore \quad T = 6.39 \text{ (3 s.f.)}$

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Further dynamics
Exercise A, Question 9

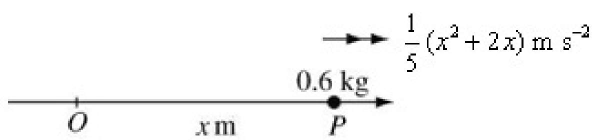
Question:

A particle of mass 0.6 kg moves in a straight line through a fixed point O. At time t seconds after passing through O the distance of P from O is x metres and the acceleration of P is $\frac{1}{5}(x^2+2x)$ m s^{-2}.

a Write down, in terms of x, an expression for the force acting on P.

b Calculate the work done by the force in moving P from $x=0$ to $x=4$.

Solution:



a $F = ma$

$F = 0.6 \times \frac{1}{5}(x^2+2x)$

$F = 0.12(x^2+2x)$

Use $F = ma$.

b Work done $= \int_0^4 F \, dx$

$= \int_0^4 0.12(x^2+2x)dx$

$= 0.12\left[\frac{x^3}{3}+x^2\right]_0^4$

$= 0.12\left[\frac{64}{3}+16-0\right]$

$= 4.48$

work done $= \int_{x_1}^{x_2} F \, dx$

The work done is 4.48 J.

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Further dynamics
Exercise B, Question 1

Question:

Above the Earth's surface, the magnitude of the force on a particle due to the Earth's gravitational force is inversely proportional to the square of the distance of the particle from the centre of the Earth. The acceleration due to gravity on the surface of the Earth is g and the Earth can be modelled as a sphere of radius R. A particle P of mass m is a distance $(x-R)$ (where $x \geq R$) above the surface of the Earth. Prove that the magnitude of the gravitational force acting on P is $\frac{mgR^2}{x^2}$.

Solution:

$F = \frac{k}{d^2}$ where $d =$ distance from centre

distance $(x-R)$ above surface

$\Rightarrow$ distance x from centre

$\therefore F = \frac{k}{x^2}$

On surface $F = mg, x = R$

$\therefore mg = \frac{k}{R^2}$

$k = mgR^2$

The magnitude of the gravitational force on a particle on the surface of the earth is the magnitude of the weight of the particle.

$\therefore$ Magnitude of the gravitational force is $\frac{mgR^2}{x^2}$.

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Further dynamics
Exercise B, Question 2

Question:

The Earth can be modelled as a sphere of radius R. At a distance x (where $x \ge R$) from the centre of the Earth the magnitude of the acceleration due to the Earth's gravitational force is A. On the surface of the Earth, the magnitude of the acceleration due to the Earth's gravitational force is g. Prove that $A = \frac{gR^2}{x^2}$.

Solution:

For a particle of mass m, distance x from the centre of the earth:

$F = ma$

$\frac{k}{x^2} = mA$ ← Use the inverse square law.

On the surface of the earth, $x = R, A = g$

$\therefore mg = \frac{k}{R^2}$

$k = mgR^2$

$\therefore mA = \frac{mgR^2}{x^2}$

$A = \frac{gR^2}{x^2}$

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Further dynamics
Exercise B, Question 3

Question:

A spacecraft S is fired vertically upwards from the surface of the Earth. When it is at a height R, where R is the radius of the Earth, above the surface of the Earth its speed is $\sqrt{gR}$. Model the spacecraft as a particle and the Earth as a sphere of radius R and find, in terms of g and R, the speed with which S was fired. (You may assume that air resistance can be ignored and that the rocket's engine is turned off immediately after the rocket fired.)

Solution:

$$F = ma$$

$$\frac{mgR^2}{x^2} = -m\ddot{x}$$

S is moving away from the earth, so the acceleration is in the direction of decreasing x.

where x is the distance of S from the centre of the Earth.

$$v\frac{dv}{dx} = -g\frac{R^2}{x^2}$$

$$\int v\, dv = -g\, R^2 \int \frac{1}{x^2}\, dx$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\frac{1}{2}v^2 = g\frac{R^2}{x} + C$$

$$x = 2R \quad v = \sqrt{g\, R}$$

$$\frac{1}{2}gR = \frac{gR^2}{2R} + C$$

$$C = 0$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x}$$

$$x = R \quad \frac{1}{2}v^2 = \frac{gR^2}{R}$$

$$v^2 = 2g\, R$$

$$v = \sqrt{2gR}$$

S was fired with speed $\sqrt{2gR}$.

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Further dynamics
Exercise B, Question 4

Question:

A rocket of mass m is fired vertically upwards from the surface of the Earth with initial speed U. The Earth is modelled as a sphere of radius R and the rocket as a particle. Find an expression for the speed of the rocket when it has travelled a distance X metres. (You may assume that air resistance can be ignored and that the rocket's engine is turned off immediately after the rocket is fired.)

Solution:

$F = ma$

$$\frac{mgR^2}{x^2} = -m\ddot{x}$$

The acceleration is in the direction of decreasing x.

where x is the distance of the rocket from the centre of the Earth.

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{gR^2}{x^2}$$

Use $\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$ as the acceleration is a function of x.

$$\int v\,\mathrm{d}v = -gR^2\int\frac{1}{x^2}\,\mathrm{d}x$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

$x = R, \quad v = U$

On the Earth's surface.

$$\frac{1}{2}U^2 = g\frac{R^2}{R} + C$$

$$C = \frac{1}{2}U^2 - gR$$

$x = (X + R)$

After travelling a distance X, the rocket is a distance $(X + R)$ from the centre of the Earth.

$$\frac{1}{2}v^2 = \frac{gR^2}{(X+R)} + \frac{1}{2}U^2 - gR$$

$$v^2 = \frac{2gR^2 + U^2(X+R) - 2gR(X+R)}{(X+R)}$$

$$v = \sqrt{\left[\frac{U^2X + U^2R - 2gRX}{(X+R)}\right]}$$

When it has travelled X metres, the speed of the rocket is $\sqrt{\left[\frac{U^2X + U^2R - 2gRX}{(X+R)}\right]}$

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Further dynamics
Exercise B, Question 5

Question:

A particle is fired vertically upwards from the Earth's surface. The initial speed of the particle is u where $u^2 = 3gR$ and R is the radius of the Earth. Find, in terms of g and R, the speed of the particle when it is at a height $4R$ above the Earth's surface. (You may assume that air resistance can be ignored.)

Solution:

$$\ddot{x} = -\frac{gR^2}{x^2}$$

The acceleration is in the direction of decreasing x.

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\int v\, dv = -gR^2\int \frac{1}{x^2}\, dx$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$$

$x = R \quad v^2 = 3gR$

$$\therefore \frac{1}{2} \times 3gR = \frac{gR^2}{R} + C$$

$$C = \frac{1}{2}gR$$

$$\therefore v^2 = \frac{2gR^2}{x} + gR$$

When $x = 5R$

At a height $4R$ above the Earth's surface, $x = 5R$.

$$v^2 = \frac{2gR^2}{5R} + gR$$

$$v^2 = \frac{7gR}{5}$$

$\therefore$ The speed at a height $4R$ above the Earth's surface is $\sqrt{\frac{7gR}{5}}$.

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Further dynamics
Exercise B, Question 6

Question:

A particle is moving in a straight line towards the centre of the Earth, which is assumed to be a sphere of radius R. The particle starts from rest when its distance from the centre of the Earth is $3R$. Find the speed of the particle as it hits the surface of the Earth. (You may assume that air resistance can be ignored.)

Solution:

$$m\ddot{x} = \frac{-mgR^2}{x^2}$$

The acceleration is in the direction of decreasing x.

$$v\frac{dv}{dx} = \frac{-gR^2}{x^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\int v\,dv = -gR^2\int \frac{1}{x^2}\,dx$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

$x = 3R, \quad v = 0$

$$C = \frac{-gR^2}{3R} = \frac{-gR}{3}$$

$$\therefore \frac{1}{2}v^2 = \frac{gR^2}{x} - \frac{gR}{3}$$

$$x = R \quad \frac{1}{2}v^2 = \frac{gR^2}{R} - \frac{gR}{3}$$

$x = R$ on the surface of the Earth.

$$\frac{1}{2}v^2 = \frac{2}{3}gR$$

$$v^2 = \frac{4}{3}gR$$

$$v = 2\sqrt{\frac{gR}{3}}$$

The particle hits the surface of the Earth with speed $2\sqrt{\frac{gR}{3}}$.

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Further dynamics
Exercise C, Question 1

Question:

A particle P is moving in a straight line with simple harmonic motion. The amplitude of the oscillation is 0.5 m and P passes through the centre of the oscillation O with speed 2 m s^{-1}. Calculate

a the period of the oscillation,

b the speed of P when $OP = 0.2$ m.

Solution:

a $v^2 = \omega^2(a^2 - x^2)$

$a = 0.5, \quad x = 0 \quad v = 2$

$$2^2 = \omega^2 \times 0.5^2$$

$$\omega = \frac{2}{0.5} = 4$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$$

The period is $\frac{\pi}{2}$ s.

b $x = 0.2 \text{ m} \quad v^2 = 4^2(0.5^2 - 0.2^2)$

$$v = 1.833\ldots$$

When $OP = 0.2$ m the speed of P is 1.83 m s^{-1} (3 s.f.)

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Further dynamics
Exercise C, Question 2

Question:

A particle P is moving in a straight line with simple harmonic motion. The period is $\frac{\pi}{3}$ s and P's maximum speed is 6 m s^{-1}. The centre of the oscillation is O. Calculate

a the amplitude of the motion,

b the speed of P 0.3s after passing through O.

Solution:

a $\text{period} = \frac{2\pi}{\omega} = \frac{\pi}{3}$

$\therefore \omega = 6$

$v^2 = \omega^2(a^2 - x^2)$

$6^2 = 6^2(a^2 - 0^2)$ ← Maximum speed occurs when $x = 0$.

$\therefore a = 1$

The amplitude is 1 m.

b $x = a\sin\omega t$ ← The speed 0.3 s after passing O is required.

$v = a\omega\cos\omega t$ ← Differentiate the line above to obtain v.

$t = 0.3\text{ s}\quad v = 1 \times 6\cos(6 \times 0.3)$

$v = 6\cos 1.8$

$v = -1.363$

When $t = 0.3$, P has speed 1.36 m s^{-1} (3 s.f.) ← Speed is positive.

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Further dynamics
Exercise C, Question 3

Question:

A particle is moving in a straight line with simple harmonic motion. Its maximum speed is 10 m s^{-1} and its maximum acceleration is 20 m s^{-2}. Calculate

a the amplitude of the motion,
b the period of the motion.

Solution:

a $v^2 = \omega^2(a^2 - x^2)$

$x = 0,\ v = 10 \text{ m s}^{-1} \quad 10^2 = \omega^2 a^2$ ①

Maximum speed occurs when $x = 0$.

$\ddot{x} = -\omega^2 x$

$x = -a, \ddot{x} = 20 \text{ m s}^{-2}$

$20 = +\omega^2 a$ ②

Maximum acceleration occurs when $x = -a$.

① ÷ ② $\frac{100}{20} = +\frac{\omega^2 a^2}{\omega^2 a}$

$a = 5$

The amplitude is 5 m.

b Using ① $10 = a\omega$

$10 = 5\omega$

$\omega = 2$

$\text{period} = \frac{2\pi}{\omega} = \pi$

The period is π s.

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Further dynamics
Exercise C, Question 4

Question:

A particle is moving in a straight line with simple harmonic motion. The period of the motion is $\frac{3\pi}{5}$ s and the amplitude is 0.4 m. Calculate the maximum speed of the particle.

Solution:

$$\text{period} = \frac{2\pi}{\omega} = \frac{3\pi}{5}$$

$$\omega = \frac{10}{3}$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = \left(\frac{10}{3}\right)^2 (0.4^2 - 0)$$

Maximum speed occurs when $x = 0$.

$$v = \frac{10}{3} \times 0.4 = \frac{4}{3}$$

The maximum speed is $\frac{4}{3}$ m s^{-1}.

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Further dynamics
Exercise C, Question 5

Question:

A particle is moving in a straight line with simple harmonic motion. Its maximum acceleration is 15 m s^{-2} and its maximum speed is 18 m s^{-1}. Calculate the speed of the particle when it is 2.5 m from the centre of the oscillation.

Solution:

$$\ddot{x} = -\omega^2 x$$

$\ddot{x} = 15\text{ m s}^{-2}$, $x = a$

$$15 = \omega^2 a \quad ①$$

$$v^2 = \omega^2(a^2 - x^2)$$

First find a and ω. (See question 3.)

$v = 18\text{ m s}^{-1}$, $x = 0$ $\quad 18^2 = \omega^2 a^2 \quad ②$

② ÷ ① $\quad \dfrac{18^2}{15} = \dfrac{\omega^2 a^2}{\omega^2 a}$

$$a = \frac{18^2}{15} = 21.6$$

Using ② $\quad a\omega = 18$

$$\omega = \frac{18}{21.6} = 0.8333\ldots$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = 0.833\ldots^2(21.6^2 - 2.5^2)$$

$$v = 17.87\ldots$$

The speed is 17.9 m s^{-1} (3 s.f.)

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Further dynamics
Exercise C, Question 6

Question:

A particle P is moving in a straight line with simple harmonic motion. The centre of the oscillation is O and the period is $\frac{\pi}{2}$ s. When $OP = 1.2$ m, P has speed 1.5 m s^{-1}.

a Find the amplitude of the motion.

At time t seconds the displacement of P from O is x metres. When $t = 0$, P is passing through O.

b Find an expression for x in terms of t.

Solution:

a $\text{period} = \frac{2\pi}{\omega} = \frac{\pi}{2}$

$\omega = 4$

$v^2 = \omega^2(a^2 - x^2)$

$x = 1.2 \text{ m} \quad v = 1.5 \text{ m s}^{-1}$

$1.5^2 = 4^2(a^2 - 1.2^2)$

$a^2 = \frac{1.5^2}{4^2} + 1.2^2$

$a = 1.257\ldots$

The amplitude is 1.26 m (3 s.f.).

Use the period to find ω.

Then use $v^2 = \omega^2(a^2 - x^2)$ with $x = 1.2$ and $v = 1.5$ to find a.

b $x = a \sin \omega t$

$x = 1.26 \sin 4t$

Use $x = a \sin \omega t$ as $x = 0$ when $t = 0$.

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Further dynamics
Exercise C, Question 7

Question:

A particle is moving in a straight line with simple harmonic motion. The particle performs 6 complete oscillations per second and passes through the centre of the oscillation, O, with speed 5 m s^{-1}. When P passes through the point A the magnitude of P's acceleration is 20 m s^{-1}. Calculate

a the amplitude of the motion,
b the distance OA.

Solution:

a $\text{period} = \frac{2\pi}{\omega} = \frac{1}{6}$

$\omega = 12\pi$

$v^2 = \omega^2(a^2 - x^2)$

$5^2 = (12\pi)^2(a^2 - 0)$

$a = \frac{5}{12\pi} = 0.1326\ldots$

The amplitude is 0.133 m (3 s.f.).

The period is the time for one complete oscillation.

b $\ddot{x} = -\omega^2 x$

$20 = |-12\pi^2|\, x$

$x = \frac{20}{12\pi^2}$

$x = 0.01407\ldots$

$OA = 0.0141 \text{ m (3 s.f.)}$

You are told the magnitude of the acceleration at A.

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Further dynamics
Exercise C, Question 8

Question:

A particle P is moving on a straight line with simple harmonic motion between two points A and B. The mid-point of AB is O. When $OP = 0.6$ m, the speed of P is 3 m s^{-1} and when $OP = 0.2$ m the speed of P is 6 m s^{-1}. Find

a the distance AB,

b the period of the motion.

Solution:

a $v^2 = \omega^2(a^2 - x^2)$

$x = 0.6\text{ m}, v = 3\text{ m s}^{-1}$

$3^2 = \omega^2(a^2 - 0.6)^2$ ①

$x = 0.2\text{ m}, v = 6\text{ m s}^{-1}$

$6^2 = \omega^2(a^2 - 0.2)^2$ ②

② ÷ ① $\dfrac{6^2}{3^2} = \dfrac{\omega^2(a^2 - 0.2^2)}{\omega^2(a^2 - 0.6^2)}$

$4(a^2 - 0.6^2) = a^2 - 0.2^2$

$3a^2 = 4 \times 0.6^2 - 0.2^2$

$a^2 = \dfrac{4 \times 0.6^2 - 0.2^2}{3}$

$a = 0.6831\ldots$

The distance AB is 1.37 m (3 s.f) ← AB is twice the amplitude.

b Using ① $9 = \omega^2(0.6831^2 - 0.6^2)$

$\omega^2 = \dfrac{9}{(0.6831^2 - 0.6^2)}$

$\omega = 9.187$

$\text{period} = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{9.187} = 0.6838\ldots$

The period is 0.684 s (3 s.f.).

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Further dynamics
Exercise C, Question 9

Question:

A particle is moving in a straight line with simple harmonic motion. When the particle is 1 m from the centre of the oscillation, O, its speed is $0.1\,\text{m s}^{-1}$. The period of the motion is 2π seconds. Calculate

a the maximum speed of the particle,

b the speed of the particle when it is 0.4 m from O.

Solution:

a

$$\text{period} = \frac{2\pi}{\omega} = 2\pi$$

$$\omega = 1$$

First find ω.

$$v^2 = \omega^2(a^2 - x^2)$$

Now use $v^2 = \omega^2(a^2 - x^2)$ to find a.

$$x = 1\text{ m}, v = 0.1\text{ m s}^{-1}$$

$$0.1^2 = 1^2(a^2 - 1^2)$$

$$a^2 = 0.1^2 + 1^2$$

$$a = 1.004\ldots$$

$$v_{\max} = \omega a$$

$$= 1 \times 1.004\ldots$$

Maximum speed occurs when $x = 0$.

The maximum speed is 1.00 m s^{-1} (3 s.f.).

b

$$v^2 = 1(1.004^2 - 0.4^2)$$

$$v = 0.9219\ldots$$

The speed is 0.922 m s^{-1} (3 s.f.).

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Further dynamics
Exercise C, Question 10

Question:

A piston of mass 1.2 kg is moving with simple harmonic motion inside a cylinder. The distance between the end points of the motion is 2.5 m and the piston is performing 30 complete oscillations per minute. Calculate the maximum value of the kinetic energy of the piston.

Solution:

$$a = \frac{2.5}{2} = 1.25$$

$$\text{Period} = \frac{2\pi}{\omega} = \frac{60}{30} = 2$$

30 oscillations per minute $\Rightarrow$ 2s for 1 oscillation

$$\omega = \pi$$

$$v_{max} = a\omega$$

$$= 1.25 \times \pi$$

$$\text{maximum K.E.} = \frac{1}{2}mv_{max}^2$$

$$= \frac{1}{2} \times 1.2 \times 1.25^2 \times \pi^2$$

$$= 9.252\ldots$$

The maximum K.E. is 9.25 J (3 s.f.).

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Further dynamics
Exercise C, Question 11

Question:

A marker buoy is moving in a vertical line with simple harmonic motion. The buoy rises and falls through a distance of 0.8 m and takes 2 s for each complete oscillation.
Calculate

a the maximum speed of the buoy,

b the time taken for the buoy to fall a distance 0.6 m from its highest point.

Solution:

a $a = 0.8 \div 2 = 0.4\,\text{m}$

The amplitude is half the distance between the highest and lowest points.

$$\text{period} = \frac{2\pi}{\omega} = 2$$
$$\omega = \pi$$
$$v^2 = \omega^2(a^2 - x^2)$$
$$x = 0 \quad v = \omega a$$
$$v = \pi \times 0.4$$
$$v = 1.256\ldots$$

The maximum speed is $1.26\,\text{m s}^{-1}$ (3 s.f.).

b 0.6 m from highest point

$\Rightarrow x = -0.2\,\text{m}$

The buoy is now below the centre.

$x = a\cos\omega t$

You want the time from the highest point.

$$-0.2 = 0.4\cos\pi t$$
$$\cos\pi t = -0.5$$
$$t = \frac{1}{\pi}\cos^{-1}(-0.5)$$
$$t = \frac{1}{\pi}\times\left(\pi - \frac{\pi}{3}\right)$$
$$t = \frac{2}{3}$$

The buoy takes $\frac{2}{3}$ s to fall 0.6 m.

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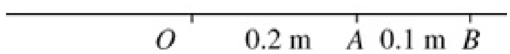
Further dynamics
Exercise C, Question 12

Question:

Points O, A and B lie in that order in a straight line. A particle P is moving on the line with simple harmonic motion. The motion has period 2 s and amplitude 0.5 m. The point O is the centre of the oscillation, $OA = 0.2\text{ m}$ and $OB = 0.3\text{ m}$. Calculate the time taken by P to move directly from A to B.

Solution:



$$\text{period} = \frac{2\pi}{\omega} = 2$$

$$\therefore \omega = \pi$$

$$x = a \sin \omega t$$

$$x = 0.5 \sin \pi t$$

Use $x = a \sin \omega t$ to find the time to go from O to A and the time to go from O to B.

$x = 0.2\text{ m}$ $\quad 0.2 = 0.5 \sin \pi t_1$

$$\pi t_1 = \sin^{-1}\left(\frac{0.2}{0.5}\right) = \sin^{-1}\left(\frac{2}{5}\right)$$

$x = 0.3$ $\quad \pi t_2 = \sin^{-1}\left(\frac{3}{5}\right)$

$$\text{time } A \to B = t_2 - t_1$$

$$= \frac{1}{\pi}\left(\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{2}{5}\right)\right)$$

$$= 0.07384\ldots$$

The time to move directly from A to B is 0.0738 (3 s.f.).

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Further dynamics
Exercise C, Question 13

Question:

A particle P is moving along the x-axis. At time t seconds the displacement, x metres, of P from the origin O is given by $x = 4\sin 2t$.

a Prove that P is moving with simple harmonic motion.

b Write down the amplitude and period of the motion.

c Calculate the maximum speed of P.

d Calculate the least value of $t\,(t > 0)$ for which P's speed is 4 m s^{-1}.

e Calculate the least value of $t\,(t > 0)$ for which $x = 2$.

Solution:

a $x = 4\sin 2t$

$\dot{x} = 8\cos 2t$

$\ddot{x} = -16\sin 2t$

$\ddot{x} = -4(4\sin 2t)$

$\ddot{x} = -4x$

$\therefore$ S.H.M.

Differentiate the given equation twice.

b amplitude $= 4$ m

period $= \dfrac{2\pi}{2} = \pi$ s

Compare $x = 4\sin 2t$ with $x = a\sin\omega t$ to obtain a and ω.

c $v^2 = \omega^2(a^2 - x^2)$

$x = 0 \quad v^2 = 4(4^2 - 0)$

$v = 8$

The maximum speed is 8 m s^{-1}.

d $x = 4\sin 2t$

$\dot{x} = 8\cos 2t$

$\dot{x} = 4\text{ m s}^{-1} \quad 4 = 8\cos 2t$

$\cos 2t = 0.5$

$t = \dfrac{1}{2}\cos^{-1}0.5$

$t = \dfrac{1}{2}\times\dfrac{\pi}{3}$

The least value of t is $\dfrac{\pi}{6}$.

From **a**.

e $x = 4\sin 2t$

$x = 2 \quad 2 = 4\sin 2t$

$\sin 2t = 0.5$

$t = \dfrac{1}{2}\sin^{-1}0.5$

$t = \dfrac{1}{2}\times\dfrac{\pi}{6}$

The least value of t is $\dfrac{\pi}{12}$.

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Further dynamics
Exercise C, Question 14

Question:

A particle P is moving along the x-axis. At time t seconds the displacement, x metres, of P from the origin O is given by $x = 3\sin\left(4t + \frac{1}{2}\right)$.

a Prove that P is moving with simple harmonic motion.
b Write down the amplitude and period of the motion.
c Calculate the value of x when $t = 0$.
d Calculate the value of $t (t > 0)$ the first time P passes through O.

Solution:

a $x = 3\sin\left(4t + \frac{1}{2}\right)$

$\dot{x} = 12\cos\left(4t + \frac{1}{2}\right)$

$\ddot{x} = -48\sin\left(4t + \frac{1}{2}\right)$

$\ddot{x} = -16x$

$\therefore$ S.H.M.

b amplitude $= 3$ m

period $= \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ s

Compare with $x = a\sin(\omega t + \varepsilon)$ to obtain a and ω.

c $t = 0 \quad x = 3\sin\left(\frac{1}{2}\right)$

$= 1.438\ldots$

When $t = 0, \quad x = 1.44$ (3 s.f.)

d $x = 0 \quad 0 = 3\sin\left(4t + \frac{1}{2}\right)$

$\sin\left(4t + \frac{1}{2}\right) = 0$

$4t + \frac{1}{2} = 0, \pi, \ldots$

$4t = \left(0 - \frac{1}{2}\right), \left(\pi - \frac{1}{2}\right), \ldots$

$t = -\frac{1}{8}$ (not applicable)

$t = \frac{1}{4}\left(\pi - \frac{1}{2}\right) = 0.6603\ldots$

The value of t is 0.660 (3 s.f.).

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Further dynamics
Exercise C, Question 15

Question:

On a certain day, low tide in a harbour is at 10 a.m. and the depth of the water is 5 m. High tide on the same day is at 4.15 p.m. and the water is then 15 m deep. A ship which needs a depth of water of 7 m needs to enter the harbour. Assuming that the water can be modelled as rising and falling with simple harmonic motion, calculate

a the earliest time, to the nearest minute, after 10 a.m. at which the ship can enter the harbour,

b the time by which the ship must leave.

Solution:

a amplitude $= \frac{(15-5)}{2} = 5$ m

The difference between high and low tides is twice the amplitude.

period $= (10 \text{ am} \rightarrow 4.15 \text{ pm}) \times 2$

$= 6.25 \times 2$ hr

$= 12.5$ hr

The time from low to high tide is half the period.

You can work through this question using hours as the unit for time.

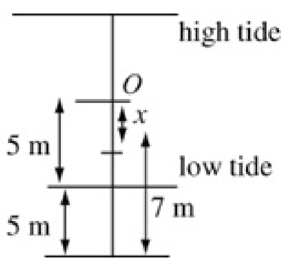
period $= \frac{2\pi}{\omega} = 12.5$

$\omega = \frac{2\pi}{12.5}$

$x = a\cos\omega t$

Start at low tide.

$x = 5\cos\left(\frac{2\pi}{12.5}t\right)$



$x = 3$ m

$3 = 5\cos\left(\frac{2\pi}{12.5}t\right)$

$\cos\left(\frac{2\pi}{12.5}t\right) = 0.6$

$t = \frac{12.5}{2\pi}\cos^{-1}0.6$

$t = 1.844\ldots$

The diagram shows that when the water is 7 m deep, $x = 3$.

Time after 10 am.

The ship can enter the harbour at 11.51 am. (nearest minute).

Remember to change the decimal part of 1.844 into minutes (0.844×60).

b The water is once more 7 m deep at $(12.5 - 1.844)$ hours after 10 am

$= 10.656$ hrs after 10 am

$= 10$ hr $39.3\ldots$ min.

$\therefore$ Ship must leave by 8.39 pm (nearest minute).

Use the symmetry of S.H.M. to find the time required.

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Further dynamics
Exercise C, Question 16

Question:

Points A, O and B lie in that order in a straight line. A particle P is moving on the line with simple harmonic motion with centre O. The period of the motion is 4 s and the amplitude is 0.75 m. The distance OA is 0.4 m and AB is 0.9 m. Calculate the time taken by P to move directly from B to A.

Solution:

$$\text{period} = \frac{2\pi}{\omega} = 4$$

$$\omega = \frac{\pi}{2}$$

$$x = a\sin\omega t$$

$$x = 0.75\sin\frac{\pi}{2}t$$

Find the time taken from O to B (using $x = 0.5\text{ m}$) and from O to the point where $x = 0.4\text{ m}$.

$x = 0.5\text{ m}$ $\quad 0.5 = 0.75\sin\frac{\pi}{2}t$

$$\sin\frac{\pi t}{2} = \frac{0.5}{0.75}$$

$$t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.5}{0.75}\right)$$

$x = 0.4\text{ m}$ $\quad t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.4}{0.75}\right)$

Time $B \to A$

$$= \frac{2}{\pi}\left[\sin^{-1}\left(\frac{0.5}{0.75}\right) + \sin^{-1}\left(\frac{0.4}{0.75}\right)\right]$$

$$= 0.8226\ldots$$

Adding these times will give the time to go directly from B to A due to the symmetry of S.H.M.

P takes 0.823s to travel directly from B to A (3 s.f.).

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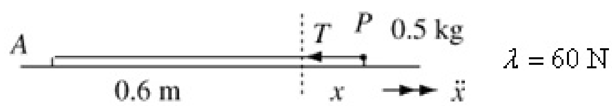
Further dynamics
Exercise D, Question 1

Question:

A particle P of mass 0.5 kg is attached to one end of a light elastic spring of natural length 0.6 m and modulus of elasticity 60 N. The other end of the spring is fixed to a point A on the smooth horizontal surface on which P rests. The particle is held at rest with $AP = 0.9$ m and then released.

a Show that P moves with simple harmonic motion.
b Find the period and amplitude of the motion.
c Calculate the maximum speed of P.

Solution:



a $F = ma$

$-T = 0.5\ddot{x}$

Hooke's law: $T = \frac{\lambda x}{l}$

$T = \frac{60x}{0.6} = 100x$

$-100x = 0.5\ddot{x}$

$\ddot{x} = -\frac{100}{0.5}x$

$\ddot{x} = -200x$

$\therefore$ S.H.M.

The equation of motion must reduce to the form $\ddot{x} = -\omega^2 x$.

b $\omega^2 = 200 \quad \omega = \sqrt{200} = 10\sqrt{2}$

period $= \frac{2\pi}{\omega} = \frac{2\pi}{10\sqrt{2}} = \frac{\pi}{10}\sqrt{2}$

$\therefore$ period is $\frac{\pi}{10}\sqrt{2}$s (or 0.444s (3 s.f.))

amplitude $= 0.9 - 0.6 = 0.3$

$\therefore$ amplitude is 0.3 m

The amplitude is the same as the initial extension.

c $v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a = 10\sqrt{2} \times 0.3$

$= 3\sqrt{2}$

The maximum speed is $3\sqrt{2}$ m s^{-1} or 4.24 m s^{-1} (3 s.f.)

Use $x = 0$ for the maximum speed.

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Further dynamics
Exercise D, Question 2

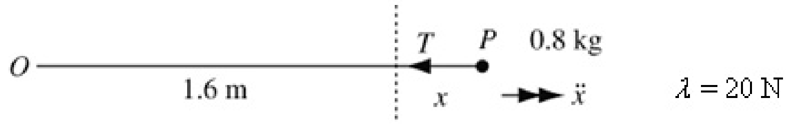
Question:

A particle P of mass 0.8 kg is attached to one end of a light elastic string of natural length 1.6 m and modulus of elasticity 20 N. The other end of the string is fixed to a point O on the smooth horizontal surface on which P rests. The particle is held at rest with $OP = 2.6$ m and then released.

a Show that, while the string is taut, P moves with simple harmonic motion.

b Calculate the time from the instant of release until P returns to its starting point for the first time.

Solution:



a $F = ma$

$-T = 0.8\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{20}{1.6}x$$

$$-\frac{20}{1.6}x = 0.8\ddot{x}$$

$$\ddot{x} = -\frac{20x}{1.6 \times 0.8} = -\frac{10x}{0.8^2}$$

$\therefore$ S.H.M.

b $$\omega = \frac{\sqrt{10}}{0.8}$$

$$\therefore \text{ period } = \frac{2\pi}{\omega} = 2\pi \times \frac{0.8}{\sqrt{10}} = \frac{1.6\pi}{\sqrt{10}}$$

amplitude $= 2.6 - 1.6 = 1\text{ m}$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a = 1 \times \frac{\sqrt{10}}{0.8}$$

total distance at this speed $= 4 \times 1.6$

$= 6.4\text{ m}$

The oscillation is split into 2 parts which are twice the natural length apart

$$\text{time} = 6.4 \times \frac{0.8}{\sqrt{10}}$$

$$\therefore \text{ total time} = 6.4 \times \frac{0.8}{\sqrt{10}} + \frac{1.6\pi}{\sqrt{10}} = 3.208\ldots$$

the total time is 3.21 s (3 s.f.)

For the middle section the particle moves at a constant speed (= the maximum speed of the S.H.M.)

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Further dynamics
Exercise D, Question 3

Question:

A particle P of mass 0.4 kg is attached to one end of a light elastic string of modulus of elasticity 24 N and natural length 1.2 m. The other end of the string is fixed to a point A on the smooth horizontal table on which P rests. Initially P is at rest with $AP = 1$ m. The particle receives an impulse of magnitude 1.8 N s in the direction AP.

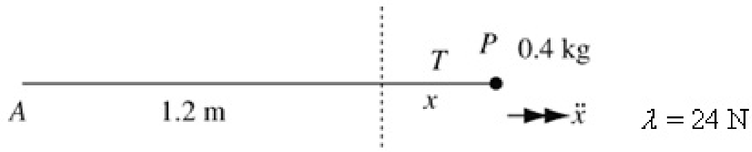
a Show that, while the string is taut, P moves with simple harmonic motion.

b Calculate the time that elapses between the moment P receives the impulse and the next time the string becomes slack.

The particle comes to instantaneous rest for the first time at the point B.

c Calculate the distance AB.

Solution:



a $F = ma$

$-T = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{\lambda x}{l}$

$T = \frac{24x}{1.2} = 20x$

$\therefore -20x = 0.4\ddot{x}$

$\ddot{x} = -\frac{20}{0.4}x$

$\ddot{x} = -50x$

$\therefore$ S.H.M.

b For the impact $I = mv - mu$

$1.8 = 0.4v$

$v = \frac{1.8}{0.4} = 4.5$ ← This is the speed of P while the string is slack. It is also the maximum speed for the S.H.M.

$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{5\sqrt{2}}$ ← The required time includes half a period.

$\therefore$ time for half an oscillation $= \frac{\pi}{5\sqrt{2}}$ s

time at constant speed

$= \frac{0.2}{4.5} = \frac{2}{45}$ s ← P travels 0.2 m before the string becomes taut.

$\text{total time} = \frac{\pi}{5\sqrt{2}} + \frac{2}{45} = 0.4887\ldots$

time is 0.489 s (3 s.f.)

c $v^2 = \omega^2(a^2 - x^2)$ ← ω and the maximum speed are known so the amplitude can be found.

$v_{max} = 4.5 \text{ m s}^{-1}$

$\therefore 4.5 = a\omega$

$a = \frac{4.5}{5\sqrt{2}}$

$AB = 1.2 + \frac{4.5}{5\sqrt{2}}$ ← AB is the natural length of the string plus the amplitude of the S.H.M.

$= 1.836$

Distance AB is 1.84 m (3 s.f.)

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Further dynamics
Exercise D, Question 4

Question:

A particle P of mass 0.8 kg is attached to one end of a light elastic spring of natural length 1.2 m and modulus of elasticity 80 N. The other end of the spring is fixed to a point O on the smooth horizontal surface on which P rests. The particle is held at rest with $OP = 0.6$ m and then released.

a Show that P moves with simple harmonic motion.
b Find the period and amplitude of the motion.
c Calculate the maximum speed of P.

Solution:

$\lambda = 80$ N

a $F = ma$

$-T = 0.8\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{80x}{1.2}$

$0.8\ddot{x} = -\frac{80}{1.2}x$

$\ddot{x} = -\frac{100}{1.2}x$

$\therefore$ SHM

b $\omega = \sqrt{\frac{100}{1.2}} = \frac{10}{\sqrt{1.2}}$

period $= \frac{2\pi}{\omega} = \frac{2\pi}{10}\sqrt{1.2}$

$= 0.6882\ldots$

period is 0.688 s (3 s.f.)

amplitude $= 1.2 - 0.6 = 0.6$ m

c $v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$= \frac{10}{\sqrt{1.2}} \times 0.6$

$= 5.477\ldots$

The max speed is 5.48 m s^{-1} (3 s.f.)

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Further dynamics
Exercise D, Question 5

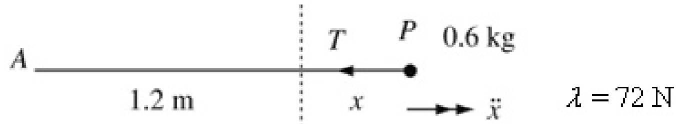
Question:

A particle P of mass 0.6 kg is attached to one end of a light elastic spring of modulus of elasticity 72 N and natural length 1.2 m. The other end of the spring is fixed to a point A on the smooth horizontal table on which P rests. Initially P is at rest with $AP = 1.2$ m. The particle receives an impulse of magnitude 3 N s in the direction AP.
Given that t seconds after the impulse the displacement of P from its initial position is x metres

a find an equation for x in terms of t,

b calculate the maximum magnitude of the acceleration of P.

Solution:



a $F = ma$

$-T = 0.6\ddot{x}$

Use $F = ma$ and Hooke's Law to obtain the value of ω.

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{72x}{1.2} = 60x$

$\therefore -60x = 0.6\ddot{x}$

$\ddot{x} = -\frac{60}{0.6}x$

$\ddot{x} = -100x$

$\therefore \omega^2 = 100,\ \omega = 10$

For the impact: $I = mv - mu$

$3 = 0.6v - 0$

$v = \frac{3}{0.6} = 5$

Use impulse = change of momentum to obtain the maximum speed.

$\therefore$ maximum speed is 5 m s^{-1}

$v^2 = \omega^2(a^2 - x^2)$

Now the amplitude can be obtained.

$v_{max} = \omega a$

$5 = 10a$

$a = \frac{5}{10} = 0.5$

$x = a \sin \omega t$

$\therefore x = 0.5 \sin 10t$

P is at the centre of the oscillation when $t = 0$.

b $\ddot{x} = -100x$

$|\ddot{x}| = 100|x|$

$|\ddot{x}|_{max} = 100 \times 0.5 = 50$

The amplitude gives the maximum value of $|x|$.

The maximum magnitude of the acceleration is 50 m s^{-2}.

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Further dynamics
Exercise D, Question 6

Question:

A particle of mass 0.9 kg rests on a smooth horizontal surface attached to one end of a light elastic string of natural length 1.5 m and modulus of elasticity 24 N. The other end of the string is attached to a point on the surface. The particle is pulled so that the string measures 2 m and released from rest.

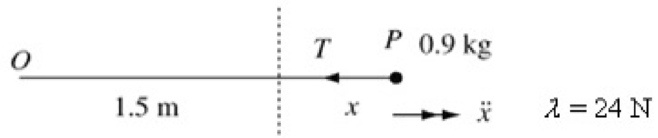
a State the amplitude of the resulting oscillation.

b Calculate the speed of the particle when the string becomes slack.

Before the string becomes taut again the particle hits a vertical surface which is at right angles to the particle's direction of motion. The coefficient of restitution between the particle and the vertical surface is $\frac{3}{5}$.

c Calculate **i** the period and **ii** the amplitude of the oscillation which takes place when the string becomes taut once more.

Solution:



a amplitude $= (2 - 1.5)\text{ m} = 0.5\text{ m}$

b energy: K.E. gained $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.9v^2$

$$\text{E.P.E. lost} = \frac{\lambda x^2}{2l} = \frac{24 \times 0.5^2}{2 \times 1.5}$$

$$\frac{1}{2} \times 0.9v^2 = 24 \times \frac{0.5^2}{2 \times 1.5}$$

$$v^2 = \frac{2 \times 24 \times 0.5^2}{0.9 \times 2 \times 1.5}$$

$$v = 2.108\ldots$$

The speed is 2.11 m s^{-1} (3 s.f.).

b can be solved by using conservation of energy or by S.H.M. methods, finding the maximum speed for the oscillation.

c Impact with the wall:

Newton's law of impact: $eu = v$

$$\therefore v = \frac{3}{5} \times 2.108\ldots$$

$$= 1.264\ldots$$

$\therefore$ maximum speed for the new oscillation is 1.264 m s^{-1}

$$F = ma$$

$$-T = 0.9\ddot{x}$$

S.H.M. methods essential for this part.

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{24}{1.5}x = 16x$$

$$\therefore -16x = 0.9\ddot{x}$$

$$\ddot{x} = -\frac{16}{0.9}x$$

$$\therefore \omega = \frac{4}{\sqrt{0.9}}$$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\frac{\sqrt{0.9}}{4} = 1.490\ldots$$

The period is 1.49s (3 s.f.).

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

Now ω is known you can find the amplitude using $v^2 = \omega^2(a^2 - x^2)$ with the maximum speed.

$$1.264 = \frac{4}{\sqrt{0.9}}a$$

$$a = 1.264 \times \frac{\sqrt{0.9}}{4}$$

$$a = 0.2997$$

The amplitude is 0.300 m (3 s.f.)

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Further dynamics
Exercise D, Question 7

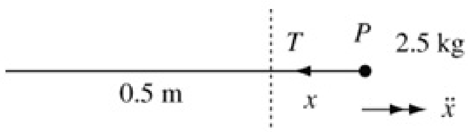
Question:

A smooth cylinder is fixed with its axis horizontal. A piston of mass 2.5 kg is inside the cylinder, attached to one end of the cylinder by a spring of modulus of elasticity 400 N and natural length 50 cm. The piston is held at rest in the cylinder with the spring compressed to a length of 42 cm. The piston is then released. The spring can be modelled as a light elastic spring and the piston can be modelled as a particle.

a Find the period of the resulting oscillations.

b Find the maximum value of the kinetic energy of the piston.

Solution:



a $F = ma$

$-T = 2.5\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{400x}{0.5} = 800x$$

$$-800x = 2.5\ddot{x}$$

$$\ddot{x} = -\frac{800}{2.5}x$$

$$\ddot{x} = -320x$$

$$\omega = \sqrt{320}$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{320}} = 0.3512\ldots$$

The period is 0.351 s (3 s.f.)

b amplitude $= (50 - 42)\text{cm}$

$= 0.08\text{ m}$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

$$= \sqrt{320} \times 0.08$$

$$\text{maximum K.E} = \frac{1}{2} \times 2.5 \times (\sqrt{320} \times 0.08)^2$$

$$= 2.56$$

The maximum K.E. is 2.56 J.

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Further dynamics
Exercise D, Question 8

Question:

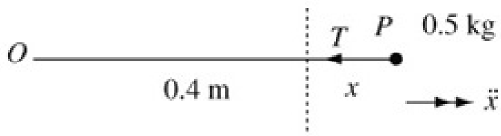
A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 0.4 m and modulus of elasticity 30 N. The other end of the string is attached to a point on the smooth horizontal surface on which P rests. The particle is pulled until the string measures 0.6 m and then released from rest.

a Calculate the speed of P when the string becomes slack for the first time.

When P has travelled a distance 0.3 m from the point of release the surface becomes rough. The coefficient of friction between P and the surface is 0.25. The particle comes to rest T seconds after it was released.

b Find the value of T.

Solution:



a $F = ma$

$-T = 0.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{30}{0.4}x = 75x$

$\therefore\ 0.5\ddot{x} = -75x$

$\ddot{x} = -\frac{75}{0.5}x$

$\ddot{x} = -150x$

$\therefore\ \omega = \sqrt{150}$

amplitude $= 0.6 - 0.4 = 0.2$ m

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = a\omega$

$= \sqrt{150} \times 0.2$

$= 2.449\ldots$

When the string becomes slack P's speed is 2.45 m s^{-1} (3 s.f.).

> **a** can be done by conservation of energy but the period of the oscillation is needed for **b**.

b period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{150}}$

> The first part of the motion is $\frac{1}{4}$ of an oscillation.

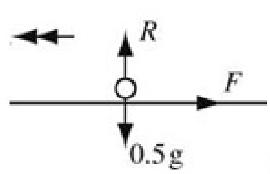
On the smooth floor:

time $= \frac{0.1}{2.449}$

> For the first 0.2 m the string is taut.

On the rough floor:

a



$\mu = 0.25$

$$-F = 0.5a$$
$$F = \mu R = 0.25 \times 0.5g$$
$$\therefore\ 0.5a = -0.25 \times 0.5g$$
$$a = -0.25g$$

Find the acceleration.

$$v = u + at$$
$$0 = 2.449 - 0.25gt$$
$$t = \frac{2.449}{0.25 \times 9.8}$$

Use $v = u + at$ with $u = 2.449$ and $a = -0.25\,g$ to find the time taken to come to rest.

$$\text{total time} = \frac{1}{4} \times \frac{2\pi}{\sqrt{150}} + \frac{0.1}{2.449} + \frac{2.449}{0.25 \times 9.8}$$
$$= 1.168\ldots$$
$$\therefore\ T = 1.17\ (3\text{ s.f.})$$

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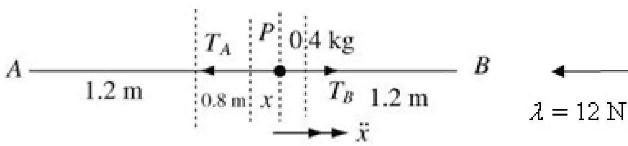
Further dynamics
Exercise D, Question 9

Question:

A particle P of mass 0.4 kg is attached to two identical light elastic springs each of natural length 1.2 m and modulus of elasticity 12 N. The free ends of the strings are attached to points A and B which are 4 m apart on a smooth horizontal surface. The point C lies between A and B with $AC = 1.4$ m and $CB = 2.6$ m. The particle is held at C and released from rest.

a Show that P moves with simple harmonic motion.
b Calculate the maximum value of the kinetic energy of P.

Solution:



The centre of the oscillation is at the mid-point of AB.

a $F = ma$

$T_B - T_A = 0.4\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

AP: extension $= (0.8 + x)$

$\therefore\ T_A = \dfrac{12(0.8+x)}{1.2} = 10(0.8+x)$

BP: extension $= (0.8 - x)$

$\therefore\ T_B = \dfrac{12(0.8-x)}{1.2} = 10(0.8-x)$

$\therefore\ 10(0.8-x) - 10(0.8+x) = 0.4\ddot{x}$

$-20x = 0.4\ddot{x}$

$\ddot{x} = -\dfrac{20}{0.4}x = -50x$

$\therefore$ P moves with S.H.M.

The tensions in the two parts of the string are different.

b $\omega^2 = 50$

amplitude $= 0.6\ m$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max}^2 = \omega^2 a^2$

$= 50 \times 0.6^2$

maximum K.E. $= \dfrac{1}{2} m v_{max}^2$

$= \dfrac{1}{2} \times 0.4 \times 50 \times 0.6^2$

$= 3.6$

The maximum K.E. is 3.6 J.

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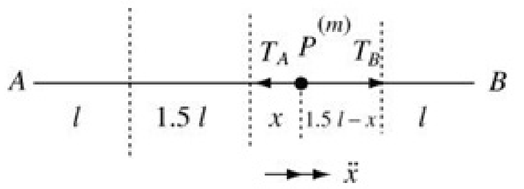
Further dynamics
Exercise D, Question 10

Question:

A particle P of mass m is attached to two identical light strings of natural length l and modulus of elasticity $3mg$. The free ends of the strings are attached to fixed points A and B which are $5l$ apart on a smooth horizontal surface. The particle is held at the point C, where $AC = l$ and A, B and C lie on a straight line, and is then released from rest.

a Show that P moves with simple harmonic motion.
b Find the period of the motion.
c Write down the amplitude of the motion.
d Find the speed of P when $AP = 3l$.

Solution:



$\lambda = 3\,mg$

The centre of the oscillation is at the mid-point of AB.

a $F = ma$

$T_B - T_A = m\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

extension $= 1.5l + x$

AP: $\quad T_A = \dfrac{3mg(1.5l + x)}{l}$

PB: extension $= 1.5l - x$

$T_B = \dfrac{3mg(1.5l - x)}{l}$

$$\therefore \frac{3mg(1.5l - x)}{l} - \frac{3mg(1.5l + x)}{l} = m\ddot{x}$$

$$-\frac{6mgx}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{6g}{l}x$$

$\therefore$ S.H.M.

b $\omega^2 = \dfrac{6g}{l} \quad \omega^2 = \sqrt{\dfrac{6g}{l}}$

period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{6g}}$

c Amplitude $= 1.5l$

d $\quad v^2 = \omega^2(a^2 - x^2)$

$AP = 3l \Rightarrow x = \dfrac{l}{2}$

$$\therefore v^2 = \frac{6g}{l}\left(\left(\frac{3l}{2}\right)^2 - \left(\frac{l}{2}\right)^2\right)$$

$$v^2 = \frac{6g}{l}\left(\frac{9l^2}{4} - \frac{l^2}{4}\right)$$

$$v^2 = \frac{6g}{l} \times \frac{8l^2}{4}$$

$$v^2 = 12gl$$

When $AP = 3l$, P's speed is $\sqrt{12gl}$ (or $2\sqrt{3gl}$).

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Further dynamics
Exercise D, Question 11

Question:

A light elastic string has natural length 2.5 m and modulus of elasticity 15 N. A particle P of mass 0.5 kg is attached to the string at the point K where K divides the unstretched string in the ratio 2 : 3. The ends of the string are then attached to the points A and B which are 5 m apart on a smooth horizontal surface. The particle is then pulled aside and held at rest in contact with the surface at the point C where $AC = 3$ m and ACB is a straight line. The particle is then released from rest.

a Show that P moves with simple harmonic motion of period $\frac{\pi}{5}\sqrt{2}$.

b Find the amplitude of the motion.

Solution:

a When P is in equilibrium:

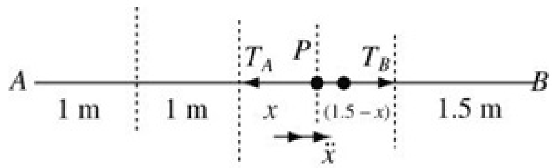
$AP = \frac{2}{5} \times 5 = 2\text{ m}$

$BP = 3\text{ m}$

Natural lengths: $AP = 1\text{ m}$

$BP = 1.5\text{ m}$

Use the ratio condition to obtain the necessary lengths for the two parts of the string.



$F = ma$

$T_B - T_A = 0.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

AP: extension $= 1 + x$

$T_A = \frac{15(1+x)}{1}$

BP: extension $= 1.5 - x$

$T_B = \frac{15(1.5 - x)}{1.5} = 10(1.5 - x)$

$\therefore 10(1.5 - x) - 15(1 + x) = 0.5\ddot{x}$

$-25x = 0.5\ddot{x}$

$\ddot{x} = -50x$

$\therefore$ S.H.M.

$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{50}} = \frac{2\pi}{5\sqrt{2}} = \frac{\pi}{5}\sqrt{2}$

b Amplitude $= (3 - 2)\text{m} = 1\text{ m}$.

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Further dynamics
Exercise E, Question 1

Question:

A particle P of mass 0.75 kg is hanging in equilibrium attached to one end of a light elastic spring of natural length 1.5 m and modulus of elasticity 80 N. The other end of the spring is attached to a fixed point A vertically above P.

a Calculate the length of the spring.

The particle is pulled downwards and held at a point B which is vertically below A. The particle is then released from rest.

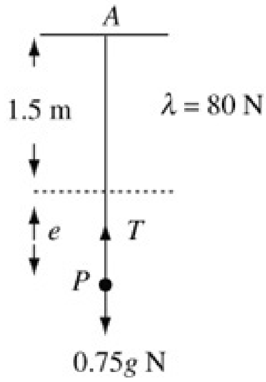
b Show that P moves with simple harmonic motion.

c Calculate the period of the oscillations.

The particle passes through its equilibrium position with speed 2.5 m s^{-1}.

d Calculate the amplitude of the oscillations.

Solution:



a In equilibrium:

$R(\uparrow)\ T = 0.75\,\text{g}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

Resolve and use Hooke's Law with the equilibrium extension (e).

$$T = \frac{80e}{1.5}$$

$$\therefore\ 0.75\text{g} = \frac{80e}{1.5}$$

$$e = 0.75 \times 9.8 \times \frac{1.5}{80}$$

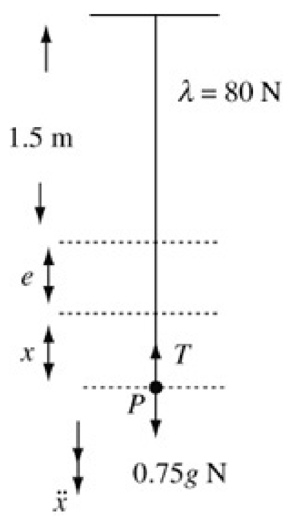
$$= 0.1378\ldots$$

$$e + l = 1.637\ldots$$

The total length of the spring is required.

The length of the spring is 1.64 m (3 s.f.)

b



$$F = ma$$

$$0.75\text{g} - T = 0.75\ddot{x}$$

x is measured from the equilibrium level, and $\ddot{x}$ is in the direction of increasing x.

Hooke's Law:

$$T = \frac{80(x+e)}{1.5}$$

$$\therefore\ 0.75\text{g} - \frac{80(x+e)}{1.5} = 0.75\ddot{x}$$

To avoid decimals use $(e+x)$ for the extension where $\dfrac{80e}{1.5} = 0.75\text{g}$ (from **a**)

from **a** $0.75\,\text{g} = \dfrac{80\ \text{e}}{1.5}$

$$\therefore\ 0.75\,\ddot{x} = -\frac{80}{1.5}x$$

$$\ddot{x} = -\frac{80}{1.5 \times 0.75}x$$

of the form $\ddot{x} = -\omega^2 x$

$\therefore$ S.H.M.

c

$$\omega^2 = \frac{80}{1.5 \times 0.75}$$

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1.5 \times 0.75}{80}}$$

$$= 0.7450\ldots$$

The period is 0.745s (3 s.f.)

d $v^2 = \omega^2(a^2 - x^2)$

$x = 0$ at the equilibrium level.

$$2.5^2 = \frac{80}{1.5 \times 0.75}a^2$$

$$a^2 = \frac{2.5^2 \times 1.5 \times 0.75}{80}$$

$$a = 0.2964\ldots$$

The amplitude is 0.296 m (3 s.f.)

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Further dynamics
Exercise E, Question 2

Question:

A particle P of mass 0.5 kg is attached to the free end of a light elastic spring of natural length 0.5 m and modulus of elasticity 50 N. The other end of the spring is attached to a fixed point A and P hangs in equilibrium vertically below A.

a Calculate the extension of the spring.

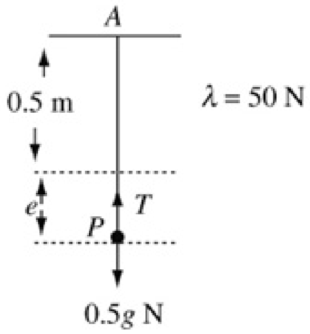
The particle is now pulled vertically down a further 0.2 m and released from rest.

b Calculate the period of the resulting oscillations.

c Calculate the maximum speed of the particle.

Solution:

a



$R(\uparrow)\ T = 0.5g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{50e}{0.5}$$

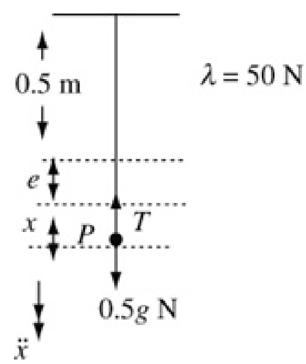
$$T = 100e$$

$$\therefore\ 100e = 0.5g$$

$$e = \frac{0.5 \times 9.8}{100} = 0.049$$

The extension is 0.049 m (or 4.9 cm)

b



$$F = ma$$

$$0.5g - T = 0.5\,\ddot{x}$$

Use $F = ma$ and Hooke's Law to find ω.

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{50(e + x)}{0.5}$$

$$T = 100(e + x)$$

$$\therefore\ 0.5g - 100(e + x) = 0.5\,\ddot{x}$$

from **a** $100e = 0.5g$

$$\therefore\ -100x = 0.5\,\ddot{x}$$

$$\ddot{x} = -200x$$

$$\omega^2 = 200$$

Compare previous line with $\ddot{x} = -\omega^2 x$.

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{200}} = \frac{2\pi}{10\sqrt{2}} = \frac{\pi}{10}\sqrt{2}$$

The period is $\dfrac{\pi}{10}\sqrt{2}$ s (or 0.444s (3 s.f.)).

c amplitude = 0.2 m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

$$= \sqrt{200} \times 0.2$$

$$= 2\sqrt{2}$$

The maximum speed occurs at the equilibrium level (i.e. when $x = 0$).

The maximum speed is $2\sqrt{2}$ m s^{-1} (or 2.83 m s^{-1} (3 s.f.)).

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Further dynamics
Exercise E, Question 3

Question:

A particle P of mass 2 kg is hanging in equilibrium attached to the free end of a light elastic spring of natural length 1.5 m and modulus of elasticity λ N. The other end of the spring is fixed to a point A vertically above P. The particle receives an impulse of magnitude 3 Ns in the direction AP.

a Find the speed of P immediately after the impact.

b Show that P moves with simple harmonic motion.

The period of the oscillations is $\frac{\pi}{2}$ s.

c Find the value of λ.

d Find the amplitude of the oscillations.

Solution:

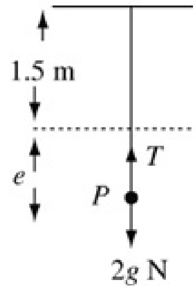
a For the impact: $I = mv - mu$

$$3 = 2v$$

$$v = 1.5$$

The speed immediately after the impact is 1.5 m s^{-1}.

b



In equilibrium

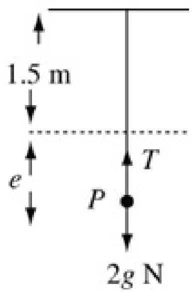
$R(\uparrow) \quad T = 2g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{\lambda e}{1.5}$$

$$\frac{\lambda e}{1.5} = 2g$$

λ is unknown so will remain in this expression.



When oscillating:

$$F = ma$$

$$2g - T = 2\ddot{x}$$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{\lambda(e + x)}{1.5}$$

$$\therefore\ 2g - \frac{\lambda(e + x)}{1.5} = 2\ddot{x}$$

From above: $\dfrac{\lambda e}{1.5} = 2g$

$$\therefore\ -\frac{\lambda x}{1.5} = 2\ddot{x}$$

$$\ddot{x} = -\frac{\lambda}{3}x$$

as $\lambda > 0$, this is S.H.M.

c period $= \dfrac{2\pi}{\omega} = \dfrac{\pi}{2}$

$$\therefore\ \omega = 4$$

From $\ddot{x} = -\dfrac{\lambda}{3}x$, $\omega^2 = \dfrac{\lambda}{3}$

$$\therefore\ \frac{\lambda}{3} = 16$$

$$\lambda = 48$$

d maximum speed $= 1.5\text{ m s}^{-1}$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

$$1.5 = 4a$$

$$a = \frac{1.5}{4} = 0.375$$

Maximum speed occurs when $x = 0$.

The amplitude is 0.375 m.

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Further dynamics
Exercise E, Question 4

Question:

A light elastic spring has one end A fixed and hangs vertically with a particle P of mass 0.6 kg attached to its free end. Initially P is hanging freely in equilibrium. The particle is then pulled vertically downwards and released from rest.

a Show that P moves with simple harmonic motion.

The period of the motion is $\frac{\pi}{5}$ s and the maximum and minimum distances of P below A are 1.2 m and 0.8 m respectively. Calculate

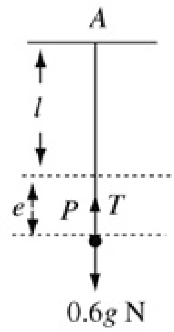
b the amplitude of the oscillation,

c the maximum speed of P,

d the maximum magnitude of the acceleration of P.

Solution:

a



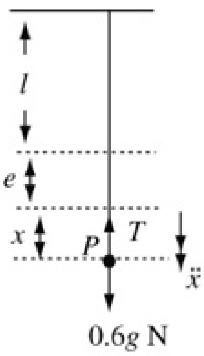
In equilibrium:

$R(\uparrow)\ T = 0.6g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{\lambda e}{l}$$

$$\therefore \frac{\lambda e}{l} = 0.6g$$



For the oscillation:

$F = ma$

$0.6g - T = 0.6\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{\lambda(e+x)}{l}$$

$$\therefore 0.6g - \frac{\lambda(e+x)}{l} = 0.6\ddot{x}$$

from above $\frac{\lambda e}{l} = 0.6g$

$$\therefore 0.6\ddot{x} = -\frac{\lambda x}{l}$$

$$\ddot{x} = -\frac{\lambda x}{0.6l}$$

As l and λ are both positive this is S.H.M.

The equation of motion must reduce to the form $\ddot{x} = -\omega^2 x$ but ω^2 can be expressed algebraically.

b amplitude $= \frac{1}{2}(1.2 - 0.8)$

$= 0.2$

The amplitude is 0.2 m.

The difference between the maximum and minimum distances below A is twice the amplitude c.

c period $= \frac{2\pi}{\omega} = \frac{\pi}{5}$

$\therefore \omega = 10$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$= 10 \times 0.2 = 2$

The maximum speed is 2 m s^{-1}.

d $\ddot{x} = -\omega^2 x$

$\ddot{x} = -100x$

Take maximum value of x.

$\therefore$ maximum magnitude of the acceleration $= 100 \times 0.2 \text{ m s}^{-2}$

$= 20 \text{ m s}^{-2}$

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Further dynamics
Exercise E, Question 5

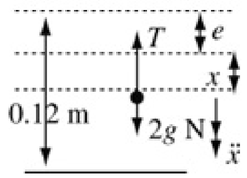
Question:

A piston of mass 2 kg moves inside a smooth cylinder which is fixed with its axis vertical. The piston is attached to the base of the cylinder by a spring of natural length 12 cm and modulus of elasticity 500 N. The piston is released from rest at a point where the spring is compressed to a length of 8 cm. Assuming that the spring can be modelled as a light elastic spring and the piston as a particle, calculate

a the period of the resulting oscillations,
b the maximum speed of the piston.

Solution:

a



$\lambda = 500$ N

In equilibrium:

$R(\uparrow)\,T = 2g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{500e}{l}$

$\therefore\ 2g = \dfrac{500e}{0.12}$ ← Change cm to m.

For the oscillations:

$F = ma$

$2g - T = 2\ddot{x}$

Hooke's Law $T = \dfrac{\lambda x}{l}$

$T = \dfrac{500(e + x)}{0.12}$

$\therefore\ 2g - \dfrac{500(e+x)}{0.12} = 2\ddot{x}$

From above: $\dfrac{500e}{0.12} = 2g$

$\therefore\ -\dfrac{500x}{0.12} = 2\ddot{x}$

$\ddot{x} = -\dfrac{250}{0.12}x$

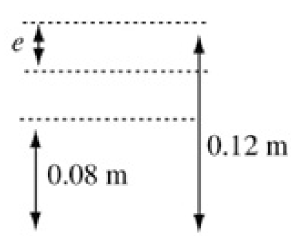
$\omega^2 = \dfrac{250}{0.12}$ ← Compare line above with $\ddot{x} = -\omega^2 x$.

$\text{period} = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{0.12}{250}}$

$= 0.1376\ldots$

The period is 0.138 s (3 s.f.)

b $e = \dfrac{2g \times 0.12}{500}$



$\text{amplitude} = 0.04 - e$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = a\omega$ ← Maximum speed occurs when $x = 0$.

$= \sqrt{\dfrac{250}{0.12}} \times (0.04 - e)$

$= \sqrt{\dfrac{250}{0.12}} \times \left(0.04 - \dfrac{2g \times 0.12}{500}\right)$

$= 1.611\ldots$

The maximum speed is $1.61\,\text{m s}^{-1}$ (3 s.f.).

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Further dynamics
Exercise E, Question 6

Question:

A light elastic string of natural length 40 cm has one end A attached to a fixed point. A particle P of mass 0.4 kg is attached to the free end of the string and hangs freely in equilibrium vertically below A. The distance AP is 45 cm.

a Find the modulus of elasticity of the string.

The particle is now pulled vertically downwards until AP measures 52 cm and then released from rest.

b Show that, while the string is taut, P moves with simple harmonic motion.

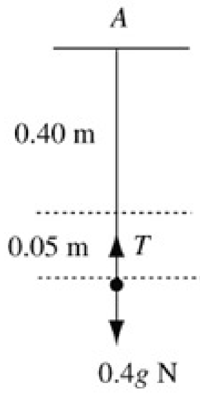
c Find the period and amplitude of the motion.

d Find the greatest speed of P during the motion.

e Find the time taken by P to rise 11 cm from the point of release.

Solution:

a



In equilibrium:

$R(\uparrow) T = 0.4g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{\lambda 0.05}{0.4}$

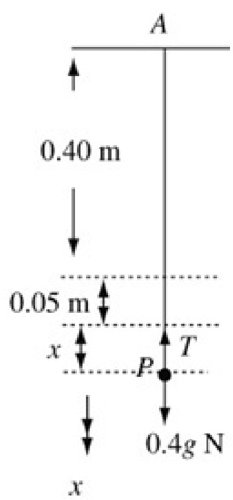
$\therefore \lambda \times \frac{0.05}{0.4} = 0.4g$

$\lambda = \frac{0.4^2}{0.05} g$

$\lambda = 3.2g = 31.36$

The modulus of elasticity is 31.4 N (3 s.f.)

b



For oscillations:

$F = ma$

$0.4g - T = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{31.36(x + 0.05)}{0.4}$

$0.4g - \frac{31.36(x+0.05)}{0.4} = 0.4\ddot{x}$

$\ddot{x} = -\frac{31.36}{0.4^2} x$

$\therefore$ S.H.M.

c From $\ddot{x} = -\frac{31.36}{0.4^2} x$

$\omega = \frac{\sqrt{31.36}}{0.4}$

$\text{period} = \frac{2\pi}{\omega} = 2\pi \times \frac{0.4}{\sqrt{31.36}} = 0.4487\ldots$

The period is 0.449 s.

amplitude $= 52 - 45 = 7$(cm)

The amplitude is 0.07 m.

d $v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$= \frac{\sqrt{31.36}}{0.4} \times 0.07$

$= 0.98$

The maximum speed is 0.98 m s^{-1}.

e 11 cm from the lowest point

$\Rightarrow AP = 41$ cm.

$\therefore\ x = -4$ cm $= -0.04$ m

$$x = a\cos\omega t$$

P starts from an end point.

$$-0.04 = 0.07\cos\omega t$$

$$\omega t = \cos^{-1}\left(-\frac{0.04}{0.07}\right) = \cos^{-1}\left(-\frac{4}{7}\right)$$

$$t = \frac{1}{\omega}\cos^{-1}\left(-\frac{4}{7}\right) = \frac{0.4}{\sqrt{31.36}}\cos^{-1}\left(-\frac{4}{7}\right)$$

$$= 0.1556\ldots$$

P takes 0.156s to rise 11 cm (3 s.f.).

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Further dynamics
Exercise E, Question 7

Question:

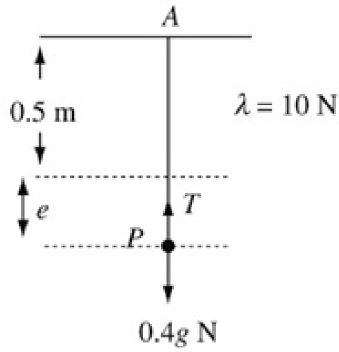
A particle P of mass 0.4 kg is attached to one end of a light elastic string of natural length 0.5 m and modulus of elasticity 10 N. The other end of the string is attached to a fixed point A and P is initially hanging freely in equilibrium vertically below A. The particle is then pulled vertically downwards a further 0.2 m and released from rest.

a Calculate the time from release until the string becomes slack for the first time.

b Calculate the time between the string first becoming slack and the next time it becomes taut.

Solution:

a



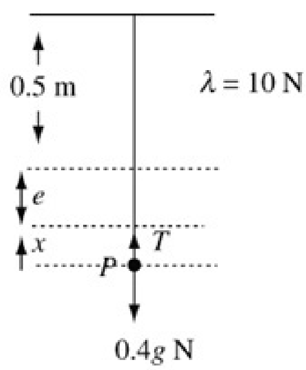
In equilibrium:

$R(\uparrow) T = 0.4g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{10e}{0.5} = 20e$

$\therefore\ 20e = 0.4g$



For the oscillations:

$F = ma$

$0.4g - T = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{10(e+x)}{0.5}$

$\therefore\ 0.4g - \frac{10(e+x)}{0.5} = 0.4\ddot{x}$

From above $0.4g = \frac{10e}{0.5}$

$\therefore\ -\frac{10x}{0.5} = 0.4\ddot{x}$

$\ddot{x} = -\frac{20x}{0.4} = -50x$

$\therefore$ S.H.M. with $\omega^2 = 50$

amplitude = 0.2 m

$x = a\cos\omega t$

$x = 0.2\cos\sqrt{50}t$

String becomes slack when $x = -e$

$-\frac{0.4g}{20} = 0.2\cos\sqrt{50}t$

$\cos\sqrt{50}t = -\frac{2g}{20} = -\frac{g}{10} = -0.98$

$\sqrt{50}t = \cos^{-1}(-0.98)$

$t = \frac{1}{\sqrt{50}}\cos^{-1}(-0.98)$

$t = 0.4159$

The string becomes slack after 0.416s (3 s.f.)

b $v^2 = \omega^2(a^2 - x^2)$

$x = -e = -\frac{0.4}{20}g$

$v^2 = 50\left(0.2^2 - \left(\frac{0.4}{20}g\right)^2\right)$

$v^2 = 0.0792$

$v = u + at$

$\sqrt{0.0792} = -\sqrt{0.0792} + 9.8t$

$t = \frac{2\sqrt{0.0792}}{9.8} = 0.05743\ldots$

The string is slack for 0.0574 s (3 s.f.)

Find the speed when the string becomes slack.

The particle moves freely under gravity while the string is slack.

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Further dynamics
Exercise E, Question 8

Question:

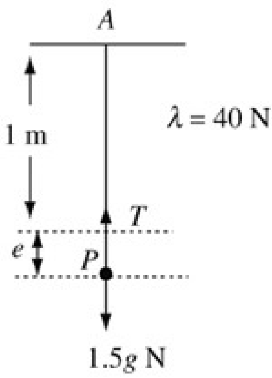
A particle P of mass 1.5 kg is hanging freely attached to one end of a light elastic string of natural length 1 m and modulus of elasticity 40 N. The other end of the string is attached to a fixed point A on a ceiling. The particle is pulled vertically downwards until AP is 1.8 m and released from rest. When P has risen a distance 0.4 m the string is cut.

a Calculate the greatest height P reaches above its equilibrium position.

b Calculate the time taken from release to reach that greatest height.

Solution:

a



In equilibrium:

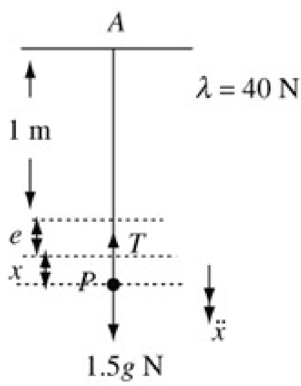
$R(\uparrow) T = 1.5g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$1.5g = \frac{40e}{1}$

$e = \frac{1.5g}{40} = 0.3675\text{ m}$

a can be done by using conservation of energy but **b** needs S.H.M. So S.H.M. has been used for both parts.



For the oscillation:

$F = ma$

$1.5g - T = 1.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$= \frac{40(x+e)}{1}$

$\therefore 1.5g - 40(x+e) = 1.5\ddot{x}$

From above $1.5g = 40e$

$\therefore 1.5\ddot{x} = -40x$

$\ddot{x} = -\frac{80}{3}x$

$\omega = \sqrt{\frac{80}{3}}$

amplitude $= 0.8 - 0.3675 = 0.4325\text{ m}$

$v^2 = \omega^2(a^2 - x^2)$

When the string is cut: $x = 0.4325 - 0.4$

$= 0.0325$

and $v^2 = \frac{80}{3}(0.4325^2 - 0.0325^2)$

$= 4.96$

Find the speed when the string is cut.

motion under gravity:

$v^2 = u^2 + 2as$

Use motion under gravity.

$0 = 4.96 - 2 \times 9.8s$

$s = \frac{4.96}{2 \times 9.8} = 0.2530\ldots$

height above equilibrium position

$= 0.2530 - 0.0325 = 0.2205$

Height is 0.221 m.

b For S.H.M. $x = a\cos\omega t$ ← Particle starts from an end-point.

$$x = 0.4325\cos\sqrt{\frac{80}{3}}t$$

$x = 0.0325$ $\quad 0.0325 = 0.4325\cos\sqrt{\frac{80}{3}}t$

$$\cos\sqrt{\frac{80}{3}}t = \frac{0.0325}{0.4325}$$

$$t = \sqrt{\frac{3}{80}}\cos^{-1}\left(\frac{0.0325}{0.4325}\right)$$

$$= 0.2896$$

Motion under gravity:

$v = u + at$

$O = \sqrt{4.96} - 9.8t$

$t = \frac{\sqrt{4.96}}{9.8}$

$\text{total time} = 0.2896\ldots + \frac{\sqrt{4.96}}{9.8} = 0.5168\ldots$

The time taken to reach the highest point is 0.517s (3 s.f.)

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Further dynamics
Exercise E, Question 9

Question:

A particle P of mass 1.5 kg is attached to the mid-point of a light elastic string of natural length 1.2 m and modulus of elasticity 15 N. The ends of the string are fixed to the points A and B where A is vertically above B and $AB = 2.8$ m.

a Given that P is in equilibrium calculate the length AP.

The particle is now pulled downwards a distance 0.15 m from its equilibrium position and released from rest.

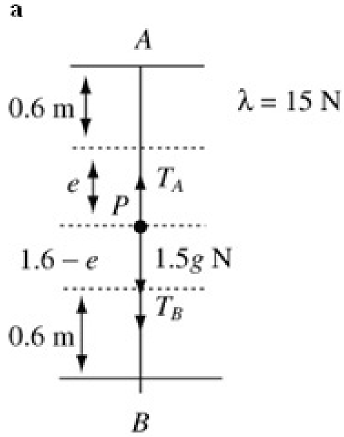
b Prove that P moves with simple harmonic motion.

T seconds after being released P is 0.1 m above its equilibrium position.

c Find the value of T.

Solution:

a



In equilibrium

$R(\uparrow)\ T_A = 1.5g + T_B$

Hooke's Law:

$$T = \frac{\lambda x}{l}$$

$$T_A = \frac{15e}{0.6} = 25e$$

$$T_B = \frac{15(1.6-e)}{0.6} = 40 - 25e$$

You must consider P to be attached to two strings. The tensions in the two parts will be different.

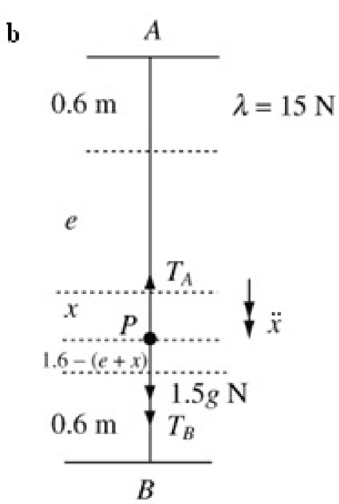
$$\therefore\ 25e = 1.5g + 40 - 25e$$

$$50e = 1.5g + 40$$

$$e = \frac{1}{50}(1.5g + 40) = 1.094$$

In equilibrium, $AP = 1.69$ m (3 s.f.)

b



For the oscillations:

$$T_A = \frac{15(e+x)}{0.6}$$

$$T_B = \frac{15(1.6-(e+x))}{0.6}$$

$F = ma$

$$1.5g + \frac{15(1.6-(e+x))}{0.6} - \frac{15(e+x)}{0.6} = 1.5\ddot{x}$$

$$1.5g + 40 - 25(e+x) - 25(e+x) = 1.5\ddot{x}$$

$$1.5g + 40 - 50e - 50x = 1.5\ddot{x}$$

from **a** $50e = 1.5g + 40$

$$\therefore\ 1.5\ddot{x} = -50x$$

$$\ddot{x} = -\frac{50}{1.5}x = -\frac{100}{3}x$$

$\therefore$ P moves with S.H.M.

c amplitude = 0.15 m

$x = a\cos\omega t = 0.15\cos\left(\frac{10}{\sqrt{3}}\right)t$

When $x = -0.1$

The equilibrium position is the centre of the oscillation.

$$-0.1 = 0.15\cos\left(\frac{10}{\sqrt{3}}T\right)$$

$$\cos\left(\frac{10}{\sqrt{3}}T\right) = -\frac{0.1}{0.15}$$

$$T = \frac{\sqrt{3}}{10}\cos^{-1}\left(-\frac{0.1}{0.15}\right)$$

$$= 0.3984\ldots$$

$$\therefore T = 0.398 \quad (3\text{ s.f.})$$

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Further dynamics
Exercise E, Question 10

Question:

A rock climber of mass 70 kg is attached to one end of a rope. He falls from a ledge which is 8 m vertically below the point to which the other end of the rope is fixed. The climber falls vertically without hitting the rock face. Assuming that the climber can be modelled as a particle and the rope as a light elastic string of natural length 16 m and modulus of elasticity 40 000 N, calculate

a the climber's speed at the instant when the rope becomes taut,
b the maximum distance of the climber below the ledge,
c the time from falling from the ledge to reaching his lowest point.

Solution:

a Until rope is taut:

$v^2 = u^2 + 2as$

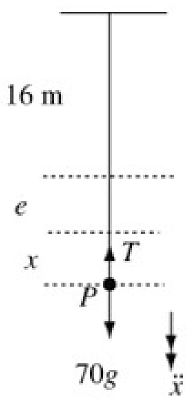
$v^2 = 0 + 2 \times 9.8 \times 8$

$v = 12.52\ldots$

When the rope becomes taut the climber's speed is 12.5 m s^{-1} (3 s.f.)

> Climber falling freely under gravity.

b



At the equilibrium level:

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{40\,000e}{16}$

$R(\uparrow)\ T = 70g$

$\therefore\ 70g = \dfrac{40\,000e}{16}$

$e = \dfrac{16 \times 70g}{40\,000} = \dfrac{7g}{250}$

> **b** can be solved by using conservation of energy. However **c** involves time, so S.H.M. methods are needed. It is more efficient to use S.H.M. for both parts.

For the oscillation:

$F = ma$

$70g - T = 70\ddot{x}$

Hooke's Law: $T = \dfrac{40\,000(x+e)}{16}$

$70g - \dfrac{40\,000(x+e)}{16} = 70\ddot{x}$

> From **a**: $70g = \dfrac{40\,000e}{16}$

$\ddot{x} = -\dfrac{4000}{16 \times 7}x = -\dfrac{250}{7}x$

$\omega^2 = \dfrac{250}{7}$

$v^2 = \omega^2(a^2 - x^2)$

> Use the result from part **a**, ie the speed when $x = e\left(= \dfrac{7g}{250}\right)$.

$156.8 = \dfrac{250}{7}\left(a^2 - \left(\dfrac{7g}{250}\right)^2\right)$

$a^2 = \dfrac{156.8 \times 7}{250} + \left(\dfrac{7g}{250}\right)^2$

$a^2 = 4.4656\ldots$

$a = 2.113\ldots$

Total distance $= 2.113 + e + 8$

$= 2.113 + \dfrac{7g}{250} + 8$

$= 10.38\ldots$

The total distance fallen is 10.4 m (3 s.f.).

> The amplitude is the greatest distance below the equilibrium level.

c $x = a\cos\omega t$

$x = 2.113\cos\sqrt{\frac{250}{7}}t$

When $x = e$

$\frac{7g}{250} = 2.113\cos\sqrt{\frac{250}{7}}t$

$t = \sqrt{\frac{7}{250}}\cos^{-1}\left(\frac{7\times 9.8}{250\times 2.113}\right)$

$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7}{250}}$

Because of the symmetry of S.H.M. there are several methods available for c.

This method assumes the oscillation is complete and finds the time from the highest point $(x = a)$ to the equilibrium level $(x = e)$. This time will be subtracted from half the period. So it does not matter that this part of the oscillation does not exist.

Time while the rope is taut:

$= \frac{2\pi}{2}\sqrt{\frac{7}{250}} - \sqrt{\frac{7}{250}}\cos^{-1}\left(\frac{7\times 9.8}{250\times 2.113}\right)$

$= 0.2846\ldots$

Time from highest point to lowest point of a complete oscillation is half the period. Subtract the time for the missing part (before the rope is taut) to obtain the time while the rope is taut.

While moving under gravity:

$s = ut + \frac{1}{2}at^2$

$8 = \frac{1}{2}\times 9.8t^2$

$t^2 = \frac{16}{9.8}$

$\text{total time} = \frac{4}{\sqrt{9.8}} + 0.2846\ldots$

$= 1.562\ldots$

The total time is 1.56 s (3 s.f.).

The time before the rope becomes taut is also needed.

Solutionbank M3

Edexcel AS and A Level Modular Mathematics

Further dynamics

Exercise F, Question 1

Question:

A particle P is moving with simple harmonic motion between two points A and B which are 0.4 m apart on a horizontal line. The mid-point of AB is O. At time $t = 0$, P passes through O, moving towards A, with speed u m s^{-1}. The next time P passes through O is when $t = 2.5$ s.

a Find the value of u.

b Find the speed of P when $t = 3$ s.

c Find the distance of P from A when $t = 3$ s.

Solution:

a amplitude $= 0.4 \div 2 = 0.2$ m

period $= 2 \times 2.5 = 5$s

$\therefore \frac{2\pi}{\omega} = 5 \quad \omega = \frac{2\pi}{5}$

$v^2 = \omega^2(a^2 - x^2)$

$u^2 = \left(\frac{2\pi}{5}\right)^2 (0.2^2 - 0)$

$u = \frac{2\pi}{5} \times 0.2 = \frac{4\pi}{50}$ (or 0.2513...)

$\therefore u = \frac{4\pi}{50}$ (or 0.251 (3 s.f.))

Find a and ω from the given information.

Now use $v^2 = \omega^2(a^2 - x^2)$ to find u.

b $x = a \sin \omega t$

$x = 0.2 \sin\left(\frac{2\pi}{5}t\right)$

$\dot{x} = \frac{2\pi}{5} \times 0.2 \cos\left(\frac{2\pi}{5}t\right)$

$t = 3 \quad \dot{x} = \frac{0.4\pi}{5} \cos \frac{6\pi}{5} = -0.2033$

When $t = 3$ P's speed is 0.203 m^{-1} (3 s.f.).

P is at the centre of oscillation when $t = 0$.

Differentiate x with respect to t to find $\dot{x}$.

Speed is positive.

c $x = a \sin \omega t$

$x = 0.2 \sin\left(\frac{2\pi}{5}t\right)$

$t = 3 \quad x = 0.2 \sin\left(\frac{6\pi}{5}\right) = -0.1175\ldots$

$\therefore$ Distance from A is $0.2 + 0.1175\ldots$

$= 0.318$ m (3 s.f.)

P is moving towards A when $t = 0$, so x is negative between O and B.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Further dynamics
Exercise F, Question 2

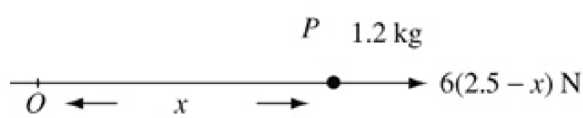
Question:

A particle P of mass 1.2 kg moves along the x-axis. At time $t = 0$, P passes through the origin O, moving in the positive x-direction. At time t seconds, the velocity of P is v m s^{-1} and $OP = x$ metres. The resultant force acting on P has magnitude $6(2.5 - x)$ N and acts in the positive x-direction. The maximum speed of P is 8 m s^{-1}.

a Write down the value of x when the speed of P is 8 m s^{-1}.

b Find an expression for v^2 in terms of x.

Solution:



a $x = 2.5$

The acceleration (and therefore the resultant force) are zero when the speed is maximum.

b

$$F = ma$$

$$6(2.5 - x) = 1.2a$$

$$6(2.5 - x) = 1.2v\frac{dv}{dx}$$

When the force is a function of x use $a = v\frac{dv}{dx}$.

$$v\frac{dv}{dx} = 5(2.5 - x)$$

$$\int v\,dv = \int 5(2.5 - x)\,dx$$

Separate the variables.

$$\frac{1}{2}v^2 = 5\left(2.5x - \frac{x^2}{2}\right) + C$$

Integrate. Don't forget the constant!

$$x = 2.5 \quad v = 8$$

a gives the initial conditions.

$$\frac{1}{2}\times 8^2 = 5\left(2.5\times 2.5 - \frac{2.5^2}{2}\right) + C$$

$$C = 32 - 5\times\frac{2.5^2}{2} = 16.375$$

$$v^2 = 10\left(2.5x - \frac{x^2}{2}\right) + 2\times 16.375$$

$$v^2 = 25x - 5x^2 + 32.75$$

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Further dynamics

Exercise F, Question 3

Question:

A particle P of mass 0.6 kg moves along the positive x-axis under the action of a single force which is directed towards the origin O and has magnitude $\frac{k}{(x+2)^2}$ N where $OP = x$ metres and k is a constant. Initially P is moving away from O. At $x = 2$ the speed of P is 8 m s^{-1} and at $x = 10$ the speed of P is 2 m s^{-1}.

a Find the value of k.

The particle first comes to instantaneous rest at the point B.

b Find the distance OB.

Solution:

a $F = ma$

$-\frac{k}{(x+2)^2} = 0.6a$

$0.6v\frac{dv}{dx} = -\frac{k}{(x+2)^2}$

The force is a function of x so use $a = v\frac{dv}{dx}$.

$0.6\int v\,dv = -\int \frac{k}{(x+2)^2}\,dx$

Separate the variables and integrate.

$0.3v^2 = \frac{k}{(x+2)} + c$

$x = 2, v = 8 \quad 0.3 \times 8^2 = \frac{k}{4} + c$

$x = 10, v = 2 \quad 0.3 \times 2^2 = \frac{k}{12} + c$

Use the given information to obtain a pair of simultaneous equations in k and c.

Subtract: $0.3(8^2 - 2^2) = \frac{k}{4} - \frac{k}{12}$

$0.3 \times 60 = \frac{k}{6}$

$k = 0.3 \times 60 \times 6 = 108$

Solve to find k.

b From above $0.3 \times 4 = \frac{k}{12} + c$

$c = 1.2 - \frac{108}{12} = -7.8$

Find c to complete the expression for v^2.

$\therefore\ 0.3v^2 = \frac{108}{(x+2)} - 7.8$

$v = 0 \quad 0 = \frac{108}{x+2} - 7.8$

$7.8(x+2) = 108$

$x = \frac{108}{7.8} - 2 = 11.84\ldots$

The distance OB is 11.8 m (3 s.f.).

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Further dynamics
Exercise F, Question 4

Question:

A particle P moves along the x-axis in such a way that at time t seconds its distance x metres from the origin O is given by $x = 3\sin\left(\frac{\pi t}{4}\right)$.

a Prove that P moves with simple harmonic motion.
b Write down the amplitude and the period of the motion.
c Find the maximum speed of P.
The points A and B are on the same side of O with $OA = 1.2\text{ m}$ and $OB = 2\text{ m}$.
d Find the time taken by P to travel directly from A to B.

Solution:

a $x = 3\sin\left(\frac{\pi}{4}t\right)$

$\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

$\ddot{x} = -3\left(\frac{\pi}{4}\right)^2 \sin\left(\frac{\pi}{4}t\right)$

Differentiate $x = 3\sin\left(\frac{\pi}{4}t\right)$ twice.

$\ddot{x} = -\left(\frac{\pi}{4}\right)^2 x$

Obtain an equation of the form $\ddot{x} = -\omega^2 x$.

$\therefore$ S.H.M.

b amplitude $= 3\,m$

period $= \frac{2\pi}{\omega} = 2\pi \times \frac{4}{\pi} = 8s$

c From **a** $\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

Or use $v_{max} = a\omega$.

$\Rightarrow$ maximum speed $= \frac{3\pi}{4}$ m s^{-1}

(or 2.36 m s^{-1} (3 s.f.))

d

$x = 3\sin\left(\frac{\pi}{4}t\right)$

At A, $x = 1.2$ $\quad 1.2 = 3\sin\left(\frac{\pi}{4}t_a\right)$

$t_a = \frac{4}{\pi}\sin^{-1}\left(\frac{1.2}{3}\right)$

At B, $x = 2$ $\quad t_b = \frac{4}{\pi}\sin^{-1}\left(\frac{2}{3}\right)$

Time $A \to B = \frac{4}{\pi}\left[\sin^{-1}\left(\frac{2}{3}\right) - \sin^{-1}\left(\frac{1.2}{3}\right)\right]$

Leave calculator work as late as possible.

$= 0.4051$

The time to go directly from A to B is 0.405 s (3 s.f.)

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Further dynamics
Exercise F, Question 5

Question:

A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 1.2 m and modulus of elasticity λ N. The other end of the string is attached to a fixed point A. The particle is hanging in equilibrium at the point O, which is 1.4 m vertically below A.

a Find the value of λ.

The particle is now displaced to a point B, 1.75 m vertically below A, and released from rest.

b Prove that while the string is taut P moves with simple harmonic motion.

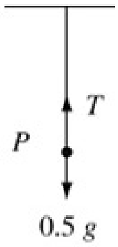
c Find the period of the simple harmonic motion.

d Calculate the speed of P at the first instant when the string becomes slack.

e Find the greatest height reached by P above O.

Solution:

a



In equilibrium:

$R(\uparrow) T = 0.5g$

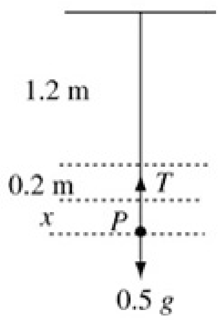
Hooke's Law: $T = \frac{\lambda x}{l}$

$0.5g = \frac{\lambda \times 0.2}{1.2}$

$\lambda = 0.5g \times \frac{1.2}{0.2}$

$\therefore \lambda = 3g \quad$ (or 29.4)

b



For oscillations:

$F = ma$

$0.5g - T = 0.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{3g(0.2+x)}{1.2}$

$\therefore 0.5g - \frac{3g(0.2+x)}{1.2} = 0.5\ddot{x}$

$\ddot{x} = -\frac{3g}{0.5 \times 1.2} x = -5gx$ ← Of form $\ddot{x} = \omega^2 x$.

$\therefore$ S.H.M.

c $\omega^2 = 5g$ ← From $\ddot{x} = -5gx$.

period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5g}} = 0.8975\ldots$

The period is 0.898s (3 s.f.).

d String becomes slack when $x = -0.2$ m.

amplitude $= 0.35$ m

$v^2 = \omega^2(a^2 - x^2)$

$v^2 = 5g(0.35^2 - 0.2^2)$ ← Use the exact value for ω^2.

$v = 2.010\ldots$

The speed is 2.01 m s^{-1} (3 s.f.).

e $v^2 = u^2 + 2as$ ← Once the string is slack the particle moves freely under gravity.

$0 = 2.010^2 - 2 \times 9.8s$

$s = \frac{2.010^2}{2 \times 9.8} = 0.2061\ldots$

Distance above $O = 0.2 + 0.2061\ldots$ ← The particle is 0.2 m above O when the string becomes slack.

$= 0.406$ m (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Further dynamics
Exercise F, Question 6

Question:

A spacecraft S of mass m is moving in a straight line towards the centre of the Earth. When the distance of S from the centre of the Earth is x metres, the force exerted by the Earth on S has magnitude $\frac{k}{x^2}$, where k is a constant, and is directed towards the centre of the Earth.

a By modelling the Erth as a sphere of radius R and S as a particle, show that $k = mgR^2$.

The spacecraft starts from rest when $x = 5R$.

b Assuming that air resistance can be ignored find the speed of S as it crashes onto the Earth's surface.

Solution:

a $F = \frac{k}{x^2}$

when $x = R, F = mg$

$\therefore \frac{k}{R^2} = mg, \quad k = mg\,R^2$

When $x = R$, S is on the surface of the Earth and the force exerted by the Earth on S is mg.

b $\text{Force} = -\frac{mg\,R^2}{x^2}$

The force is in the direction of *decreasing* x.

$F = ma$

$-\frac{mg\,R^2}{x^2} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$

The force is a function of x so use $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$.

$-\int \frac{gR^2}{x^2}\,\mathrm{d}x = \int v\,\mathrm{d}v$

$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$

$x = 5R, v = 0 \quad c = \frac{-gR^2}{5R}$

$\therefore v^2 = 2g\frac{R^2}{x} - \frac{2g\,R^2}{5R}$

When $x = R$ $v^2 = 2g\frac{R^2}{R} - \frac{2g\,R^2}{5R}$

$v^2 = \frac{8Rg}{5}$

The speed of the spacecraft is

$\sqrt{\left(\frac{8\,Rg}{5}\right)}$ or $2\sqrt{\left(\frac{2\,Rg}{5}\right)}$

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Further dynamics
Exercise F, Question 7

Question:

A particle P of mass m is attached to the mid-point of a light elastic string of natural length $4l$ and modulus of elasticity $5mg$. One end of the string is attached to a fixed point A and the other end to a fixed point B, where A and B lie on a smooth horizontal surface and $AB = 6l$. The particle is held at the point C where A, C and B are collinear and $AC = \frac{9l}{4}$, and released from rest.

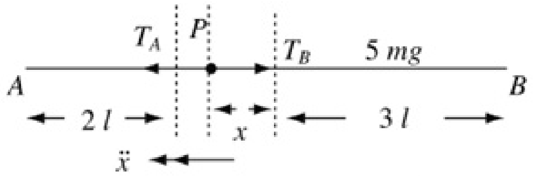
a Prove that P moves with simple harmonic motion.

Find, in terms of g and l,

b the period of the motion,

c the maximum speed of P.

Solution:



$\lambda = 5mg$

a Hooke's Law: $T = \frac{\lambda x}{l}$

$$T_A = \frac{5mg(l-x)}{2l}$$

$$T_B = \frac{5mg(l+x)}{2l}$$

Consider the particle to be attached to two strings, *AP* and *PB*, both with natural length $2l$ and modulus $5\,mg$.

$$F = ma$$

$$T_A - T_B = m\ddot{x}$$

$$\frac{5mg(l-x)}{2l} - \frac{5mg(l+x)}{2l} = m\ddot{x}$$

$$-\frac{5mgx}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{5gx}{l}$$

∴ S.H.M.

b $\omega^2 = \frac{5g}{l}$ period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{5g}}$

The period is $2\pi\sqrt{\left(\frac{l}{5g}\right)}$

c amplitude $= \frac{3l}{4}$

Find the amplitude from the given information.

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a = \sqrt{\frac{5g}{l}} \times \frac{3l}{4} = \frac{3}{4}\sqrt{5gl}$$

Maximum speed when $x = 0$.

The maximum speed is $\frac{3}{4}\sqrt{5gl}$.

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Further dynamics
Exercise F, Question 8

Question:

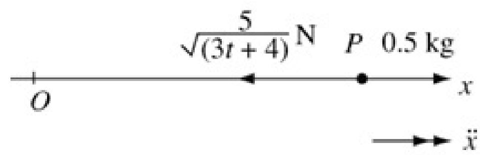
A particle P of mass 0.5 kg is moving along the x-axis, in the positive x-direction. At time t seconds (where $t > 0$) the resultant force acting on P has magnitude $\frac{5}{\sqrt{(3t+4)}}$ N and is directed towards the origin O. When $t = 0$, P is moving through O with speed 12 m s^{-1}.

a Find an expression for the velocity of P at time t seconds.

b Find the distance of P from O when P is instantaneously at rest.

Solution:

a



$$F = ma$$

$$-\frac{5}{\sqrt{(3t+4)}} = 0.5\,\ddot{x}$$

$$\ddot{x} = -10(3t+4)^{-\frac{1}{2}}$$

$$\dot{x} = -\frac{10}{\frac{1}{2}\times 3}(3t+4)^{\frac{1}{2}} + c$$

$$t = 0 \quad \dot{x} = 12 \quad 12 = -\frac{20}{3}\sqrt{4} + c$$

$$c = 12 + \frac{40}{3} = \frac{76}{3}$$

$$\therefore\ \dot{x} = -\frac{20}{3}(3t+4)^{\frac{1}{2}} + \frac{76}{3}$$

b $x = -\frac{20}{3\times\frac{3}{2}\times 3}(3t+4)^{\frac{3}{2}} + \frac{76}{3}t + A$

Integrate line above.

$$t = \ x = 0 \therefore A = \frac{40}{27}\times 4^{\frac{3}{2}} = \frac{320}{27}$$

P at rest $\Rightarrow \frac{76}{3} = \frac{20}{3}(3t+4)^{\frac{1}{2}}$

Using result from **a**.

$$\left(\frac{76}{20}\right)^2 = 3t + 4$$

$$t = \frac{1}{3}\left[\left(\frac{76}{20}\right)^2 - 4\right]$$

$$t = 3.48$$

When $t = 3.48$

$$x = -\frac{40}{27}\left(\frac{76}{20}\right)^3 + \frac{76}{3}\times 3.48 + \frac{320}{27}$$

$$x = 18.72$$

$3t + 4 = \left(\frac{76}{20}\right)^2$, so use the exact value here.

P is 18.7 m from O (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

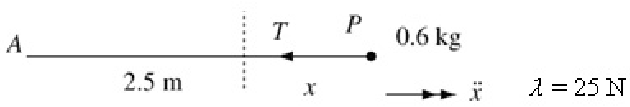
Further dynamics
Exercise F, Question 9

Question:

A particle P of mass 0.6 kg is attached to one end of a light elastic spring of natural length 2.5 m and modulus of elasticity 25 N. The other end of the spring is attached to a fixed point A on the smooth horizontal table on which P lies. The particle is held at the point B where $AB = 4$ m and released from rest.

a Prove that P moves with simple harmonic motion.
b Find the period and amplitude of the motion.
c Find the time taken for P to move 2 m from B.

Solution:



a $F = ma$

$-T = 0.6\,\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{25}{2.5}x = 10x$

$\therefore\ 0.6\,\ddot{x} = -10x$

$\ddot{x} = -\dfrac{10}{0.6}x$

$\therefore$ S.H.M.

b $\omega^2 = \dfrac{10}{0.6}$

$\text{period} = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{0.6}{10}} = 1.539\ldots$

$\text{period} = 1.54\text{ s (3 s.f.)}$

$\text{amplitude} = (4 - 2.5)\text{m} = 1.5\text{ m}$

c $x = a\cos\omega t$ ← B is an end-point.

$x = 1.5\cos\left(\sqrt{\dfrac{10}{0.6}}t\right)$

$x = -0.5\text{m} \quad -0.5 = 1.5\cos\left(\sqrt{\dfrac{10}{0.6}}t\right)$ ← D is on the other side of the centre from O so x is negative.

$t = \sqrt{\dfrac{0.6}{10}}\cos^{-1}\left(-\dfrac{0.5}{1.5}\right) = 0.4680\ldots$

P takes 0.468s to move 2 m from B (3 s.f.).

Solutionbank M3

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Further dynamics
Exercise F, Question 10

Question:

A particle P of mass 0.4 kg is attached to the mid-point of a light elastic string of natural length 1.2 m and modulus of elasticity 2.5 N. The ends of the string are attached to points A and B on a smooth horizontal table where $AB = 2$ m. The particle P is released from rest at the point C on the table, where A, C and B lie in a straight line and $AC = 0.7$ m.

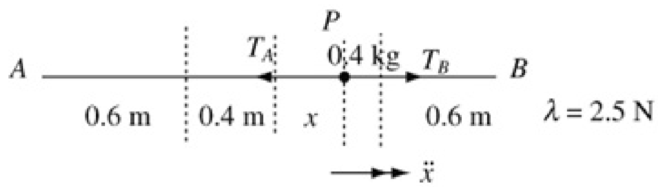
a Show that P moves with simple harmonic motion.

b Find the period of the motion.

The point D lies between A and B and $AD = 0.85$ m.

c Find the time taken by P to reach D for the first time.

Solution:



a $F = ma$

$T_B - T_A = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T_A = \frac{2.5(0.4+x)}{0.6}$

$T_B = \frac{2.5(0.4-x)}{0.6}$

$\therefore \frac{2.5(0.4-x)}{0.6} - \frac{2.5(0.4+x)}{0.6} = 0.4\ddot{x}$

$-2\times\frac{2.5x}{0.6} = 0.4\ddot{x}$

$\ddot{x} = -\frac{2\times 2.5}{0.6\times 0.4}x$

$\therefore$ S.H.M

Consider P to be attached to two strings, each of natural length 0.6 m and modulus 2.5 N.

b $\omega^2 = \frac{2\times 2.5}{0.6\times 0.4} = \frac{5}{0.24}$

period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.24}{5}} = 1.376\ldots$

The period is 1.38s (3 s.f.)

c $x = a\cos\omega t$

For time from B (an end-point)

At D, $x = 1 - 0.85 = 0.15$

$0.15 = 0.3\cos\left(\sqrt{\frac{5}{0.24}}t\right)$

D and C are on the same side of the centre, so x is positive.

$\sqrt{\frac{5}{0.24}}t = \cos^{-1}0.5$

$t = \sqrt{\frac{0.24}{5}}\cos^{-1}0.5$

$t = 0.2294\ldots$

P takes 0.229s (3 s.f.) to reach D.

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Motion in a circle
Exercise A, Question 1

Question:

Express

a an angular speed of 5 revolutions per minute in rad s^{-1},

b an angular speed of 120 revolutions per minute in rad s^{-1},

c an angular speed of 4 rad s^{-1} in revolutions per minute,

d an angular speed of 3 rad s^{-1} in revolutions per hour.

Solution:

a $5 \text{ rev min}^{-1} = 5 \times 2\pi \text{ rad min}^{-1} = \dfrac{5 \times 2\pi}{60} \text{ rad s}^{-1} \approx 0.524 \text{ rad s}^{-1}$.

b $120 \text{ rev min}^{-1} = 120 \times 2\pi \text{ rad min}^{-1} = \dfrac{120 \times 2\pi}{60} \text{ rad s}^{-1} \approx 12.6 \text{ rad s}^{-1}$.

c $4 \text{ rad s}^{-1} = 4 \times 60 \text{ rad min}^{-1} = \dfrac{4 \times 60}{2\pi} \text{ rev min}^{-1} \approx 38.2 \text{ rev min}^{-1}$.

d $3 \text{ rad s}^{-1} = 3 \times 60 \times 60 \text{ rad h}^{-1} = \dfrac{3 \times 60 \times 60}{2\pi} \text{ rev h}^{-1} \approx 1720 \text{ rev h}^{-1}$.

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Motion in a circle
Exercise A, Question 2

Question:

Find the speed in m s^{-1} of a particle moving on a circular path of radius 20 m at

a 4 rad s^{-1},

b 40 rev min^{-1}.

Solution:

a $v = r\omega$: $v = 20 \times 4 = 80 \text{ m s}^{-1}$.

b Distance per minute $= 40 \times 2\pi \times 20 = 1600\pi$ m

Distance per second $= \dfrac{1600\pi}{60} \approx 83.8 \text{ m}, v = 83.8 \text{ m s}^{-1}$.

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Motion in a circle
Exercise A, Question 3

Question:

A particle moves on a circular path of radius 25 cm at a constant speed of 2 m s^{-1}.
Find the angular speed of the particle

a in rad s^{-1},

b in rev min^{-1} .

Solution:

a $\omega = \frac{v}{r} = \frac{2}{0.25} = 8 \text{ rad s}^{-1}$ ← Need to convert cm to m.

b $8 \text{ rad s}^{-1} = 8 \times 60 \text{ rad min}^{-1} = \frac{8 \times 60}{2\pi} \approx 76.4 \text{ rev min}^{-1}$.

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Motion in a circle
Exercise A, Question 4

Question:

Find the speed in m s^{-1} of a particle moving on a circular path of radius 80 cm at

a 2.5 rad s^{-1},

b 25 rev min^{-1} .

Solution:

a $v = r\omega : v = 0.8 \times 2.5 = 2 \text{ m s}^{-1}$.

b $25 \text{ rev min}^{-1} = 25 \times 2\pi \text{ rad min}^{-1} = \dfrac{25 \times 2\pi}{60} \text{ rad s}^{-1} = 2.617\ldots \text{ rad s}^{-1}$.

$v = r\omega : v = 0.8 \times 2.617\ldots \simeq 2.09 \text{ m s}^{-1}$

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Motion in a circle
Exercise A, Question 5

Question:

An athlete is running round a circular track of radius 50 m at 7 m s^{-1}.

a How long does it take the athlete to complete one circuit of the track?

b Find the angular speed of the athlete in rad s^{-1}.

Solution:

a $\text{time} = \dfrac{\text{distance}}{\text{speed}} = \dfrac{2\pi \times 50}{7} \approx 44.9 \text{ s}$

b $\omega = \dfrac{v}{r} = \dfrac{7}{50} = 0.14 \text{ rad s}^{-1}$

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Motion in a circle
Exercise A, Question 6

Question:

A disc of radius 12 cm rotates at a constant angular speed, completing one revolution every 10 seconds. Find

a the angular speed of the disc in rad s^{-1},

b the speed of a particle on the outer rim of the disc in m s^{-1},

c the speed of a particle at a point 8 cm from the centre of the disc in m s^{-1}.

Solution:

a $1 \text{ rev in } 10 \text{ s} = 0.1 \text{ rev s}^{-1} = 0.1 \times 2\pi \text{ rad s}^{-1} \approx 0.628 \text{ rad s}^{-1}$.

b $v = r\omega : v = 0.12 \times 0.628\ldots \approx 0.0754 \text{ m s}^{-1}$.

c $v = r\omega : v = 0.08 \times 0.628\ldots \approx 0.0503 \text{ m s}^{-1}$

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Motion in a circle
Exercise A, Question 7

Question:

A cyclist completes two circuits of a circular track in 45 seconds. Calculate

a his angular speed in rad s^{-1},

b the radius of the track given that his speed is 40 km h^{-1}.

Solution:

a 2 circuits $= 2 \times 2\pi$ radians in 45 seconds $= \dfrac{4\pi}{45} \text{ rad s}^{-1} \approx 0.279 \text{ rad s}^{-1}$.

b $40 \text{ km h}^{-1} = \dfrac{40 \times 1000}{3600} = 11.1\ldots \text{ m s}^{-1}$.

$$r = \frac{v}{\omega} = \frac{11.111}{0.279} \approx 39.8 \text{ m}$$

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Motion in a circle
Exercise A, Question 8

Question:

Anish and Bethany are on a fairground roundabout. Anish is 3 m from the centre and Bethany is 5 m from the centre. If the roundabout completes 10 revolutions per minute, calculate the speeds with which Anish and Bethany are moving.

Solution:

$$10 \text{ rev min}^{-1} = 10 \times 2\pi \text{ rad min}^{-1} = \frac{10 \times 2\pi}{60} \text{ rad s}^{-1} \approx 1.05 \text{ rad s}^{-1}.$$

$v = r\omega$: Anish's speed $= 3 \times 1.047\ldots \approx 3.14 \text{ m s}^{-1}$,

Bethany's speed $= 5 \times 1.047\ldots \approx 5.24 \text{m s}^{-1}$.

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Motion in a circle
Exercise A, Question 9

Question:

A model train completes one circuit of a circular track of radius 1.5 m in 26 seconds.
Calculate

a the angular speed of the train in rad s^{-1},

b the linear speed of the train in m s^{-1}.

Solution:

a 1 circuit in 26 seconds $= \frac{2\pi}{26} \text{ rad s}^{-1} \approx 0.242 \text{ rad s}^{-1}$.

b $v = r\omega : v = 1.5 \times 0.24166\ldots = 0.362 \text{ m s}^{-1}$

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Motion in a circle
Exercise A, Question 10

Question:

A train is moving at 150 km h^{-1} round a circular bend of radius 750 m. Calculate the angular speed of the train in rad s^{-1}.

Solution:

$$150 \text{ km h}^{-1} = \frac{150 \times 1000}{3600} \text{ m s}^{-1} \approx 41.7 \text{ m s}^{-1}$$

$$\omega = \frac{v}{r} = \frac{41.66666}{750} \approx 0.056 \text{ rad s}^{-1}$$

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Motion in a circle
Exercise A, Question 11

Question:

The hour hand on a clock has radius 10 cm, and the minute hand has radius 15 cm.
Calculate

a the angular speed of the end of each hand,

b the linear speed of the end of each hand.

Solution:

a Hour hand: 2π radians in 12 hours $= \dfrac{2\pi}{12 \times 3600}$ rad s^{-1} ≈ 0.000145 rad s^{-1}.

Minute hand: 2π radians in 1 hour $= \dfrac{2\pi}{3600}$ rad s^{-1} ≈ 0.00175 rad s^{-1}.

b $v = r\omega$: End of hour hand moves at $0.1 \times 0.000145 = 1.45 \times 10^{-5}$ m s^{-1}.

End of minute hand moves at $0.15 \times 0.00175 = 2.62 \times 10^{-4}$ m s^{-1}.

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Motion in a circle
Exercise A, Question 12

Question:

The drum of a washing machine has diameter 50 cm. The drum spins at 1200 rev min^{-1}. Find the linear speed of a point on the drum.

Solution:

$1200 \text{ rev min}^{-1} = 1200 \times 2\pi \text{ rad min}^{-1} = \frac{1200 \times 2\pi}{60} \text{ rad s}^{-1} \approx 126 \text{ rad s}^{-1}$.

$v = r\omega : v = 125.66.. \times 0.5 = 62.8 \text{ m s}^{-1}$.

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Motion in a circle
Exercise A, Question 13

Question:

A gramophone record rotates at 45 rev min^{-1}.

a Find the angular speed of the record in rad s^{-1}.

b Find the distance from the centre of a point moving at 12 cm s^{-1}.

Solution:

a $45 \text{ rev min}^{-1} = 45 \times 2\pi \text{ rad min}^{-1} = \dfrac{45 \times 2\pi}{60} \text{ rad s}^{-1} \approx 4.71 \text{ rad s}^{-1}$.

b $r = \dfrac{v}{\omega} = \dfrac{12}{4.712} \approx 2.55 \text{ cm}$ ← Working is in cm, not m here.

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Motion in a circle
Exercise A, Question 14

Question:

The Earth completes one orbit of the sun in a year. Taking the orbit to be a circle of radius 1.5×10^{11} m, and a year to be 365 days, calculate the speed at which the Earth is moving.

Solution:

Distance travelled in one year $= 2\pi \times 1.5\times 10^{11}$ m,

So speed $= \dfrac{2\pi \times 1.5\times 10^{11}}{365\times 24\times 3600} \approx 30\,000$ m s^{-1}.

We can not achieve more than 2 s.f. accuracy in the answer because one of the original values only contained 2 s.f.

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Motion in a circle
Exercise B, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle is moving on a horizontal circular path of radius 16 cm with a constant angular speed of 5 rad s^{-1}. Calculate the acceleration of the particle.

Solution:

$a = r\omega^2 : a = 0.16 \times 25 = 4 \text{ m s}^{-2}$.

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Motion in a circle
Exercise B, Question 2

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle is moving on a horizontal circular path of radius 0.3 m at a constant speed of 2.5 m s^{-1}. Calculate the acceleration of the particle.

Solution:

$$a = \frac{v^2}{r} : a = \frac{2.5^2}{0.3} \approx 20.8 \text{ m s}^{-2}.$$

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Motion in a circle
Exercise B, Question 3

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle is moving on a horizontal circular path of radius 3 m. Given that the acceleration of the particle is 75 m s^{-2} towards the centre of the circle, find

a the angular speed of the particle,

b the linear speed of the particle.

Solution:

a $a = r\omega^2 : 75 = 3\omega^2, \omega^2 = 25, \omega = 5 \text{ rad s}^{-1}$.

b $a = \frac{v^2}{r} : 75 = \frac{v^2}{3}, v^2 = 3 \times 75 = 225, v = 15 \text{ m s}^{-1}$.

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Motion in a circle
Exercise B, Question 4

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A particle is moving on a horizontal circular path of diameter 1.2 m. Given that the acceleration of the particle is 100 m s^{-2} towards the centre of the circle, find

a the angular speed of the particle,
b the linear speed of the particle.

Solution:

a $a = r\omega^2 : 100 = 0.6\omega^2, \omega^2 \approx 166.7, \omega \approx 12.9\text{ rad s}^{-1}$.

b $a = \dfrac{v^2}{r} : 100 = \dfrac{v^2}{0.6}, v^2 = 100 \times 0.6 = 60, v \approx 7.75\text{ m s}^{-1}$.

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Motion in a circle
Exercise B, Question 5

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car is travelling round a bend which is an arc of a circle of radius 90 m. The speed of the car is 50 km h^{-1}. Calculate its acceleration.

Solution:

$$50 \text{ km h}^{-1} = \frac{50 \times 1000}{3600} \approx 13.89 \text{ m s}^{-1}.$$

$$a = \frac{v^2}{r} : a = \frac{13.89^2}{90} \approx 2.14 \text{ m s}^{-2}.$$

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Motion in a circle
Exercise B, Question 6

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car moving along a horizontal road which follows an arc of a circle of radius 75 m has an acceleration of 6 m s^{-2} directed towards the centre of the circle. Calculate the angular speed of the car.

Solution:

$a = r\omega^2 : 6 = 75\omega^2, \omega^2 = 0.08, \omega \approx 0.283 \text{ rad s}^{-1}$.

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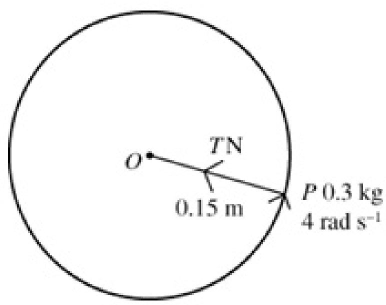
Motion in a circle
Exercise B, Question 7

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
One end of a light inextensible string of length 0.15 m is attached to a particle P of mass 300 g. The other end of the string is attached to a fixed point O on a smooth horizontal table. P moves in a horizontal circle centre O at constant angular speed 4 rad s^{-1}. Find the tension in the string.

Solution:



Suppose that the tension in the string is T.
Using $F = ma$
$T = 0.3 \times 0.15 \times 4^2 = 0.72 \text{ N}$

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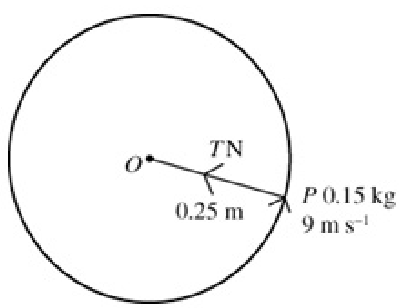
Motion in a circle
Exercise B, Question 8

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
One end of a light inextensible string of length 25 cm is attached to a particle P of mass 150 g. The other end of the string is attached to a fixed point O on a smooth horizontal table. P moves in a horizontal circle centre O at constant speed 9 m s^{-1}. Find the tension in the string.

Solution:



Suppose that the tension in the string is T.
Using $F = ma$

$$T = \frac{0.15 \times 9^2}{0.25} = 48.6 \text{ N}$$

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Motion in a circle
Exercise B, Question 9

Question:

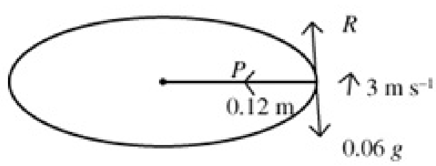
Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A smooth wire is formed into a circle of radius 0.12 m. A bead of mass 60 g is threaded onto the wire. The wire is horizontal and the bead is made to move along it with a constant speed of 3 m s^{-1}. Find

a the vertical component of the force on the bead due to the wire,

b the horizontal component of the force on the bead due to the wire.

Solution:



a Suppose the vertical component of the force is R:

$R(\updownarrow) R = 0.06g = 0.59 \text{ N}$

b Suppose the horizontal component of the force is F:

Using $F = ma$,

$$F = \frac{0.06 \times 3^2}{0.12} = 4.5 \text{ N}$$

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Motion in a circle
Exercise B, Question 10

Question:

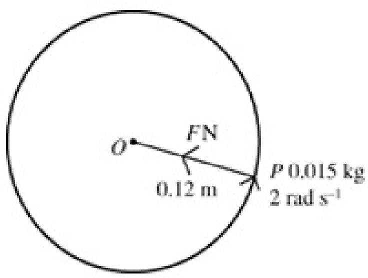
Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle P of mass 15 g rests on a rough horizontal disc at a distance 12 cm from the centre. The disc rotates at a constant angular speed of 2 rad s^{-1}, and the particle does not slip. Calculate

a the linear speed of the particle,

b the force due to the friction acting on the particle.

Solution:



a $v = r\omega$: $v = 0.12 \times 2 = 0.24 \text{ m s}^{-1}$

b Using $F = ma$

$F = 0.015 \times 0.12 \times 2^2 = 0.0072 \text{ N}$

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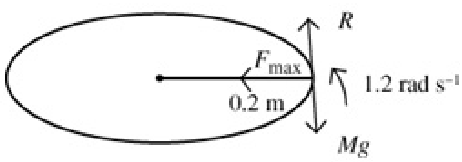
Motion in a circle
Exercise B, Question 11

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle P rests on a rough horizontal disc at a distance 20 cm from the centre. When the disc rotates at constant angular speed of 1.2 rad s^{-1}, the particle is just about to slip. Calculate the value of the coefficient of friction between the particle and the disc.

Solution:



Let R be the normal reaction between the particle and the disc, F the frictional force, M the mass of the particle, and μ be the coefficient of friction between the particle and the disc.

R(↕): $R = Mg$

The particle is about to slip, so $F = F_{max} = \mu R = \mu Mg$.

Using $F = ma$, $\mu Mg = M \times 0.2 \times 1.2^2 = M \times 0.288$,

$$\mu = \frac{0.288}{g} \approx 0.029$$

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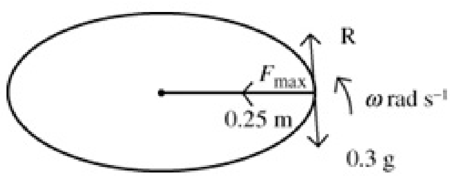
Motion in a circle
Exercise B, Question 12

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle P of mass 0.3 kg rests on a rought horizontal disc at a distance 0.25 m from the centre of the disc. The coefficient of friction between the particle and the disc is 0.25. Given that P is on the point of slipping, find the angular speed of the disc.

Solution:



Let R be the normal reaction between the particle and the disc, F the frictional force.

Given $\mu = 0.25$ and $F = F_{max} = 0.25R$.

R(↕): $R = 0.3g$, so $F_{max} = 0.25 \times 0.3g$.

Using $F = ma$, $0.25 \times 0.3\, g = 0.3 \times 0.25 \times \omega^2$,

$g = \omega^2$

$\omega \approx 3.1 \text{ rad s}^{-1}$.

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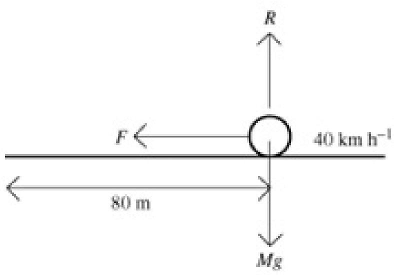
Motion in a circle
Exercise B, Question 13

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car is travelling round a bend in the road which is an arc of a circle of radius 80 m. The greatest speed at which the car can travel round the bend without slipping is 40 km h^{-1}. Find the coefficient of friction between the tyres of the car and the road.

Solution:



$40 \text{ km h}^{-1} = \frac{40 \times 1000}{3600} \approx 11.11 \text{ m s}^{-1}$.

Let the mass of the car be M. Let F be the force due to friction between the car tyres and the road, μ the coefficient of friction, and R the normal reaction between the car and the road.

At maximum speed the car is about to slip, so $F = F_{max}$

R(↕) $R = Mg$, so $F = F_{max} = \mu R = \mu Mg$

R(↔) Using $F = ma$, $\mu Mg = \frac{M \times 11.11^2}{80}$, $\mu = \frac{11.11^2}{80 \times 9.8} \approx 0.16$

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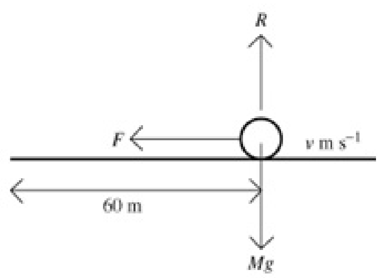
Motion in a circle
Exercise B, Question 14

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car is travelling round a bend in the road which is an arc of a circle of radius 60 m. The coefficient of friction between the tyres of the car and the road is $\frac{1}{3}$. Find the greatest angular speed at which the car can travel round the bend without slipping.

Solution:



Let the mass of the car be M.
Let F be the force due to friction between the car tyres and the road, and R the normal reaction between the car and the road.
Max speed $\Rightarrow F = F_{max}$.

R(↕) $R = ma$, so $F = F_{max} = \mu R = \frac{1}{3}Mg$.

R(↔) Using $F = ma$, $\frac{1}{3}Mg = M \times 60 \times \omega^2$, $\omega^2 = \frac{g}{180} \approx 0.054$, $\omega \approx 0.23 \text{ rad s}^{-1}$.

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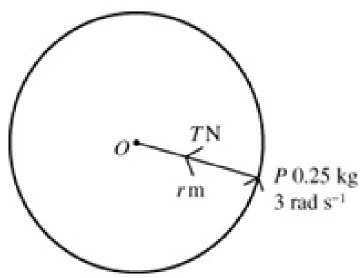
Motion in a circle
Exercise B, Question 15

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
One end of a light extensible string of natural length 0.3 m and modulus of elasticity 10 N is attached to a particle P of mass 250 g. The other end of the string is attached to a fixed point O on a smooth horizontal table. P moves in a horizontal circle centre O at constant angular speed 3 rad s^{-1}. Find the radius of the circle.

Solution:



If the extension in the string is x m, then the radius of the circle is $(0.3 + x)$ m, and the tension in the string is given by

$$T = \frac{\lambda x}{a} = \frac{10x}{0.3}$$

Using $F = ma$, $\frac{10x}{0.3} = 0.25 \times (0.3 + x) \times 3^2$, $10x = \frac{9}{4} \times \frac{3}{10}(0.3 + x)$

$400x = 8.1 + 27x$, $373x = 8.1$, $x \approx 0.0217 \text{ m}$

$\Rightarrow$ the radius of the circle ≈ 0.322 m

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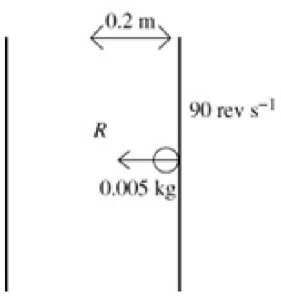
Motion in a circle
Exercise B, Question 16

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A centrifuge consists of a vertical hollow cylinder of radius 20 cm rotating about a vertical axis through its centre at 90 rev s^{-1}. Calculate the magnitude of the normal reaction between the cylinder and a particle of mass 5 g on the inner surface of the cylinder.

Solution:



$$90 \text{ rev s}^{-1} = 90 \times 2\pi \text{ rad s}^{-1}$$
$$= 180\pi \text{ rad s}^{-1}.$$

Let the normal reaction between the particle and the cylinder be R.

$(\leftrightarrow)$ Using $F = ma$, $R = 0.005 \times 0.2 \times (180\pi)^2 \approx 320 \text{ N}$

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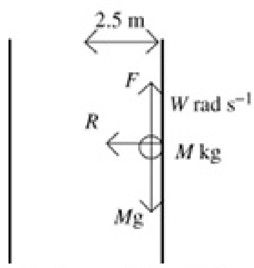
Motion in a circle
Exercise B, Question 17

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A fairground ride consists of a vertical hollow cylinder of diameter 5 m which rotates about a vertical axis through its centre. When the ride is rotating at W rad s^{-1} the floor of the cylinder opens. The people on the ride remain, without slipping, in contact with the inner surface of the cylinder. Given that the coefficient of friction between a person and the inner surface of the cylinder is $\frac{2}{3}$, find the minimum value for W.

Solution:



Suppose that the person has mass M. Let the normal reaction between the person and the cylinder be R. F is the frictional force between the person and the wall of the cylinder.

Minimum $W \Rightarrow$ the person is about to slip $\Rightarrow F = F_{max} = \frac{2}{3}R$

Also, R(↕) $\Rightarrow F_{max} = Mg$, so $Mg = \frac{2}{3}R, R = \frac{3Mg}{2}$.

↔Using $F = ma$,

$$R = \frac{3Mg}{2} = M \times 5 \times W^2, W^2 = \frac{3g}{5}, W = \sqrt{\frac{3g}{5}} \approx 2.42 \text{ rad s}^{-1}.$$

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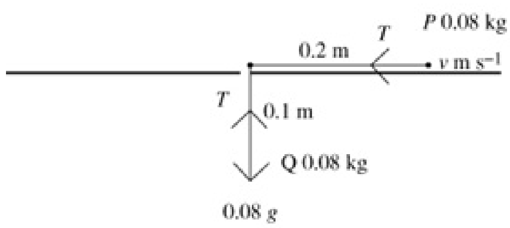
Motion in a circle
Exercise B, Question 18

Question:

Whenever a numerical value of g is required take $g = 9.8\ \text{m s}^{-2}$.
Two particles P and Q, both of mass 80 g, are attached to the ends of a light inextensible string of length 30 cm. Particle P is on a smooth horizontal table, the string passes through a small smooth hole in the centre of the table, and particle Q hangs freely below the table at the other end of the string. P is moving on a circular path about the centre of the table at constant linear speed. Find the linear speed at which P must move if Q is in equilibrium 10 cm below the table.

Solution:



Let the tension in the string be T N, and the speed of P be $v\ \text{m s}^{-1}$.
Q is in equilibrium, so
R(↕) at $Q \Rightarrow T = 0.08\ g$

For P, Using $F = ma$,

$$T = 0.08\ g = \frac{0.08v^2}{0.2}, v^2 = 0.2 \times g \approx 1.96, v \approx 1.4\ \text{m s}^{-1}.$$

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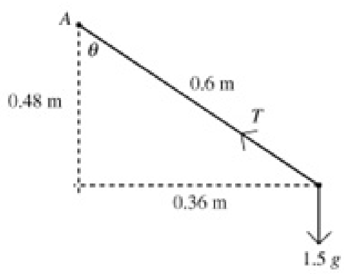
Motion in a circle
Exercise C, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle of mass 1.5 kg is attached to one end of a light inextensible string of length 60 cm. The other end of the string is attached to a fixed point A. The particle moves with constant angular speed in a horizontal circle of radius 36 cm. The centre of the circle is vertically below A. Calculate the tension in the string and the angular speed of the particle.

Solution:



Let the tension in the string be T, and the angular speed be ω. The angle between the string and the vertical is θ. Since the triangle is right angled, the third side will have length 0.48 m (3, 4, 5 triangle).

$R(\updownarrow): T\cos\theta = 1.5\,g$

$$\Rightarrow \frac{4}{5}T = \frac{3g}{2}, T = \frac{15\,g}{8} \approx 18 \text{ N}$$

$R(\leftrightarrow): T\sin\theta = mr\omega^2$

$$\Rightarrow \frac{3}{5}T = \frac{3}{2}\times 0.36\times\omega^2, T = 0.9\omega^2$$

Equating the two expressions for T:

$$\frac{15g}{8} = 0.9\omega^2$$

$$\omega^2 = \frac{15g}{0.9\times 8} \approx 20.41, \omega \approx 4.5 \text{ rad s}^{-1}.$$

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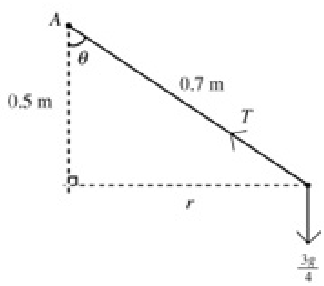
Motion in a circle
Exercise C, Question 2

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle of mass 750 g is attached to one end of a light inextensible string of length 0.7 m. The other end of the string is attached to a fixed point A. The particle moves with constant angular speed in a horizontal circle whose centre is 0.5 m vertically below A. Calculate the tension in the string and the angular speed of the particle.

Solution:



Let the tension in the string be T, and the angular speed be ω. The angle between the string and the vertical is θ, and the radius of the circle is r.

$$R(\updownarrow): T\cos\theta = \frac{3g}{4}$$

$$\Rightarrow \frac{5}{7}T = \frac{3g}{4}, T = \frac{21g}{20} \approx 10 \text{ N}$$

$$R(\leftrightarrow): T\sin\theta = mr\omega^2$$

$$\Rightarrow \frac{r}{0.7}T = \frac{3}{4}\times r\times\omega^2, T = \frac{3}{4}\times 0.7\omega^2$$

Equating the two expressions for T:

$$\frac{21g}{20} = \frac{3}{4}\times 0.7\omega^2$$

$$\omega^2 = \frac{7g}{5\times 0.7} \approx 19.6, \omega \approx 4.4 \text{ rad s}^{-1}.$$

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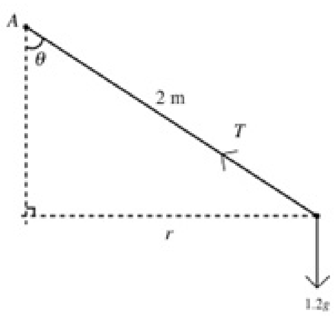
Motion in a circle
Exercise C, Question 3

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle of mass 1.2 kg is attached to one end of a light inextensible string of length 2 m. The other end of the string is attached to a fixed point A. The particle moves in a horizontal circle with constant angular speed. The centre of the circle is vertically below A. The particle takes 2 seconds to complete one revolution. Calculate the tension in the string and the angle between the string and the vertical.

Solution:



Let the tension in the string be T, and the angular speed be ω. The angle between the string and the vertical is θ, and the radius of the circle is r.

2 seconds to complete 2π radians $\Rightarrow$ angular speed is $\pi \text{ rad s}^{-1}$.

$R(\updownarrow): T\cos\theta = 1.2g$

$R(\leftrightarrow): T\sin\theta = mr\omega^2$

$$\Rightarrow T \times \frac{r}{2} = 1.2 \times r \times \pi^2, T = 2.4\pi^2$$

$$= 23.7 \text{ N}$$

and using this value in the first equation gives

$$\theta = \cos^{-1}\left(\frac{1.2g}{T}\right) \approx \cos^{-1} 0.496 \approx 60°.$$

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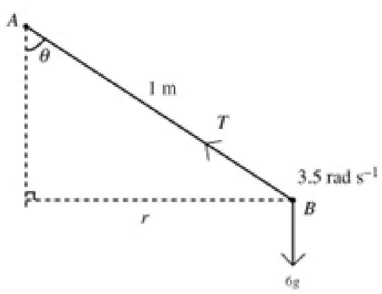
Motion in a circle
Exercise C, Question 4

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A conical pendulum consists of a light inextensible string AB of length 1 m, fixed at A and carrying a small ball of mass 6 kg at B. The particle moves in a horizontal circle, with centre vertically below A, at constant angular speed 3.5 rad s^{-1}. Find the tension in the string and the radius of the circle.

Solution:



Let the tension in the string be T. The angle between the string and the vertical is θ, and the radius of the circle is r.

$R(\updownarrow): T\cos\theta = 6g$

$R(\leftrightarrow): T\sin\theta = 6 \times r \times 3.5^2$

$$T \times \frac{r}{1} = 73.5r, \; T = 73.5 \text{ N}$$

and using this value in the first equation gives

$73.5\cos\theta = 6g$, $\cos\theta = 0.8$ radius $= \sin\theta = 0.6$ m

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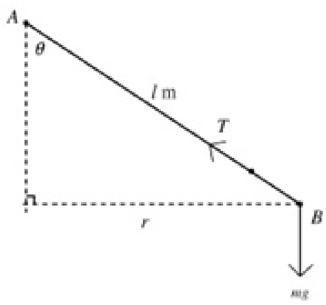
Motion in a circle
Exercise C, Question 5

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A conical pendulum consists of a light inextensible string AB of length l, fixed at A and carrying a small ball of mass m at B. The particle moves in a horizontal circle, with centre vertically below A, at constant angular speed ω. Find, in terms of m, l and ω, the tension in the string.

Solution:



Let the tension in the string be T. The angle between the string and the vertical is θ, and the radius of the circle is r.

$$R(\leftrightarrow): T\sin\theta = m \times r \times \omega^2$$

$$T\frac{r}{l} = m \times r \times \omega^2$$

$$T = ml\omega^2$$

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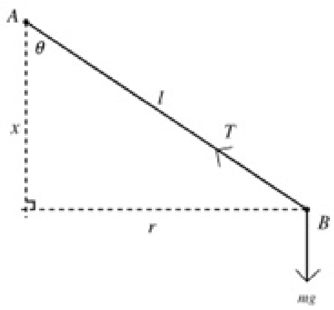
Motion in a circle
Exercise C, Question 6

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A conical pendulum consists of a light inextensible string AB fixed at A and carrying a small ball of mass m at B. With the string taut the particle moves in a horizontal circle at constant angular speed ω. The centre of the circle is at distance x vertically below A. Show that $\omega^2 x = g$.

Solution:



Let the tension in the string be T. The angle between the string and the vertical is θ, and the radius of the circle is r.

$R(\leftrightarrow): T\sin\theta = m \times r \times \omega^2$

$R(\updownarrow): T\cos\theta = mg$

Dividing the first equation by the second

$$\Rightarrow \tan\theta = \frac{mr\omega^2}{mg}$$

$$\frac{r}{x} = \frac{r\omega^2}{g}, \omega^2 x = g$$

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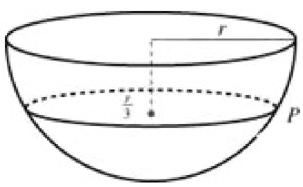
Motion in a circle
Exercise C, Question 7

Question:

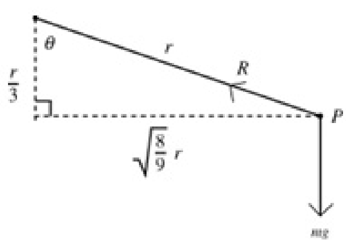
Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A hemispherical bowl of radius r is resting in a fixed position with its rim horizontal.
A particle P of mass m is moving in a horizontal circle around the smooth inside surface of the bowl.

The centre of the circle is $\frac{r}{3}$ below the centre of the bowl.

Find the angular speed of the particle and the magnitude of the reaction between the bowl and the particle.



Solution:



R is the normal reaction at P.
Using geometry, we know that the radius at P is perpendicular to the tangent at P, so R acts along this radius.
θ is the angle between the radius and the vertical.
Using Pythagoras' theorem we know that the radius of the circle is $\sqrt{\frac{8}{9}}r$.

$$\text{R}(\uparrow): R\cos\theta = mg$$

$$\frac{R}{3} = mg, R = 3mg$$

$$\text{R}(\leftarrow): R\sin\theta = m\sqrt{\frac{8}{9}}r\omega^2$$

$$R\frac{\sqrt{\frac{8}{9}}r}{r} = m\sqrt{\frac{8}{9}}r\omega^2$$

Substituting for R and simplifying:

$$3mg = mr\omega^2, \omega = \sqrt{\frac{3g}{r}}$$

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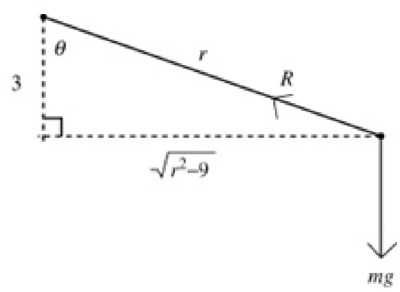
Motion in a circle
Exercise C, Question 8

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A hemispherical bowl of radius r cm is resting in a fixed position with its rim horizontal. A small marble of mass m is moving in a horizontal circle around the smooth inside surface of the bowl. The plane of the circle is 3 cm below the plane of the rim of the bowl. Find the angular speed of the marble.

Solution:



R is the normal reaction at the marble.
Using geometry, we know that the radius at the marble is perpendicular to the tangent at that point, so R acts along this radius.
θ is the angle between the radius and the vertical.
Using Pythagoras' theorem we know that the radius of the circle is $\sqrt{r^2-9}$.

$$\text{R}(\leftrightarrow): R\sin\theta = m\sqrt{r^2-9}\,\omega^2$$

$$R\frac{\sqrt{r^2-9}}{r} = m\sqrt{r^2-9}\,\omega^2$$

$$\omega^2 = \frac{R}{mr}$$

$$\text{R}(\updownarrow): R\cos\theta = mg = R\times\frac{3}{r},\ R = \frac{mgr}{3}$$

Substituting this expression for R in the first equation:

$$\omega^2 = \frac{mgr}{3mr} = \frac{g}{3},\ \omega = \sqrt{\frac{g}{3}}$$

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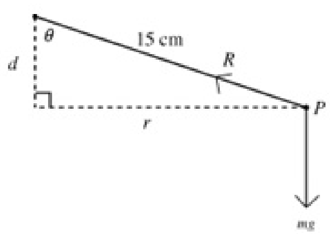
Motion in a circle
Exercise C, Question 9

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A hemispherical bowl of radius 15 cm is resting in a fixed position with its rim horizontal. A particle P of mass m is moving at 14 rad s^{-1} in a horizontal circle around the smooth inside surface of the bowl. Find the distance of the plane of the circle below the plane of the rim of the bowl.

Solution:



R is the normal reaction at the marble. Using geometry, we know that the radius at P is perpendicular to the tangent at that point, so R acts along this radius.
θ is the angle between the radius of the bowl and the vertical.
The particle moves on a circle of radius r m, depth d m below the rim of the bowl.

$R(\updownarrow): R\cos\theta = mg$ ①

$R(\leftrightarrow): R\sin\theta = mr\omega^2$ ②

Dividing ② ÷ ① to eliminate R,

$$\tan\theta = \frac{r\omega^2}{g} = \frac{r}{d}$$

$$\Rightarrow d = \frac{g}{\omega^2} \approx \frac{9.8}{196} = 0.05 \text{ m} = 5 \text{ cm}$$

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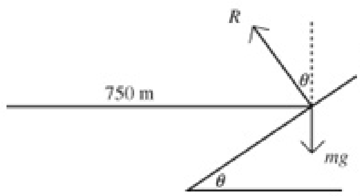
Motion in a circle
Exercise C, Question 10

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car travels round a bend of radius 750 m on a road which is banked at angle θ to the horizontal. The car is assumed to be moving at constant speed in a horizontal circle. If there is no frictional force acting on the car when it is travelling at 126 km h^{-1}, find the value of θ.

Solution:



No friction, so just the normal reaction, R, between the car and the road with a horizontal component.

$$126 \text{ km h}^{-1} = \frac{126 \times 1000}{3600} = 35 \text{ m s}^{-1}$$

$$\text{R}(\updownarrow): R\cos\theta = mg \quad ①$$

$$\text{R}(\leftrightarrow): R\sin\theta = m\frac{v^2}{r} = m \times \frac{35^2}{750} \quad ②$$

Dividing ② ÷ ① to eliminate R and m

$$\Rightarrow \tan\theta = \frac{35^2}{750g} \approx 0.167, \theta \approx 9.5°$$

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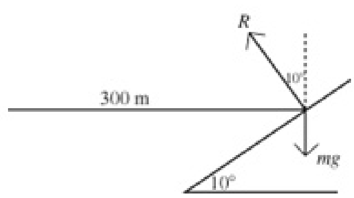
Motion in a circle
Exercise C, Question 11

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car travels round a bend of radius 300 m on a road which is banked at an angle of 10° to the horizontal. The car is assumed to be moving at constant speed in a horizontal circle. At what speed does the car move if there is no frictional force?

Solution:



No friction, so just the normal reaction, R, between the car and the road with a horizontal component.

$R(\updownarrow): R\cos 10^\circ = mg$

$R(\leftrightarrow): R\sin 10^\circ = \frac{mv^2}{r} = \frac{mv^2}{300}$

Dividing to eliminate R

$\Rightarrow \tan 10^\circ = \frac{mv^2}{300\,mg}$

$v^2 = 300\,g\tan 10^\circ = 518.4\ldots$

$v \approx 23 \text{ m s}^{-1}$

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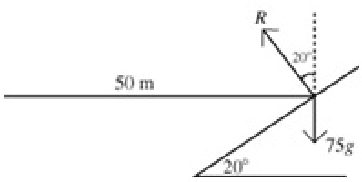
Motion in a circle
Exercise C, Question 12

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A boy rides his cycle round a circular track of diameter 50 m. The track is banked at $20°$ to the horizontal. There is no force due to friction. By modelling the boy and his cycle as a particle of mass 75 kg, find the speed at which the cycle is moving.

Solution:



No friction, so just the normal reaction, R, between the cycle and the road with a horizontal component.

$\text{R}(\updownarrow): R\cos 20° = 75g$

$\text{R}(\leftrightarrow): R\sin 20° = \dfrac{75 \times v^2}{50}$

Dividing to eliminate R

$\Rightarrow \tan 20° = \dfrac{75v^2}{50 \times 75g} = \dfrac{v^2}{50g}$

$v^2 = 50g\tan 20° = 178.3\ldots$

$v \approx 13 \text{ m s}^{-1}$.

It was not necessary to know the value of the mass because it cancels out at the stage when the two equations are combined to find $\tan 20°$.

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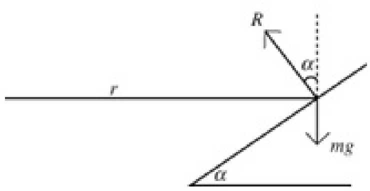
Motion in a circle
Exercise C, Question 13

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A bend in the road is a horizontal circular arc of radius r. The surface of the bend is banked at an angle α to the horizontal. When a vehicle is driven round the bend there is no tendency to slip. Show that the speed of the vehicle is $\sqrt{rg\tan\alpha}$.

Solution:



No friction, so just the normal reaction, R, between the vehicle and the road with a horizontal component.

$\text{R}(\updownarrow): R\cos\alpha = mg$

$\text{R}(\leftrightarrow): R\sin\alpha = \frac{mv^2}{r}$

Dividing to eliminate R

$\Rightarrow \tan\alpha = \frac{mv^2}{rmg} = \frac{v^2}{rg}$

$$v^2 = rg\tan\alpha, v = \sqrt{rg\tan\alpha}$$

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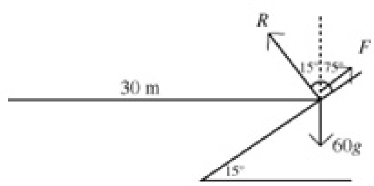
Motion in a circle
Exercise C, Question 14

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A girl rides her cycle round a circular track of diameter 60 m. The track is banked at $15°$ to the horizontal. The coefficient of friction between the track and the tyres of the cycle is 0.25. Modelling the girl and her cycle as a particle of mass 60 kg moving in a horizontal circle, find the minimum speed at which she can travel without slipping.

Solution:



R is the normal reaction between the cycle and the track. F is the force due to friction. At minimum speed the force due to friction is acting up the slope to stop the cycle from sliding down. (At maximum speed the friction will act down the slope to prevent sliding up the slope.)

As slipping is about to occur, $F = \mu R$.

$$\text{R}(\updownarrow): R\cos 15° + F\cos 75° = 60\text{ g}$$

$$R\left(\cos 15° + \frac{\cos 75°}{4}\right) = 60\text{ g}$$

$$\text{R}(\leftrightarrow): R\cos 75° - F\cos 15° = 60 \times \frac{v^2}{30}$$

$$R\left(\cos 75° - \frac{\cos 15°}{4}\right) = 2v^2$$

Dividing to eliminate R

$$\Rightarrow \frac{\cos 75° - \frac{\cos 15°}{4}}{\cos 15° + \frac{\cos 75°}{4}} = \frac{2v^2}{60\text{ g}}$$

$$v^2 = \frac{\cos 75° - 0.25 \times \cos 15°}{\cos 15° + 0.25 \times \cos 75°} \times 30\text{ g}$$

$$v^2 = 4.94\ldots, \quad v \approx 2.2 \text{ m s}^{-1}.$$

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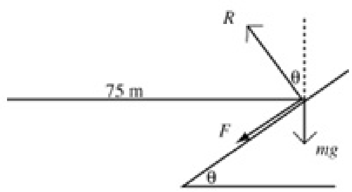
Motion in a circle
Exercise C, Question 15

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A van is moving on a horizontal circular bend in the road of radius 75 m. The bend is banked at $\tan^{-1}\frac{1}{3}$ to the horizontal. The maximum speed at which the van can be driven round the bend without slipping is 90 km h^{-1}. Calculate the coefficient of friction between the road surface and the tyres of the van.

Solution:



R is the normal reaction between the van and the track. F is the force due to friction. At maximum speed the force due to friction is acting down the slope to stop the van from sliding up.
μ is the coefficient of friction between the tyres and the road.
As slipping is about to occur, $F = \mu R$.

$$90 \text{ km h}^{-1} = \frac{90 \times 1000}{3600} = 25 \text{ m s}^{-1}.$$

$$\text{R}(\updownarrow): F\sin\theta + mg = R\cos\theta$$

$$\text{R}(\leftrightarrow): F\cos\theta + R\sin\theta = m \times \frac{25^2}{75}$$

Substituting $F = \mu R$

$$\Rightarrow mg = R(\cos\theta - \mu\sin\theta)$$

$$\frac{25m}{3} = R(\mu\cos\theta + \sin\theta)$$

Dividing to eliminate m

$$\Rightarrow \frac{25}{3g} = \frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta} = \frac{\mu + \tan\theta}{1 - \mu\tan\theta}$$

(on dividing top and bottom by $\cos\theta$)

$$= \frac{\mu + \frac{1}{3}}{1 - \frac{\mu}{3}} \left(\text{using } \tan\theta = \frac{1}{3} \right)$$

$$= \frac{3\mu + 1}{3 - \mu}$$

$$\Rightarrow 25(3 - \mu) = 3g(3\mu + 1)$$

Rearranging this equation gives

$$\mu(9g + 25) = 75 - 3g$$

$$\mu = \frac{75 - 3g}{9g + 25} \approx 0.40$$

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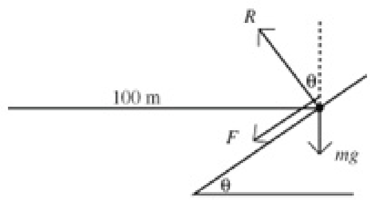
Motion in a circle
Exercise C, Question 16

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A car moves on a horizontal circular path round a banked bend in a race track. The radius of the path is 100 m. The coefficient of friction between the car tyres and the track is 0.3. The maximum speed at which the car can be driven round the bend without slipping is 144 km h^{-1}. Find the angle at which the track is banked.

Solution:



R is the normal reaction between the car and the track. F is the force due to friction. At maximum speed the force due to friction is acting down the slope to stop the car from sliding up.
The track is banked at θ to the horizontal.
As slipping is about to occur, $F = \mu R$.

$$144 \text{ km h}^{-1} = \frac{144 \times 1000}{3600} = 40 \text{ m s}^{-1}.$$

$$\text{R}(\updownarrow): mg = R\cos\theta - F\sin\theta$$

$$\text{R}(\leftrightarrow): R\sin\theta + F\cos\theta = m \times \frac{40^2}{100}$$

Substituting $F = \mu R$ and dividing to eliminate m

$$\Rightarrow \frac{\sin\theta + 0.3\cos\theta}{\cos\theta - 0.3\sin\theta} = \frac{40^2}{100g}$$

Dividing top and bottom of the left hand side by $\cos\theta$

$$\Rightarrow \frac{\tan\theta + 0.3}{1 - 0.3\tan\theta} = \frac{1600}{100g} = \frac{16}{g}$$

$$g(\tan\theta + 0.3) = 16(1 - 0.3\tan\theta)$$

$$\tan\theta(g + 4.8) = 16 - 0.3g$$

$$\tan\theta = \frac{16 - 0.3g}{g + 4.8} = 0.894\ldots$$

$$\theta \approx 42^\circ$$

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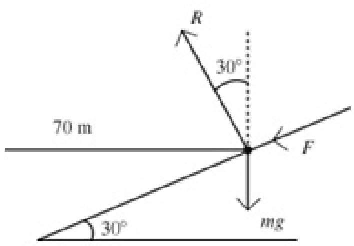
Motion in a circle
Exercise C, Question 17

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A bend in a race track is banked at $30°$. A car will follow a horizontal circular path of radius 70 m round the bend. The coefficient of friction between the car tyres and the track surface is 0.4. Find the maximum and minimum speeds at which the car can be driven round the bend without slipping.

Solution:



R is the normal reaction between the car and the track, F is force due to friction. At maximum speed F acts down the slope and is equal to $\mu R = 0.4R$

$\text{R}(\updownarrow) R\cos 30° - F\sin 30° = mg$

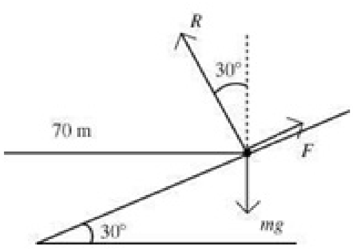
$\text{R}(\leftrightarrow) F\cos 30° + R\sin 30° = \frac{mv}{70}$

Substituting $F = 0.4R$ and dividing

$\Rightarrow \frac{v^2}{70g} = \frac{0.4\cos 30° + \sin 30°}{\cos 30° - 0.4\sin 30°}$

$\Rightarrow v^2 = 871.7\ldots$

$\Rightarrow v \simeq 29.5 \text{ m s}^{-1}$



At minimum speed, F acts up the slope

$\text{R}(\updownarrow) R\cos 30° + F\sin 30° = mg$

$\text{R}(\leftrightarrow) R\sin 30° - F\cos 30° = \frac{mv^2}{70}$

which leads to $\frac{v^2}{70\ g} = \frac{\sin 30° - 0.4\cos 30°}{\cos 30° + 0.4\sin 30°}$

giving $v^2 = 98.83\ldots \Rightarrow v \approx 9.9 \text{ m s}^{-1}$

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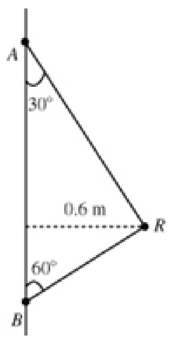
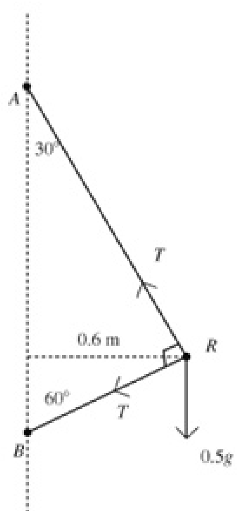
Motion in a circle
Exercise C, Question 18

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
The diagram shows a small smooth ring R of mass 500 g threaded on a light inextensible string. The ends of the string are attached to fixed points A and B, where A is vertically above B. The string is taut and the system rotates about AB. The ring moves with constant angular speed on a horizontal circle of radius 0.6 m.
$\angle ABR = 60°$ and $\angle BAR = 30°$. Modelling the ring as a particle, calculate the tension in the string and the angular speed of the particle.



Solution:



Because the ring is smooth and the string is continuous, we have only one value for tension, T.

$$R(\updownarrow): T\cos 30° - T\cos 60° = 0.5g$$
$$T(\cos 30° - \cos 60°) = 0.5g$$
$$T = \frac{0.5g}{\cos 30° - \cos 60°} = \frac{g}{\sqrt{3}-1}$$
$$\approx 13.4 \text{ N}$$
$$R(\leftrightarrow): T\cos 60° + T\cos 30° = 0.5 \times 0.6 \times \omega^2$$
$$T(\cos 60° + \cos 30°) = \frac{0.6\omega^2}{2}$$

Substituting for T and the trigonometric ratios:

$$\frac{g}{\sqrt{3}-1} \times \left(\frac{1+\sqrt{3}}{2}\right) = \frac{0.6\omega^2}{2}$$

Rearranging:

$$\omega^2 = \frac{g(1+\sqrt{3})}{0.6(\sqrt{3}-1)} = 60.9\ldots$$
$$\omega \approx 7.8 \text{ rad s}^{-1}.$$

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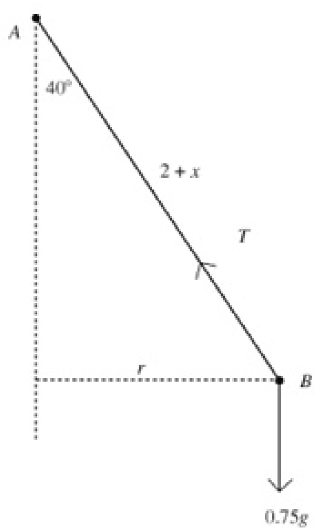
Motion in a circle
Exercise C, Question 19

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A light elastic string AB has natural length 2 m and modulus of elasticity 30 N. The end A is attached to a fixed point. A particle of mass 750 g is attached to the end B. The particle is moving in a horizontal circle below A with the string inclined at $40°$ to the vertical. Find the angular speed of the particle.

Solution:



T is the tension in AB. AB is an elastic string, if the extension in the string is x then $T = \frac{\lambda x}{l} = \frac{30x}{2}$

$R(\updownarrow): T\cos 40° = 0.75\ g$

$R(\leftrightarrow): T\sin 40° = 0.75 \times r \times \omega^2$

The radius of the circle is $(2 + x)\sin 40°$, so substituting for r and T gives

$15x \sin 40° = 0.75 \times (2 + x)\sin 40° \times \omega^2$

$$\Rightarrow \omega^2 = \frac{15x}{0.75(2+x)} = \frac{20x}{2+x}$$

From the first equation:

$15x\cos 40° = 0.75\ g$

$$x = \frac{0.75g}{15\cos 40°} = 0.639\ldots$$

$$\Rightarrow \omega^2 = 4.846, \omega \approx 2.2 \text{ rad s}^{-1}.$$

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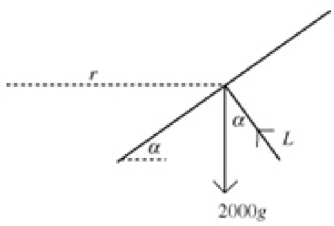
Motion in a circle
Exercise C, Question 20

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
An aircraft of mass 2 tonnes flies at 400 km h^{-1} on a path which follows a horizontal circular arc in order to change course from a bearing of 060° to a bearing of 015°. It takes 25 seconds to change course, with the aircraft banked at $\alpha°$ to the horizontal. Calculate the two possible values of α and the corresponding values of the magnitude of the lift force perpendicular to the surface of the aircraft's wings.

Solution:



L is the lift force, and r is the radius of the circular arc.

$400 \text{ km h}^{-1} = \frac{400 \times 1000}{3600} = \frac{1000}{9} \text{ m s}^{-1}$.

In 25 seconds the aircraft travels $\frac{25\,000}{9}$ m .

The direction changes by 45° or 315° depending on the direction of turning.

For 45°, $\frac{25\,000}{9} = \frac{1}{8} \times 2\pi r, r = 3536.7.. \text{ m}$

For 315°, $\frac{25\,000}{9} = \frac{7}{8} \times 2\pi r, r = 505.25 \text{ m}$

$R(\updownarrow) : L\cos\alpha = 2000\text{ g}$

$R(\leftrightarrow) : L\sin\alpha = \frac{2000v^2}{r}$

Dividing the second equation by the first

$\Rightarrow \tan\alpha = \frac{v^2}{rg}$

When $r = 3536.7\ldots$

$\tan\alpha = \frac{1000^2}{9^2 \times 3537 \times 9.8} \approx 0.356$

$\alpha \approx 20°$, and $L \approx 21\,000$ N

When $r = 505.2\ldots$

$\tan\alpha = \frac{1000^2}{9^2 \times 505.3 \times 9.8} \approx 2.493$

$\alpha \approx 68°$, and $L \approx 53\,000$ N

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Motion in a circle
Exercise D, Question 1

Question:

At time t seconds the position vector, relative to the centre of the circle, of a particle moving in a horizontal circle, centre O, at constant angular speed ω rad s^{-1} is given by

$\mathbf{r} = r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j}$

a Differentiate $\mathbf{r}$ with respect to t to obtain the velocity, $\mathbf{v}$, of the particle.

b Hence calculate the linear speed of the particle and deduce that $\mathbf{v} = r\omega$.

Solution:

a

$$\mathbf{r} = \mathbf{r}\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\omega t\mathbf{i} - r\sin\omega t\mathbf{j}) = \mathbf{v}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\omega t)\mathbf{i} + \frac{\mathrm{d}}{\mathrm{d}t}(r\sin\omega t)\mathbf{j}$$

$$= -r\omega\sin\omega t\mathbf{i} + r\omega\cos\omega t\mathbf{j}$$

b

$$\text{speed} = v = |\mathbf{v}| = \sqrt{(-\omega r\sin\omega t)^2 + (\omega r\cos\omega t)^2}$$

$$= \omega r\sqrt{\sin^2\omega t + \cos^2\omega t} = \omega r, \text{ as required.}$$

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Motion in a circle
Exercise D, Question 2

Question:

a By considering the gradients of the vectors **r** and **v**, or by taking the scalar product of **r** and **v**, find the angle between these two vectors.
b What does this tell you about the velocity of the particle?

Solution:

a $r.\mathbf{v} = (r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j}).(-r\omega\sin\omega t\mathbf{i} + r\omega\cos\omega t\mathbf{j})$
$= (r\cos\omega t)\times(-r\omega\sin\omega t) + (r\sin\omega t)\times(r\omega\cos\omega t)$
$= -r^2\omega\cos\omega t.\sin\omega t + r^2\omega\sin\omega t.\cos\omega t = 0$, so **r** is perpendicular to **v**.

b The direction of **v** is perpendicular to the direction of **r**. Since **r** is always directed from the origin to the particle, the direction of the velocity at any instant is along the tangent to the circular path.

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Motion in a circle
Exercise D, Question 3

Question:

a Differentiate **v** with respect to t to obtain the acceleration, **a**, of the particle.
b Express **a** in terms of **r**. What does this tell you about the direction of the acceleration?
c Calculate the magnitude of **a**.

Solution:

a $\mathbf{v} = -r\omega \sin \omega t\mathbf{i} + r\omega \cos \omega t\mathbf{j}$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}(-r\omega \sin \omega t\mathbf{i} + r\omega \cos \omega t\mathbf{j}) = \mathbf{a}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}(-r\omega \sin \omega t)\mathbf{i} + \frac{\mathrm{d}}{\mathrm{d}t}(r\omega \cos \omega t)\mathbf{j}$$
$$= -r\omega^2 \cos \omega t\mathbf{i} - r\omega^2 \sin \omega t\mathbf{j}$$
$$= -\omega^2(r \cos \omega t\mathbf{i} + r \sin \omega t\mathbf{j})$$

b $\mathbf{a} = -\omega^2(r \cos \omega t\mathbf{i} + r \sin \omega t\mathbf{j}) = -\omega^2\mathbf{r}$

a is parallel to **r** but in the opposite direction, so the direction of the acceleration is towards the centre of the circle.

c $|\mathbf{a}| = |-\omega^2\mathbf{r}|$, so $|\mathbf{a}| = \omega^2 \times |\mathbf{r}|$, $a = r\omega^2$.

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Motion in a circle

Exercise E, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

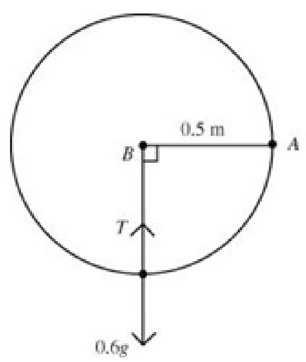
A particle of mass 0.6 kg is attached to end A of a light rod AB of length 0.5 m. The rod is free to rotate in a vertical plane about B. The particle is held at rest with AB horizontal. The particle is released. Calculate

a the speed of the particle as it passes through the lowest point of the path,

b the tension in the rod at this point.

Solution:

a



Let the speed of the particle at the lowest point be $v \text{ m s}^{-1}$, and the tension in the rod be T N.

At the lowest point the particle has fallen a distance 0.5 m, so the P.E. lost $= 0.6 \times g \times 0.5$

and the K.E. gained $= \frac{1}{2} \times 0.6 \times v^2$

$$\therefore 0.6 \times g \times 0.5 = \frac{1}{2} \times 0.6 \times v^2$$

$$v^2 = g, v \approx 3.1 \text{ m s}^{-1}$$

b At the lowest point, the force towards the centre of the circle

$$= T - 0.6\,g = \frac{0.6v^2}{0.5}$$

$$\Rightarrow T = 0.6\,g + \frac{0.6\,g}{0.5} = 1.8\,g \approx 17.6 \text{ N}$$

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Motion in a circle
Exercise E, Question 2

Question:

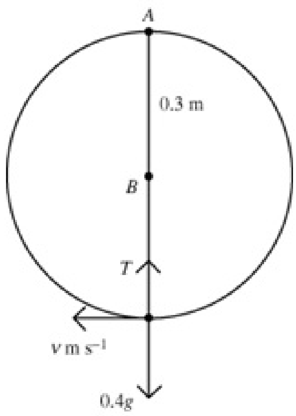
Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle of mass 0.4 kg is attached to one end A of a light rod AB of length 0.3 m. The rod is free to rotate in a vertical plane about B. The particle is held at rest with A vertically above B. The rod is slightly displaced so that the particle moves in a vertical circle. Calculate

a the speed of the particle as it passes through the lowest point of the path,

b the tension in the rod at this point.

Solution:

a



Let the speed of the particle at the lowest point be $v \text{ m s}^{-1}$, and the tension in the rod be T N.
At the lowest point the particle has fallen a distance 0.6 m, so the P.E. lost $= 0.4 \times g \times 0.6$,

and the K.E. gained $= \frac{1}{2} \times 0.4 \times v^2$.

$\therefore 0.4 \times g \times 0.6 = \frac{1}{2} \times 0.4 \times v^2$

$v^2 = 2 \times g \times 0.6 = 1.2g, v \approx 3.4 \text{ m s}^{-1}$

b At the lowest point, the force towards the centre of the circle

$= T - 0.4\,g = \frac{0.4v^2}{0.3}$

$\Rightarrow T = 0.4g + \frac{0.4 \times 1.2\,g}{0.3} = 2\,g \approx 19.6 \text{ N}$

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Motion in a circle
Exercise E, Question 3

Question:

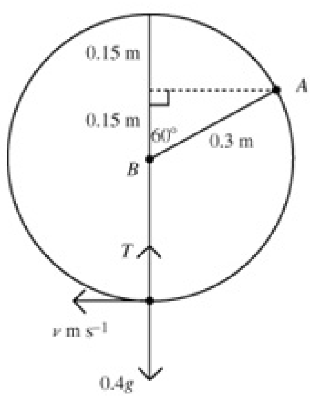
Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle of mass 0.4 kg is attached to end A of a light rod AB of length 0.3 m. The rod is free to rotate in a vertical plane about B. The particle is held at rest with AB at $60°$ to the upward vertical. The particle is released. Calculate

a the speed of the particle as it passes through the lowest point of the path,

b the tension in the rod at this point.

Solution:

a



Let the speed of the particle at the lowest point be $v \text{ m s}^{-1}$, and the tension in the rod be T N.
At the lowest point the particle has fallen a distance $0.3\cos 60° + 0.3 = 0.45 \text{ m}$, so the P.E. lost $= 0.4 \times g \times 0.45$, and the

K.E. gained $= \frac{1}{2} \times 0.4 \times v^2$.

$$\therefore 0.4 \times g \times 0.45 = \frac{1}{2} \times 0.4 \times v^2$$

$$v^2 = 2 \times g \times 0.45 = 0.9\,g, v \approx 3.0 \text{ m s}^{-1}$$

b At the lowest point, the force towards the centre of the circle

$$= T - 0.4\,g = \frac{0.4v^2}{0.3}$$

$$\Rightarrow T = 0.4\,g + \frac{0.4 \times 0.9g}{0.3} = 1.6\,g \approx 15.7 \text{ N}$$

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Motion in a circle
Exercise E, Question 4

Question:

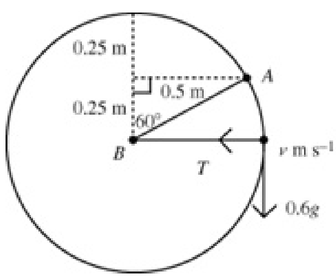
Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A particle of mass 0.4 kg is attached to end A of a light rod AB of length 0.3 m. The rod is free to rotate in a vertical plane about B. The particle is held at rest with AB at $60°$ to the upward vertical. The particle is released. Calculate

a the speed of the particle as it passes through the lowest point of the path,

b the tension in the rod at this point.

Solution:

a



Let the speed of the particle where AB is horizontal be $v\text{ m s}^{-1}$, and the tension in the rod be T N.
At the point where AB is horizontal, the particle has fallen a distance $0.5\cos 60° = 0.25\text{ m}$, so the
P.E. lost $= 0.6 \times g \times 0.25$,

and the K.E. gained $= \frac{1}{2} \times 0.6 \times v^2$.

$\therefore 0.6 \times g \times 0.25 = \frac{1}{2} \times 0.6 \times v^2$

$v^2 = 2 \times g \times 0.25 = 0.5g, v \approx 2.2\text{ m s}^{-1}$

b When AB is horizontal, the force towards the centre of the circle

$= T = \frac{0.6v^2}{0.5}$

$\Rightarrow T = \frac{0.6 \times 0.5\, g}{0.5} = 0.6\, g \approx 5.9\text{ N}$

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Motion in a circle
Exercise E, Question 5

Question:

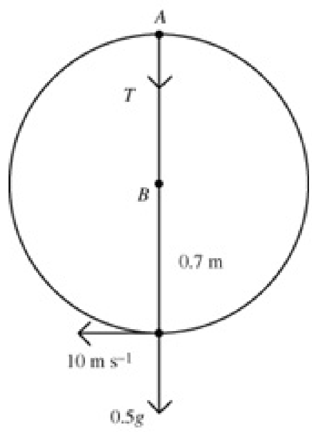
Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A particle of mass 0.5 kg is attached to end A of a light rod AB of length 0.7 m. The rod is free to rotate in a vertical plane about B. The particle is hanging with A vertically below B when it is projected horizontally with speed 10 m s^{-1}. Calculate

a the speed of the particle when it is vertically above B,

b the tension in the rod at this point.

Solution:

a



Let the speed of the particle at the highest point be $v\text{ m s}^{-1}$, and the tension in the rod be TN.
At the highest point the particle has risen a distance 1.4 m, so the
P.E. gained $= 0.5 \times g \times 1.4$, and the
K.E. lost $= \frac{1}{2} \times 0.5 \times 10^2 - \frac{1}{2} \times 0.5 \times v^2$.
$\therefore 0.5 \times g \times 1.4 = \frac{1}{2} \times 0.5 \times (100 - v^2)$
$100 - v^2 = 2 \times g \times 1.4 = 2.8g, v \approx 8.5\text{ m s}^{-1}$

b At the highest point, the force towards the centre of the circle

$$= T + 0.5\,g = \frac{0.5v^2}{0.7}$$

$$\Rightarrow T = \frac{0.5 \times (100 - 2.8g)}{0.7} - 0.5g = 46.9\text{ N}$$

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Motion in a circle
Exercise E, Question 6

Question:

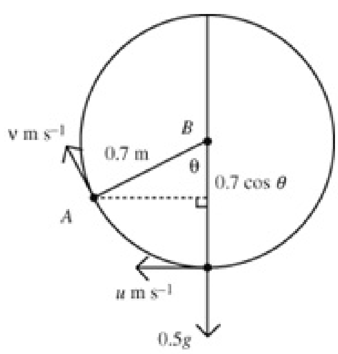
Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A particle of mass 0.5 kg is attached to end A of a light rod AB of length 0.7 m. The rod is free to rotate in a vertical plane about B. The particle is hanging with A vertically below B when it is projected horizontally with speed $u\text{ ms}^{-1}$. Find

a an expression in terms of u and θ for the speed of the particle when AB makes an angle of θ with the downward vertical through B,

b the restriction on u if the particle is to reach the highest point of the circle.

Solution:

a



When the angle between AB and the vertical is θ particle has speed $v\text{ m s}^{-1}$.

P.E. gained $= mgh = 0.5 \times g \times 0.7(1-\cos\theta)$

Loss in K.E. =

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{0.5}{2}(u^2 - v^2)$$

Energy is conserved

$$\therefore 0.5\,g \times 0.7(1-\cos\theta) = \frac{0.5}{2}(u^2 - v^2)$$

$$v^2 = u^2 - 1.4g(1-\cos\theta)$$

$$\Rightarrow v = \sqrt{u^2 - 1.4g(1-\cos\theta)}$$

b If the particle is to reach to top of the circle then we require $v > 0$ when $\theta = 180°$.

$\Rightarrow u^2 - 1.4g(1-\cos 180°) > 0$

But $\cos 180° = -1$, so $u^2 > 1.4g \times 2, u > \sqrt{2.8\,g}$

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Motion in a circle
Exercise E, Question 7

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.

A particle A of mass 1.5 kg is attached to one end of a light inextensible string of length 2 m. The other end of the string is attached to a fixed point B. The particle is hanging in equilibrium when it is set in motion with a horizontal speed of $u\text{ ms}^{-1}$.

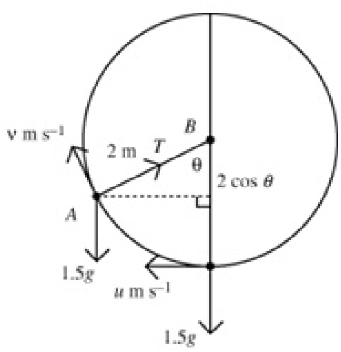
Find

a an expression for the tension in the string when it is at an angle θ to the downward vertical through B,

b the minimum value of u for which the particle will perform a complete circle.

Solution:

a



Take the lowest point of the circle as the zero level for potential energy.
At the lowest level the particle has

$\text{K.E.} = \frac{1}{2} \times 1.5 \times u^2 = 0.75u^2\text{ J}$ and $\text{P.E.} = 0\text{ J}$

When the rod is at angle θ to the vertical the particle has

$\text{K.E.} = \frac{1}{2} \times 1.5 \times v^2 = 0.75v^2\text{ J}$ and

$\text{P.E.} = 1.5 \times g \times 2(1 - \cos\theta)\text{ J}.$

Energy is conserved

$\therefore 0.75u^2 = 0.75v^2 + 3g(1 - \cos\theta)$

Resolving towards the centre of the circle:

$T - 1.5\, g\cos\theta = \frac{mv^2}{r} = \frac{1.5v^2}{2}$, so substituting for v^2 gives

$$T = 1.5g\cos\theta + \frac{3}{4}(u^2 - 4g + 4g\cos\theta)$$
$$= 4.5g\cos\theta + \frac{3u^2}{4} - 3g$$

b If the particle is to reach to top of the circle then we require $T > 0$ when $\theta = 180°$.

$$\Rightarrow -4.5g + \frac{3u^2}{4} - 3g > 0, \frac{3u^2}{4} > 7.5g, u^2 > 10g, u > \sqrt{10g}$$

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Motion in a circle

Exercise E, Question 8

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

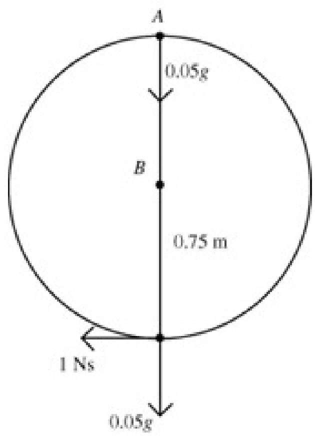
A small bead of mass 50 g is threaded on a smooth circular wire of radius 75 cm which is fixed in a vertical plane. The bead is at rest at the lowest point of the wire when it is hit with an impulse of I Ns horizontally causing it to start to move round the wire. Find the value of I if

a the bead just reaches the top of the circle,

b the bead just reaches the point where the radius from the bead to the centre of the circle makes an angle of $\tan^{-1}\frac{3}{4}$ with the upward vertical and then starts to slide back to its original position.

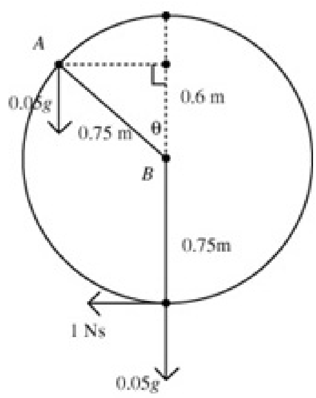
Solution:

a



Impulse = change in momentum, so if the initial speed of the bead is u m s^{-1} then
$I = 0.05u, u = 20I$.
Take the lowest point of the circle as the zero level for potential energy.
At the lowest level the particle has
$\text{K.E.} = \frac{1}{2} \times 0.05 \times u^2 = 0.025 \times 400 I^2 = 10I^2$ J and
$\text{P.E.} = 0$ J
At the highest level the particle has $\text{K.E.} = 0$ (since we are told that the bead if just reaches the top) and it has risen 1.5 m so it has
$\text{P.E.} = 0.05 \times g \times 1.5 = 0.075\, g$
Energy is conserved, $\therefore 10I^2 = 0.075g$,
$\Rightarrow I^2 = 0.0075\, g, I \approx 0.27$

b



Take the lowest point of the circle as the zero level for potential energy.
At the lowest level the particle has
$\text{K.E.} = \frac{1}{2} \times 0.05 \times u^2 = 0.025 \times 400 I^2 = 10I^2$ J and
$\text{P.E.} = 0$ J
When the rod is at angle $\tan^{-1}\frac{3}{4}$ to the vertical the particle has $\text{K.E.} = 0$ J (we are told that the particle reaches this point and then starts to slide back, so we can deduce that the speed is zero here) and
$\text{P.E.} = 0.05 \times g \times 0.75 \left\{1 + \cos\left(\tan^{-1}\frac{3}{4}\right)\right\}$
$= 0.0675\, g$ J
Energy is conserved
$10I^2 = 0.0675\, g$
$I = 0.26$

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Motion in a circle
Exercise E, Question 9

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

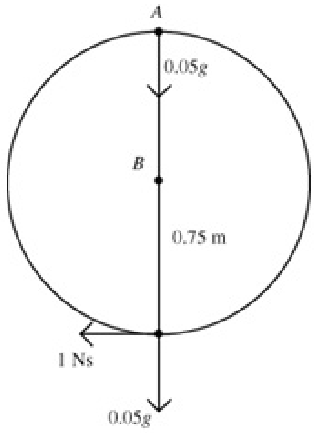
A particle of mass 50 g is attached to one end of a light inextensible string of length 75 cm. The other end of the string is attached to a fixed point. The particle is hanging at rest when it is hit with an impulse of I Ns horizontally causing it to start to move in a vertical circle. Find the value of I if

a the particle just reaches the top of the circle,

b the string goes slack at the instant when the particle reaches the point where the string makes an angle of $\tan^{-1}\frac{3}{4}$ with the upward vertical.

Solution:

a



Impulse = change in momentum, so if the initial speed of the bead is u m s^{-1} then

$I = 0.05u, u = 20I$.

Take the lowest point of the circle as the zero level for potential energy.

At the lowest level the particle has

$$\text{K.E.} = \frac{1}{2}\times 0.05\times u^2 = 0.025\times 400I^2 = 10I^2 \text{ J}$$

and P.E. = 0 J

If the bead just reaches the top of the circle then this is the point at which the tension in the string becomes zero. If the speed of the bead at this point is v m s^{-1} then

$$0.05g = \frac{mv^2}{r} = \frac{0.05v^2}{0.75}, v^2 = 0.75\,g$$

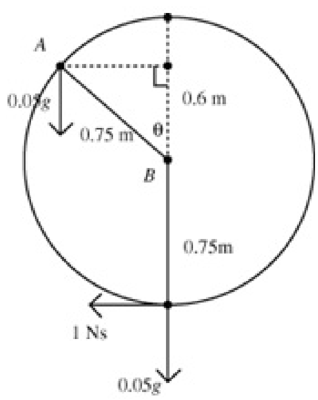
The bead has risen 1.5 m so it has

P.E. $= 0.05\times g\times 1.5 = 0.075\,g$

Energy is conserved, so

$$10I^2 = 0.075\,g + \frac{1}{2}\times 0.05\times 0.75\,g = 0.09375\,g, I \approx 0.30$$

b



Take the lowest point of the circle as the zero level for potential energy.

At the lowest level the particle has

$$\text{K.E.} = \frac{1}{2}\times 0.05\times u^2 = 0.025\times 400I^2 = 10I^2 \text{ J}$$

and P.E. = 0 J

When the bead just reaches the point where AB is at $\tan^{-1}\frac{3}{4}$ to the vertical the tension in the string becomes zero. If the speed of the bead at this point is v m s^{-1} then

$$0.05g\cos\left(\tan^{-1}\frac{3}{4}\right) = \frac{mv^2}{r} = \frac{0.05v^2}{0.75}$$

$$v^2 = 0.75\times g\times\frac{4}{5} = 0.6\,g$$

The bead has risen $0.75 + 0.6 = 1.35$ m,

so gain in P.E. $= 0.05\times g\times 1.35 = 0.0675\,g$.

Energy is conserved $\Rightarrow 10I^2 = 0.0675\,g + \frac{1}{2}\times 0.05\times 0.6\,g = 0.0825\,g$

$$I \approx 0.28$$

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Motion in a circle
Exercise E, Question 10

Question:

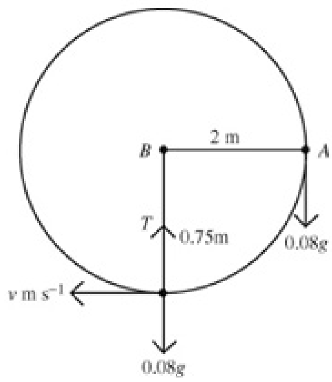
Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle of mass 0.8 kg is attached to end A of a light rod AB of length 2 m. The end B is attached to a fixed point so that the rod is free to rotate in a vertical circle with its centre at B. The rod is held in a horizontal position and then released. Calculate the speed of the particle and the tension in the rod when

a the particle is at the lowest point of the circle,

b the rod makes an angle of $\tan^{-1}\frac{3}{4}$ with the downward vertical through B.

Solution:

a



Let the speed of the particle at the lowest point be v m s^{-1}, and the tension in the rod be T N.

Take the starting level as the zero level for potential energy, the particle starts with

P.E. $= 0$ and K.E. $= 0$.

At the lowest level,

P.E. $= -0.8 \times g \times 2 = -1.6\,g$

K.E. $= \frac{1}{2} \times m \times v^2 = \frac{1}{2} \times 0.8v^2 = 0.4v^2$

Conservation of energy

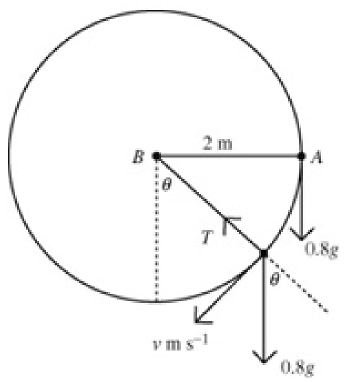
$\Rightarrow -1.6g + 0.4v^2 = 0, \quad v^2 = 4g$

$$v \approx 6.3 \text{ m s}^{-1}.$$

Force towards the centre of the circle $= T - 0.8g = \frac{mv^2}{r} = \frac{0.8 \times 4g}{2} = 1.6g$

$$T = 2.4\,g \approx 24 \text{ N}$$

b



Let the speed of the particle at the point when $\theta = \tan^{-1}\frac{3}{4}$ be v m s^{-1}, and the tension in the rod be T N.

Take the starting level as the zero level for potential energy, the particle starts with

P.E. $= 0$ and K.E. $= 0$.

When AB is at θ to the vertical the particle has fallen $2\cos\theta = 1.6$ m.

P.E. $= -0.8 \times g \times 1.6 = -1.28\,g$

K.E. $= \frac{1}{2} \times m \times v^2 = \frac{1}{2} \times 0.8v^2 = 0.4v^2$

Conservation of energy

$\Rightarrow -1.28g + 0.4v^2 = 0, \quad v^2 = 3.2g$

$$v \approx 5.6 \text{ m s}^{-1}.$$

Force towards the centre of the circle $= T - 0.8g\cos\theta = \frac{mv^2}{r} = \frac{0.8 \times 3.2g}{2} = 1.28g$

$$T = 1.28\,g + 0.8\,g \times \frac{4}{5} = 1.92g \approx 19 \text{ N}$$

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Motion in a circle
Exercise E, Question 11

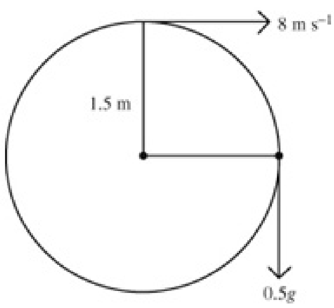
Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.
A particle of mass 500 g describes complete vertical circles on the end of a light inextensible string of length 1.5 m. Given that the speed of the particle is 8 m s^{-1} at the highest point, find

a the speed of the particle when the string is horizontal,
b the magnitude of the tangential acceleration when the string is horizontal,
c the tension in the string when the particle is at the lowest point of the circle.

Solution:

a



Let the speed of the particle when the string is horizontal be $v \text{ m s}^{-1}$.
Take the lowest point as the zero level for potential energy, the particle starts with

$\text{P.E.} = 0.5 \times g \times 3$ and $\text{K.E.} = \frac{1}{2} \times 0.5 \times 8^2$.

When the string is horizontal,

$\text{P.E.} = 0.5 \times g \times 1.5$ and $\text{K.E.} = \frac{1}{2} \times 0.5 \times v^2$.

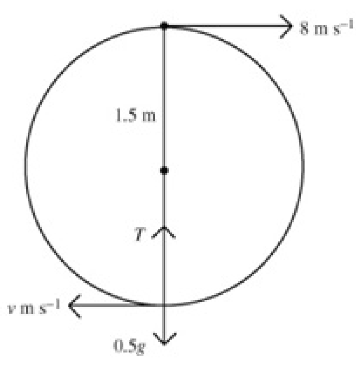
Energy is conserved

$$\Rightarrow 1.5g + 16 = 0.75g + \frac{v^2}{4}$$

$$v^2 = 4(0.75g + 16), v \approx 9.7 \text{ m s}^{-1}.$$

b The only force with a vertical component is the weight.
Acceleration $= g \text{ m s}^{-2}$.

c



Let the speed of the particle at the lowest point be $v \text{ m s}^{-1}$, and the tension in the rod be T N.
Take the lowest point as the zero level for potential energy, the particle starts with

$\text{P.E.} = 0.5 \times g \times 3$ and $\text{K.E.} = \frac{1}{2} \times 0.5 \times 8^2$.

When the string is vertical,

$\text{P.E.} = 0$ and $\text{K.E.} = \frac{1}{2} \times 0.5 \times v^2$.

Energy is conserved

$$\Rightarrow 1.5g + 16 = \frac{v^2}{4}, \quad v \approx 11.1 \text{ m s}^{-1}$$

$$T - 0.5g = \frac{0.5v^2}{1.5}, \quad T \approx 45.8 \text{ N}$$

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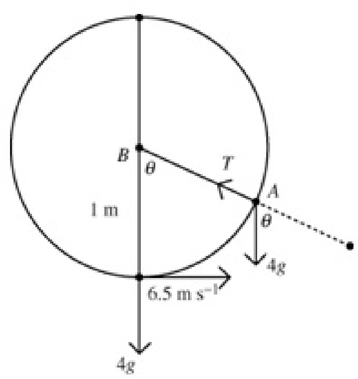
Motion in a circle
Exercise E, Question 12

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A light rod AB of length 1 m has a particle of mass 4 kg attached at A. End B is pivoted to a fixed point so that AB is free to rotate in a vertical plane. When the rod is vertical with A below B the speed of the particle rod is 6.5 m s^{-1}. Find the angle between AB and the vertical at the instant when the tension in the rod is zero, and calculate the speed of the particle at that instant.

Solution:



At the lowest point let P.E. $= 0$ J.
The K.E. at the lowest point is
$\frac{1}{2} \times 4 \times 6.5^2 = 84.5$ J.
When AB is at angle θ to the vertical, the tension in the rod is T, and the particle has speed $v\text{ m s}^{-1}$.
Particle has risen $(1 - \cos\theta)$, so
P.E. $= 4 \times g \times (1 - \cos\theta)$ J and
K.E. $= \frac{1}{2} \times 4 \times v^2 = 2v^2$

Energy is conserved $\Rightarrow 84.5 = 2v^2 + 4g(1 - \cos\theta)$,
$$v^2 = 42.25 - 2g(1 - \cos\theta)$$
Force towards the centre of the circle
$= T - 4g\cos\theta = \frac{mv^2}{r} = \frac{4(42.25 - 2g(1 - \cos\theta))}{1}$.
$\therefore T = 4g\cos\theta + 169 - 8g(1 - \cos\theta) = 169 + 12g\cos\theta - 8g = 0$ when
$\cos\theta = \frac{8g - 169}{12g} = -0.77\ldots$ giving $\theta \approx 140°$
i.e. $\theta \approx 40°$ to the upward vertical and

$v^2 = 42.25 - 2g(1 - \cos\theta) = 7.5\ldots, v \approx 2.7\text{ m s}^{-1}$

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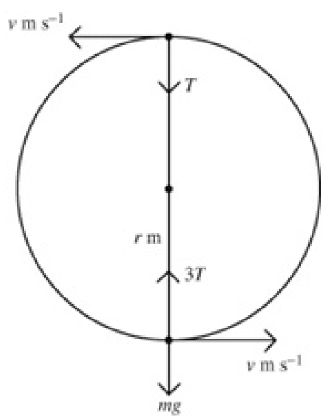
Motion in a circle
Exercise E, Question 13

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A particle P of mass m kg is attached to one end of a light rod of length r m which is free to rotate in a vertical plane about its other end. The particle describes complete vertical circles. Given that the tension at the lowest point of P's path is three times the tension at the highest point, find the speed of P at the lowest point on its path.

Solution:



Let the speed at the lowest point be $u \text{ m s}^{-1}$, and the speed at the highest point be $v \text{ m s}^{-1}$.

The gain in P.E. in moving from the lowest point to the highest is $2mgr$.

The loss in K.E. is $\frac{1}{2}mu^2 - \frac{1}{2}mv^2$

Energy is conserved

$\therefore 2mgr = \frac{1}{2}mu^2 - \frac{1}{2}mv^2, \ v^2 = u^2 - 4gr$

At the lowest point $3T - mg = \frac{mu^2}{r}$

At the highest point $T + mg = \frac{mv^2}{r}$

Substituting for T and v^2 in the first of these two equations:

$$3\left(\frac{m(u^2-4gr)}{r} - mg\right) - mg = \frac{mu^2}{r}, 3\frac{(u^2-4gr)}{r} - 4g = \frac{u^2}{r}$$

$$\frac{2u^2}{r} = 16g, u^2 = 8gr, u = \sqrt{8gr}$$

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Motion in a circle
Exercise E, Question 14

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.

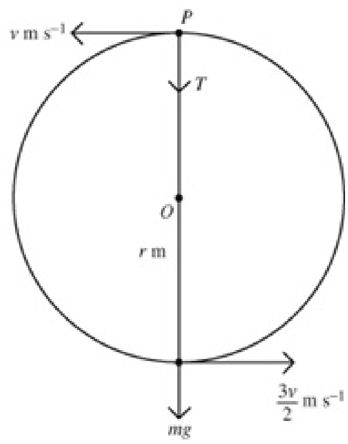
A particle P of mass m kg is attached to one end of a light inextensible string of length r m. The other end of the string is attached to a fixed point O, and P describes complete vertical circles about O. Given that the speed of the particle at the lowest point is one-and-a-half times the speed of the particle at the highest point, find

a the speed of the particle at the highest point,

b the tension in the string when the particle is at the highest point.

Solution:

a



Let the speed at the lowest point be $\frac{3v}{2}\text{ m s}^{-1}$, and the speed at the highest point be $v\text{ m s}^{-1}$.

The gain in P.E. in moving from the lowest point to the highest is $2mgr$.

The loss in K.E. is $\frac{1}{2}m\left(\frac{3v}{2}\right)^2 - \frac{1}{2}mv^2$

Energy is conserved

$$\therefore 2mgr = \frac{1}{2}m \times \frac{9v^2}{4} - \frac{1}{2}mv^2 = \frac{5}{8}mv^2$$

$$v^2 = \frac{16gr}{5}, v = \sqrt{\frac{16\,gr}{5}}$$

b At the highest point, $T + mg = \frac{mv^2}{r} = \frac{m\frac{16gr}{5}}{r} = \frac{16mg}{5}, T = \frac{11mg}{5}$

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Motion in a circle
Exercise E, Question 15

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

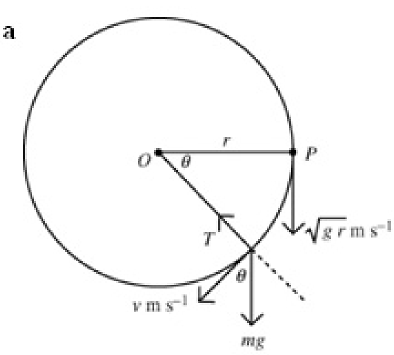
A light inelastic string of length r has one end attached to a fixed point O. A particle P of mass m kg is attached to the other end. P is held with OP horizontal and the string taut. P is then projected vertically downwards with speed $\sqrt{gr}$.

a Find, in terms of θ, m and g, the tension in the string when OP makes an angle θ with the horizontal.

b Given that the string will break when the tension in the string is $2mg$ N, find the angle between the string and the horizontal when the string breaks.

Solution:

a



With OP horizontal, the particle has

P.E. $= 0$ and K.E. $= \frac{1}{2}mv^2 = \frac{1}{2}mgr$.

When OP is $\theta°$ below the horizontal, the tension in the string is T and the speed of the particle is v.

The particle has

P.E. $= -mgr\sin\theta$ and K.E. $= \frac{1}{2}mv^2$

Energy is conserved

$$\therefore \frac{1}{2}mgr = -mgr\sin\theta + \frac{1}{2}mv^2$$

$$v^2 = gr(1 + 2\sin\theta)$$

Resolving towards O:

$$T - mg\sin\theta = \frac{mv^2}{r} = \frac{mgr(1+2\sin\theta)}{r}$$

$$T = mg(1 + 3\sin\theta)\text{N}$$

b When $T = 2mg$ N, $2mg = mg(1 + 3\sin\theta)$, $\sin\theta = \frac{1}{3}$, $\theta \approx 19.5°$

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Motion in a circle
Exercise F, Question 1

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.

A particle P of mass m is attached to one end of a light inextensible string of length l. The other end of the string is attached to a fixed point O. The particle is hanging in equilibrium at a point A, directly below O, when it is set in motion with a horizontal speed $\sqrt{3gl}$.

When OP has turned through an angle θ and the string is still taut, the tension in the string is T.

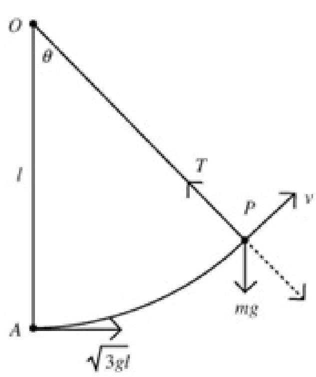
a Find an expression for T.

b Find the height of P above A at the instant when the string goes slack.

c Find the maximum height above A reached by P before it starts to fall to the ground again.

Solution:

a



Conservation of energy.

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgl(1-\cos\theta)$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mgl(1-\cos\theta)$$

$$v^2 = 3gl - 2gl(1-\cos\theta) = gl(1+2\cos\theta)$$

Resolving towards the centre of the circle:

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$T = mg\cos\theta + \frac{mgl}{l}(1+2\cos\theta) = mg + 3mg\cos\theta$$

b String slack $\Rightarrow T = 0 \Rightarrow \cos\theta = -\frac{1}{3} \Rightarrow \text{height} = l + \frac{l}{3} = \frac{4l}{3}$

c When the string goes slack, $v^2 = gl\left(1+2\times\left(-\frac{1}{3}\right)\right) = \frac{gl}{3}$

So horizontal component of velocity $= \frac{1}{3}\sqrt{\frac{gl}{3}}$

Using energy, if the maximum additional height is h, then

$$mgh + \frac{1}{2}m\times\left(\frac{1}{3}\sqrt{\frac{gl}{3}}\right)^2 = \frac{1}{2}m\left(\frac{gl}{3}\right)$$

$$h = \frac{l}{6} - \frac{l}{6\times 9} = \frac{8l}{54} = \frac{4l}{27}, \text{ height above } A = \frac{4l}{3} + \frac{4l}{27} = \frac{40l}{27}$$

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Motion in a circle
Exercise F, Question 2

Question:

Whenever a numerical value of g is required take $g = 9.8 \text{ m s}^{-2}$.

A smooth solid hemisphere with radius 6 m and centre O is resting in a fixed position on a horizontal plane with its flat face in contact with the plane. A particle P of mass 3 kg is slightly disturbed from rest at the highest point of the hemisphere.

When OP has turned through an angle θ and the particle is still on the surface of the hemisphere the normal reaction of the sphere on the particle is R.

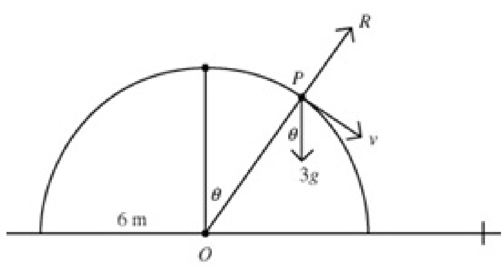
a Find an expression for R.

b Find the angle between OP and the upward vertical when the particle leaves the surface of the hemisphere.

c Find the distance of the particle from the centre of the hemisphere when it hits the ground.

Solution:

a



Conservation of energy from top to P:

$$mg \times 6 = mg \times 6\cos\theta + \frac{1}{2}mv^2$$

$$v^2 = 12g(1 - \cos\theta)$$

Resolving towards O:

$$3g\cos\theta - R = \frac{mv^2}{r} = \frac{12 \times 3g}{6}(1 - \cos\theta)$$

$$9g\cos\theta - 6g = R$$

b $R = 0 \Rightarrow \cos\theta = \frac{2}{3}, \theta \approx 48°$

c $v^2 = 12g \times \frac{1}{3} = 4g$

Speed $\rightarrow v\cos\theta$, $\quad \downarrow v\sin\theta + gt$

Distance $\rightarrow v\cos\theta t$, $\quad \downarrow v\sin\theta t + \frac{1}{2}gt^2 = 6 \times \frac{2}{3} = 4$

$$2\sqrt{g}\,\frac{\sqrt{5}}{3}t + \frac{g}{2}t^2 = 4,\ 4.9t^2 + \frac{14}{3}t - 4 = 0$$

$t \approx 0.545\ldots$

Total horizontal distance from $O = 6\sin\theta + \sqrt{4g}\cos\theta \times t \approx 6.7$ m

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Motion in a circle
Exercise F, Question 3

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A smooth solid hemisphere is fixed with its plane face on a horizontal table and its curved surface uppermost. The plane face of the hemisphere has centre O and radius r. The point A is the highest point on the hemisphere. A particle P is placed on the hemisphere at A. It is then given an initial horizontal speed u, where $u^2 = \frac{rg}{4}$. When OP makes an angle θ with OA, and while P remains on the hemisphere, the speed of P is v.

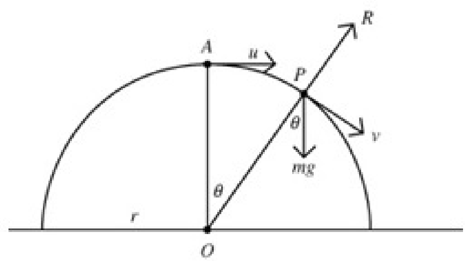
a Find an expression for v^2.
b Find the value of $\cos\theta$ when P leaves the hemisphere.
c Find the value of v when P leaves the hemisphere.

After leaving the hemisphere P strikes the table at B.

d Find the speed of P at B.
e Find the angle at which P strikes the table.

Solution:

a



Conservation of energy:

$$\frac{1}{2}mu^2 + mgr = \frac{1}{2}mv^2 + mgr\cos\theta$$

$$\frac{rg}{8} + rg = \frac{9rg}{8} = \frac{1}{2}v^2 + rg\cos\theta$$

$$v^2 = \frac{9rg}{4} - 2rg\cos\theta$$

b Resolving towards O: $mg\cos\theta - R = \frac{mv^2}{r} = mg\left(\frac{9}{4} - 2\cos\theta\right)$

$$R = 0 \Rightarrow 3mg\cos\theta = mg \times \frac{9}{4}, \cos\theta = \frac{3}{4}$$

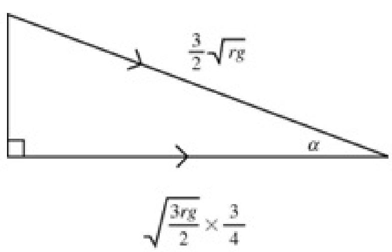
c $v^2 = \frac{9rg}{4} - 2rg \times \frac{3}{4} = \frac{3rg}{4}, \quad v = \sqrt{\frac{3rg}{4}}$

d Conservation of energy from A to the table:

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgr$$

$$v^2 = u^2 + 2gr = \frac{rg}{4} + 2gr = \frac{9rg}{4}, v = \frac{3}{2}\sqrt{rg}$$

e



After leaving the hemisphere the horizontal component of the velocity remains constant.

Direction is angle α to the ground,

$$\cos\alpha = \frac{\sqrt{\frac{3rg}{4}} \times \frac{3}{4}}{\frac{3}{2} \times \sqrt{rg}} = \frac{\sqrt{3}}{4}$$

$$\alpha = 64^\circ$$

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Motion in a circle
Exercise F, Question 4

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.

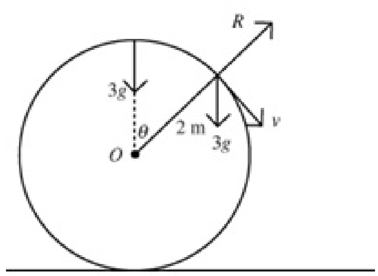
A smooth sphere with centre O and radius 2 m is fixed to a horizontal surface. A particle P of mass 3 kg is slightly disturbed from rest at the highest point of the sphere and starts to slide down the surface of the sphere.

a Find the angle between OP and the upward vertical at the instant when P leaves the surface of the sphere.

b Find the magnitude and direction of the velocity of the particle as it hits the horizontal surface.

Solution:

a



Conservation of energy:

$$mgr = \frac{1}{2}mv^2 + mgr\cos\theta$$

$$v^2 = 2mgr(1-\cos\theta) = 4mg(1-\cos\theta)$$

Resolving towards O:

$$3g\cos\theta - R = \frac{3v^2}{2} = \frac{3\times 4g(1-\cos\theta)}{2}$$

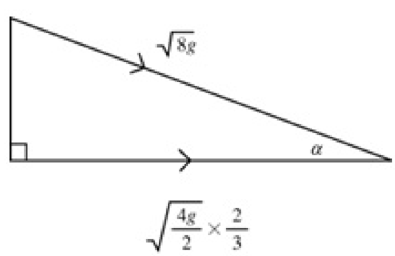
$$R = 0 \Rightarrow 9g\cos\theta = 6g, \quad \cos\theta = \frac{2}{3}$$

$$\theta \approx 48^\circ$$

b Using conservation of energy from the highest point to the ground:

$\frac{1}{2}mv^2 = mgh = mg\times 4$, $v = \sqrt{8g}$ when P hits the ground.

When P leaves the sphere $v^2 = 4mg(1-\cos\theta) = 4mg\times\frac{1}{3}, v = \sqrt{\frac{4mg}{3}}$



After leaving the hemisphere the horizontal component of the velocity remains constant. Direction is angle α to the ground,

$$\cos\alpha = \frac{\sqrt{\frac{4g}{3}}\times\frac{2}{3}}{\sqrt{8g}} = \frac{2}{3}\times\sqrt{\frac{1}{6}}$$

$$\alpha = 74^\circ$$

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Motion in a circle
Exercise F, Question 5

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.

A particle of mass m is projected with speed v from the top of the outside of a smooth sphere of radius a. In the subsequent motion the particle slides down the surface of the sphere and leaves the surface of the sphere with speed $\frac{\sqrt{3ga}}{2}$.

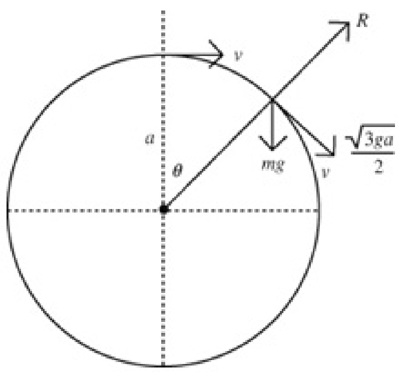
a Find the vertical distance travelled by the particle before it loses contact with the surface of the sphere.

b Find v.

c Find the magnitude and direction of the velocity of the particle when it is at the same horizontal level as the centre of the sphere.

Solution:

a



Forces acting along the radius:

$$mg\cos\theta - R = \frac{mv^2}{r} = \frac{m \times 3ga}{4a} = \frac{3mg}{4}$$

$$R = 0 \Rightarrow \cos\theta = \frac{3}{4}$$

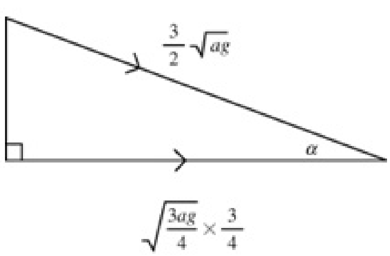
$$\text{Distance fallen} = a - a\cos\theta = \frac{a}{4}$$

b Conservation of energy from the top to the point where the particle leaves the sphere:

$$mg\frac{a}{4} = \frac{1}{2}m \times \frac{3ga}{4} - \frac{1}{2}mv^2, \quad \frac{1}{2}v^2 = \frac{3ga}{8} - \frac{ga}{4} = \frac{ga}{8}, v^2 = \frac{ga}{4}, v = \sqrt{\frac{ga}{4}}$$

c Looking at the energy at the top and level with the centre:

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga = \frac{1}{2}m\frac{ga}{4} + mga, v^2 = \frac{9ga}{4}, v = \frac{3}{2}\sqrt{ga}$$



After leaving the hemisphere the horizontal component of the velocity remains constant.

$$\text{Direction is } \alpha, \ \cos\alpha = \frac{\sqrt{\frac{3ag}{4}} \times \frac{3}{4}}{\frac{3}{2} \times \sqrt{ag}} = \frac{\sqrt{3}}{4}$$

$$\alpha = 64^\circ \text{ to the horizontal}$$

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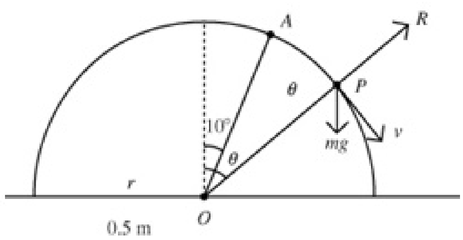
Edexcel AS and A Level Modular Mathematics

Motion in a circle
Exercise F, Question 6

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A smooth hemisphere with centre O and radius 50 cm is fixed with its plane face in contact with a horizontal surface. A particle P is released from rest at point A on the sphere, where OA is inclined at $10°$ to the upward vertical. The particle leaves the sphere at point B. Find the angle between OB and the upward vertical.

Solution:



Conservation of energy:

$$\frac{1}{2}mv^2 + mg\frac{1}{2}\cos\theta = mg\frac{1}{2}\cos 10°$$

$$v^2 = g(\cos 10° - \cos\theta)$$

Forces acting towards O:

$$mg\cos\theta - R = \frac{mv^2}{0.5} = 2mv^2$$

$$R = 0 \Rightarrow g\cos\theta = 2v^2 = 2g(\cos 10° - \cos\theta) \Rightarrow 3g\cos\theta = 2g\cos 10°$$

$$\cos\theta = \frac{2}{3}\cos 10°, \theta \approx 49°$$

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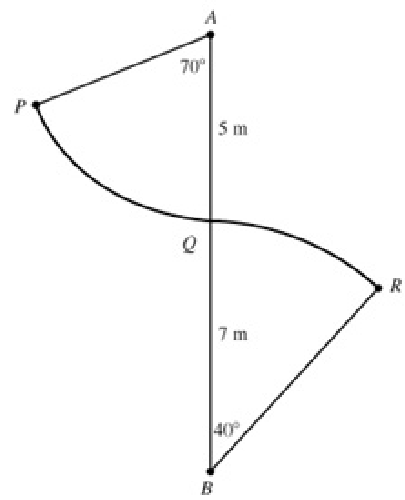
Motion in a circle
Exercise F, Question 7

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
A smooth laundry chute is built in two sections, PQ and QR. Each section is in the shape of an arc of a circle. PQ has radius 5 m and subtends an angle of $70°$ at its centre, A. QR has radius 7 m and subtends an angle of $40°$ at its centre, B. The points A, Q and B are in a vertical straight line. The laundry bags are collected in a large bin $\frac{1}{2}$ m below R. To test the chute, a small particle of mass 2 kg is released from rest at P.



a Calculate the speed with which the particle reaches the bin at the bottom of the chute.
b Determine whether or not the particle loses contact with the chute before it reaches R.

Solution:

a Total height lost $= 5(1-\cos 70°) + 7(1-\cos 40°) + 0.5 = 5.427\ldots$ m
Conservation of energy:
$\frac{1}{2}\times 2\times v^2 = 2\times g\times 5.427\ldots \quad v \approx 10.3\text{ m s}^{-1}$

b At R: $\frac{1}{2}\times 2\times v^2 = 2g\left(12 - 5\cos 70° - 7\cos 40°\right), \quad v^2 = 96.58$

Resolving ↗ towards B: $mg\cos\theta - R = \frac{mv^2}{7}$

$R = 2g\cos 40° - \frac{2v^2}{7} \approx -2.6 < 0$, which is impossible, so the particle has lost contact with the chute before this point.

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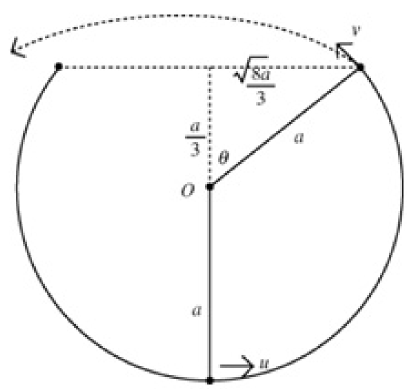
Motion in a circle
Exercise F, Question 8

Question:

Whenever a numerical value of g is required take $g = 9.8\text{ m s}^{-2}$.
Part of a hollow spherical shell, centre O and radius a, is removed to form a bowl with a plane circular rim. The bowl is fixed with the rim uppermost and horizontal. The centre of the circular rim is $\frac{4a}{3}$ vertically above the lowest point of the bowl. A marble is projected from the lowest point of the bowl with speed u. Find the minimum value of u for which the marble will leave the bowl and not fall back in to it.

Solution:



$$\text{K.E.} + \text{P.E. at lowest point} = \frac{1}{2}mu^2$$

$$\text{K.E.} + \text{P.E. at rim} = \frac{1}{2}mv^2 + mg \times \frac{4a}{3}$$

$$\Rightarrow u^2 = v^2 + \frac{8ga}{3}$$

After the particle leaves the bowl:

The vertical speed when the particle returns to the level of the rim of the bowl is $v\sin\theta$ downwards, so using $v = u + at, -v\sin\theta = v\sin\theta - gt, t = \frac{2v\sin\theta}{g}$

The horizontal distance covered in this time is $v\cos\theta \times \frac{2v\sin\theta}{g}$

The width of the top of the bowl $= 2 \times \frac{\sqrt{8}}{3}a = \frac{4\sqrt{2}a}{3}$

$$\Rightarrow 2\frac{v^2}{g}\sin\theta\cos\theta > \frac{4\sqrt{2}a}{3}, v^2 \times \frac{\sqrt{8}}{3} \times \frac{1}{3} > \frac{2\sqrt{2}ag}{3}, \quad v^2 > 3ag$$

$$\Rightarrow u^2 > 3ag + \frac{8ga}{3} = \frac{17ga}{3}$$

so minimum value of u is $\sqrt{\frac{17ag}{3}}$

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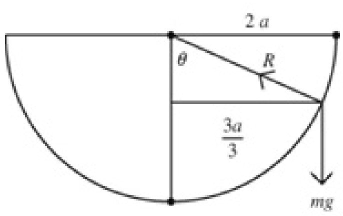
Motion in a circle
Exercise G, Question 1

Question:

A particle of mass m moves with constant speed u in a horizontal circle of radius $\frac{3a}{2}$ on the inside of a fixed smooth hollow sphere of radius $2a$. Show that $9ag = 2\sqrt{7}u^2$.

Solution:



$$R(\updownarrow)\ R\cos\theta = mg$$

$$R(\leftrightarrow)\ R\sin\theta = \frac{mv^2}{r} = \frac{2mv^2}{3a}$$

$$\text{Dividing} \Rightarrow \tan\theta = \frac{2u^2}{3ag}\text{, but}$$

$$\tan\theta = \frac{\frac{3a}{2}}{\frac{\sqrt{7}a}{2}} = \frac{3}{\sqrt{7}}\text{, so}$$

$$\frac{2u^2}{3ag} = \frac{3}{\sqrt{7}},\ 9ag = 2\sqrt{7}u^2$$

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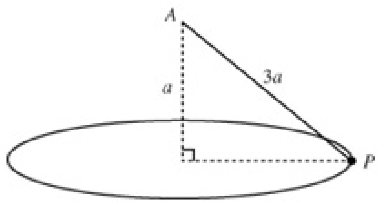
Motion in a circle
Exercise G, Question 2

Question:

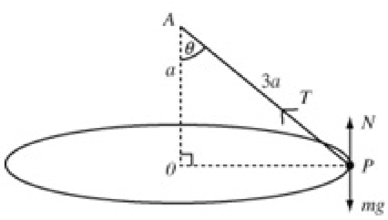


A particle P of mass m is attached to one end of a light inextensible string of length $3a$. The other end of the string is attached to a fixed point A which is a vertical distance a above a smooth horizontal table. The particle moves on the table in a circle whose centre O is vertically below A, as shown in the diagram. The string is taut and the speed of P is $2\sqrt{ag}$. Find

a the tension in the string,
b the normal reaction of the table on P.

Solution:

a



N is the normal reaction of the table on P, T is the tension in the string, and θ is the angle between the string and the vertical.
Right-angled triangle so

$OP = a\sqrt{8}$

$$R(\leftarrow): T\sin\theta = \frac{mv^2}{a\sqrt{8}}$$

$$T\frac{\sqrt{8}a}{3a} = \frac{m \times 4ga}{a\sqrt{8}}$$

$$\Rightarrow T = \frac{3mg}{2}$$

b $R(\uparrow): T\cos\theta + N = mg \Rightarrow N = mg - \frac{3}{2}mg \times \frac{1}{3} = \frac{1}{2}mg$

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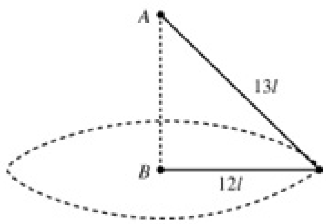
Motion in a circle
Exercise G, Question 3

Question:

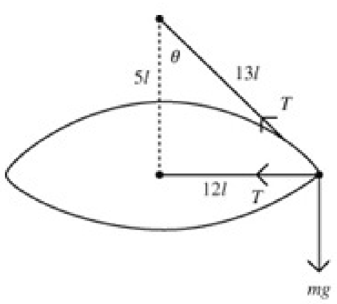


A light inextensible string of length $25l$ has its ends fixed to two points A and B, where A is vertically above B. A small smooth ring of mass m is threaded on the string. The ring is moving with constant speed in a horizontal circle with centre B and radius $12l$, as shown in the diagram. Find

a the tension in the string,
b the speed of the ring.

Solution:

a



Let θ be the angle between the string and the vertical.

We have a 5, 12, 13 triangle.

$R(\updownarrow) T\cos\theta = mg$

$$T = \frac{mg}{\cos\theta} = \frac{13mg}{5}$$

b $R(\leftrightarrow) T + T\sin\theta = \frac{mv^2}{r} \Rightarrow T\left(1+\frac{12}{13}\right) = \frac{mv^2}{12l}, \frac{25}{13}\times\frac{13mg}{5} = 5mg = \frac{mv^2}{12l}$

$\Rightarrow v^2 = 60gl, \quad v = \sqrt{60gl}$

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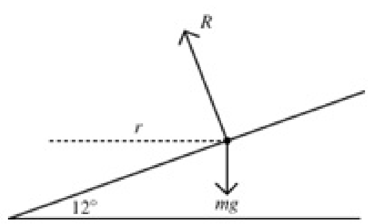
Motion in a circle
Exercise G, Question 4

Question:

A car moves round a bend which is banked at a constant angle of $12°$ to the horizontal. When the car is travelling at a constant speed of 15 m s^{-1} there is no sideways frictional force on the car. The car is modelled as a particle moving in a horizontal circle of radius r metres. Calculate the value of r.

Solution:



R is the normal reaction of the surface on the car.

No friction.

$$\text{R}(\updownarrow)\; R\cos 12° = mg$$

$$\text{R}(\leftrightarrow)\; R\sin 12° = \frac{mv^2}{r} = \frac{m \times 15^2}{r}$$

$$\text{Dividing : } \tan 12° = \frac{225}{gr}$$

$$r = \frac{225}{g\tan 12°} \approx 108 \text{ m}$$

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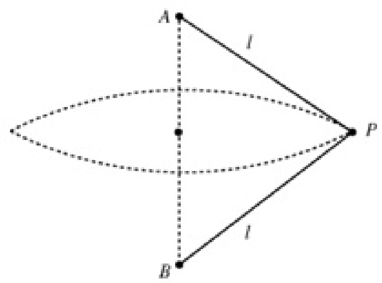
Motion in a circle
Exercise G, Question 5

Question:



A particle P of mass m is attached to the ends of two light inextensible strings AP and BP each of length l. The ends A and B are attached to fixed points, with A vertically above B and $AB = l$, as shown in the diagram. The particle P moves in a horizontal circle with constant angular speed ω. The centre of the circle is the mid-point of AB and both strings remain taut.

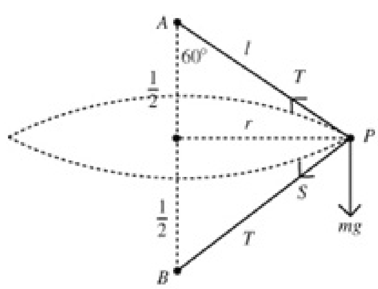
a Show that the tension in AP is $\frac{m}{2}\left(2g + l\omega^2\right)$.

b Find, in terms of m, l, ω and g, an expression for the tension in BP.

c Deduce that $\omega^2 \geq \frac{2g}{l}$.

Solution:

a



T is the tension in AP and S is the tension in BP.
The triangle is equilateral (3 equal sides).

$$R(\updownarrow): T\cos 60^\circ = mg + S\cos 60^\circ$$
$$T - S = 2mg$$
$$R(\leftrightarrow): T\cos 30^\circ + S\cos 30^\circ = mr\omega^2$$
$$(T+S)\cos 30^\circ = ml\cos 30^\circ \times \omega^2$$
$$T + S = ml\omega^2$$

Adding these two equations gives

$$2T = 2mg + ml\omega^2, T = \frac{m}{2}(2g + l\omega^2).$$

b $S = T - 2mg = \frac{m}{2}(l\omega^2 - 2g)$

c Both strings taut $\Rightarrow l\omega^2 - 2g > 0$, $\quad \omega^2 > \frac{2g}{l}$

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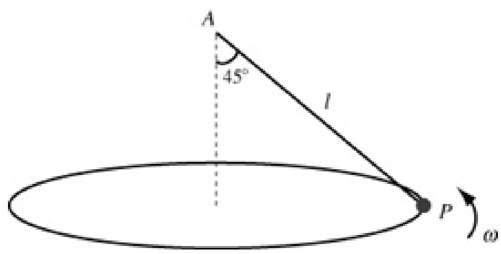
Motion in a circle
Exercise G, Question 6

Question:



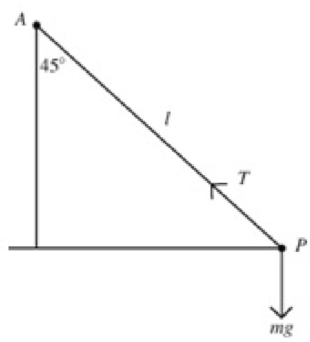
A particle P of mass m is attached to one end of a light string of length l. The other end of the string is attached to a fixed point A. The particle moves in a horizontal circle with constant angular speed ω and with the string inclined at an angle of $45°$ to the vertical, as shown in the diagram.

a Show that the tension in the string is $\sqrt{2}mg$.

b Find ω in terms of g and l.

Solution:

a



T is the tension in the string.

$R(\updownarrow): T\cos 45° = mg, T = \sqrt{2}mg$

b $R(\leftrightarrow): T\cos 45° = mr\omega^2 = ml\cos 45° \omega^2, T = ml\omega^2, \omega = \sqrt{\frac{T}{ml}} = \sqrt{\frac{\sqrt{2}g}{l}}$

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Motion in a circle
Exercise G, Question 7

Question:

A rough disc rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis. A particle P of mass m lies on the disc at a distance $\frac{3}{5}a$ from the axis. The coefficient of friction between P and the disc is $\frac{3}{7}$. Given that P remains at rest relative to the disc,

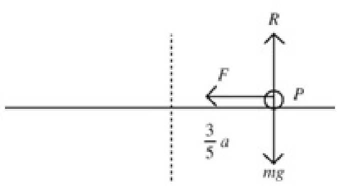
a prove that $\omega^2 \le \frac{5g}{7a}$.

The particle is now connected to the axis by a horizontal light elastic string of natural length $\frac{a}{2}$ and modulus of elasticity $\frac{5mg}{2}$. The disc again rotates with constant angular velocity ω about the axis and P remains at rest relative to the disc at a distance $\frac{3}{5}a$ from the axis.

b Find the range of possible values of ω^2.

Solution:

a



F is the force due to friction,
R is the normal reaction.

$R(\updownarrow): R = mg$

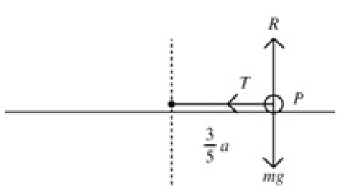
$R(\leftrightarrow): F = mr\omega^2$

If P is not to slip then

$$\frac{3}{7}mg \ge m\frac{3}{5}a\omega^2$$

$$\therefore \omega^2 \le \frac{5g}{7a}.$$

b



T is the tension in the elastic string.

$$T = \frac{\lambda x}{l} = \frac{\frac{5mg}{2}\times\left(\frac{3}{5}a - \frac{a}{2}\right)}{\frac{a}{2}} = \frac{5mg}{10} = \frac{mg}{2}$$

The limits for ω^2 depend on whether the friction is acting with the tension or against it.

$$R(\leftrightarrow): \frac{3}{7}mg + \frac{mg}{2} \ge m\frac{3}{5}a\omega^2, \omega^2 \le \frac{5}{3a}\times\frac{13g}{14} = \frac{65g}{42a}$$

$$\text{or } R(\leftrightarrow): -\frac{3}{7}mg + \frac{mg}{2} \le m\frac{3}{5}a\omega^2, \omega^2 \ge \frac{5}{3a}\times\frac{g}{14} = \frac{5g}{42a}$$

$$\frac{5g}{42a} \le \omega^2 \le \frac{65g}{42a}$$

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Motion in a circle
Exercise G, Question 8

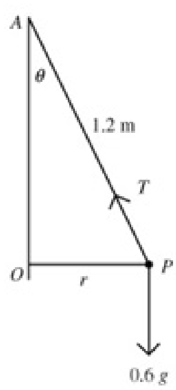
Question:

A particle P of mass 0.6 kg is attached to one end of a light inextensible string of length 1.2 m. The other end of the string is attached to a fixed point A. The particle is moving, with the string taut, in a horizontal circle with centre O vertically below A. The particle is moving with constant angular speed 3 rad s^{-1}. Find

a the tension in the string,
b the angle between AP and the downward vertical.

Solution:

a



r is the radius of the circle,
T is the tension in the string and $\angle OAP$ is θ.
From the triangle, $r = 1.2\sin\theta$.

$$R(\leftrightarrow): T\sin\theta = mr\omega^2 = 0.6\times 1.2\sin\theta\times 9$$
$$T = 0.6\times 1.2\times 9 = 6.48\text{ N}$$

b $R(\updownarrow)\, T\cos\theta = mg,\ 6.48\cos\theta = 0.6g,\ \cos\theta = \dfrac{0.6g}{6.48} \approx 0.907,\ \theta \approx 25°.$

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Motion in a circle
Exercise G, Question 9

Question:

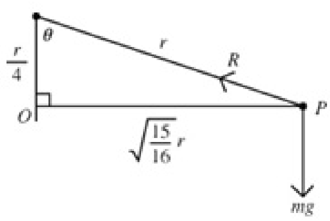
A particle P of mass m moves on the smooth inner surface of a spherical bowl of internal radius r. The particle moves with constant angular speed in a horizontal circle, which is at a depth $\frac{r}{4}$ below the centre of the bowl. Find

a the normal reaction of the bowl on P,

b the time it takes P to complete three revolutions of its circular path.

Solution:

a



The angle between the radius through P and the vertical is θ. P has angular speed ω rad s^{-1}.
R is the reaction of the bowl on P.
$R(\updownarrow): R\cos\theta = mg, R = 4mg.$

b $R(\leftrightarrow): R\sin\theta = mr\omega^2 = m \times r\sin\theta \times \omega^2, \ \omega = \sqrt{\frac{4mg}{mr}} = \sqrt{\frac{4g}{r}}$

Three revolutions is 6π radians, time taken $= \frac{6\pi}{\sqrt{\frac{4g}{r}}} = 3\pi\sqrt{\frac{r}{g}}$

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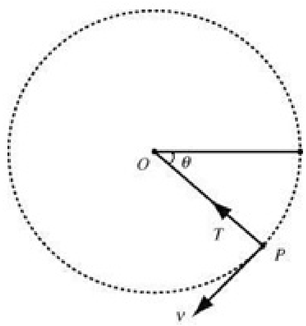
Motion in a circle
Exercise G, Question 10

Question:

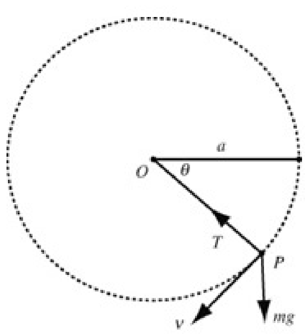


A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is fixed at a point O. The particle is held with the string taut and OP horizontal. It is then projected vertically downwards with speed u, where $u^2 = \frac{4}{3}ga$. When OP has turned through an angle θ and the string is still taut, the speed of P is v and the tension in the string is T, as shown in the diagram.

a Find an expression for v^2 in terms of a, g and θ.
b Find an expression for T in terms of m, g and θ.
c Find, to the nearest degree, the value of θ when the string becomes slack.
d Explain why P would not complete a vertical circle if the string were replaced by a light rod.

Solution:

a



Loss in P.E. = gain in K.E. so

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga\sin\theta$$

$$\Rightarrow v^2 = \frac{4}{3}ga + 2ga\sin\theta$$

b Resolving towards O: $T - mg\sin\theta = \frac{mv^2}{a}$

$$T = \frac{4}{3}mg + 2mg\sin\theta + mg\sin\theta = mg\left(\frac{4}{3} + 3\sin\theta\right)$$

c $T = 0$ when $\sin\theta = -\frac{4}{9}, \theta = 206°$

d When $v = 0, \sin\theta = -\frac{4}{6} = -\frac{2}{3}, (\theta \approx 222°)$ so the particle can not complete the circle.

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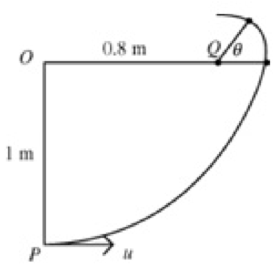
Motion in a circle
Exercise G, Question 11

Question:

A particle P of mass 0.4 kg is attached to one end of a light inelastic string of length 1 m. The other end of the string is fixed at point O. P is hanging in equilibrium below O when it is projected horizontally with speed u ms^{-1}. When OP is horizontal it meets a small smooth peg at Q, where $OQ = 0.8$ m. Calculate the minimum value of u if P is to describe a complete circle about Q.

Solution:



Consider the circle centre Q, radius 0.2 m.
When QP is at θ above the horizontal:

Energy: $\frac{1}{2}mw^2 + mg \times 0.2 \sin\theta = \frac{1}{2}mv^2$,

$w^2 = v^2 - 0.4g \sin\theta$

where v is the speed when $\theta = 0$, and w the speed at angle θ.

Circular motion: $T + mg\sin\theta = \frac{mw^2}{r} = \frac{m(v^2 - 0.4g\sin\theta)}{0.2}$

$T = \frac{m(v^2 - 0.4g\sin\theta)}{0.2} - mg\sin\theta = \frac{m(v^2 - 0.6\ g\sin\theta)}{0.2} \geq 0$

Looking at the larger circle, conservation of energy

$\Rightarrow \frac{1}{2}mv^2 + mg \times 1 = \frac{1}{2}mu^2, \quad v^2 = u^2 - 2g$

At the top of the small circle, $\sin\theta = 1$,

$\Rightarrow u^2 - 2g - 0.6g \geq 0, u^2 \geq 2.6g, u \geq \sqrt{2.6\ g}$

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Motion in a circle
Exercise G, Question 12

Question:

A smooth solid hemisphere is fixed with its plane face on a horizontal table and its curved surface uppermost. The plane face of the hemisphere has centre O and radius a. The point A is the highest point on the hemisphere. A particle P is placed on the hemisphere at A.

It is then given an initial horizontal speed u, where $u^2 = \frac{ag}{2}$. When OP makes an angle θ with OA, and while P remains on the hemisphere, the speed of P is v.

a Find an expression for v^2.

b Show that P is still on the hemisphere when $\theta = \cos^{-1} 0.9$

c Find the value of

i $\cos\theta$ when P leaves the hemisphere,

ii v when P leaves the hemisphere.

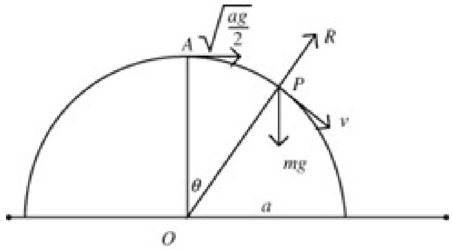
After leaving the hemisphere P strikes the table at B.

d Find the speed of P at B.

e Find the angle at which P strikes the table.

Solution:

a



R is the reaction between the particle and the surface.
If the level of P is the level of zero P.E., conservation of energy

$$\Rightarrow \frac{1}{2}m\frac{ag}{2} + mga(1-\cos\theta) = \frac{1}{2}mv^2,$$

$$v^2 = \frac{ga}{2} + 2ga(1-\cos\theta)$$

$$= \frac{ga}{2}(5-4\cos\theta)$$

b Resolving towards O: $mg\cos\theta - R = \frac{mv^2}{r} = \frac{mg}{2}(5-4\cos\theta)$

Substituting $\cos\theta = 0.9$: $R = mg \times 0.9 - \frac{mg}{2}(5-3.6) = 0.2mg > 0$

so P is still on the hemisphere.

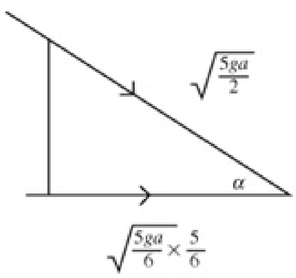
c **i** $R = 0 \Rightarrow \cos\theta = \frac{1}{2}(5-4\cos\theta), 3\cos\theta = \frac{5}{2}, \cos\theta = \frac{5}{6}.$

ii $v^2 = \frac{ga}{2}(5-4\cos\theta) = \frac{ga}{2}(5-\frac{10}{3}) = \frac{5ga}{6}, v = \sqrt{\frac{5ga}{6}}$

d By considering K.E. + P.E. at A and B, if v is the speed at B,

$$\frac{1}{2}mv^2 = \frac{1}{2}m\frac{ag}{2} + mga, v^2 = \frac{5ga}{2}, v = \sqrt{\frac{5ga}{2}}$$

e



After the particle leaves the sphere the horizontal velocity remains constant $= \sqrt{\frac{5ga}{6}} \times \frac{5}{6}$.

If α is the angle at which the particle strikes the table then $\cos\alpha = \frac{\sqrt{\frac{5ga}{6}} \times \frac{5}{6}}{\sqrt{\frac{5ga}{2}}} = \frac{5}{6\sqrt{3}}$

$$\alpha \approx 61^\circ$$

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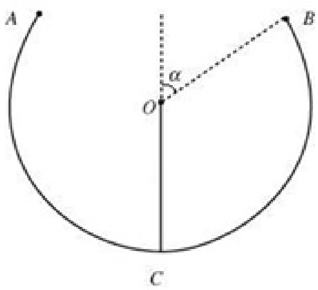
Motion in a circle
Exercise G, Question 13

Question:



Part of a hollow spherical shell, centre O and radius r, is removed to form a bowl with a plane circular rim. The bowl is fixed with the circular rim uppermost and horizontal. The point C is the lowest point of the bowl. The point B is on the rim of the bowl and OB is at an angle α to the upward vertical as shown in the diagram. Angle α satisfies $\tan\alpha = \frac{4}{3}$. A smooth small marble of mass m is placed inside the bowl at C and given an initial horizontal speed u. The direction of motion of the marble lies in the vertical plane COB. The marble stays in contact with the bowl until it reaches B.

When the marble reaches B it has speed v.

a Find an expression for v^2.

b If $u^2 = 4gr$, find the normal reaction of the bowl on the marble as the marble reaches B.

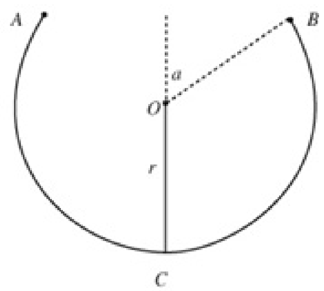
c Find the least possible value of u for the marble to reach B.

The point A is the other point of the rim of the bowl lying in the vertical plane COB.

d Find the value of u which will enable the marble to leave the bowl at B and meet it again at A.

Solution:

a



K.E. + P.E. at C = K.E. + P.E. at B.
If P.E. = 0 at C then

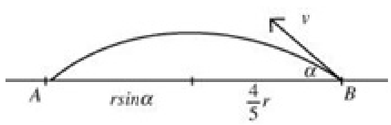
$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg(r + r\cos\alpha) = \frac{1}{2}mv^2 + \frac{8}{5}mgr$$

$$v^2 = u^2 - \frac{16}{5}gr$$

b $u^2 = 4gr \Rightarrow v^2 = \frac{4}{5}gr$. Resolving towards O: $R + \frac{3}{5}mg = \frac{mv^2}{r} = \frac{4mg}{5}, R = \frac{mg}{5}$

c $R = 0$ at $B \Rightarrow \frac{3mg}{5} = \frac{mv^2}{r} = \frac{m\left(u^2 - \frac{16gr}{5}\right)}{r}, \frac{mu^2}{r} = \frac{3mg}{5} + \frac{16mg}{5}, u = \sqrt{\frac{19gr}{5}}$

d



The particle is now moving freely under gravity.
Horizontal distance

$$= 2r\sin\alpha = \frac{8r}{5} = v\cos\alpha \times t$$

so $t = \frac{8r}{3v}$.

Vertical distance $= 0 = \frac{4v}{5}t - \frac{1}{2}gt^2 \Rightarrow t = \frac{8v}{5g} = \frac{8r}{3v}, \Rightarrow v = \sqrt{\frac{5rg}{3}}$

$$\Rightarrow u^2 = \frac{5rg}{3} + \frac{16gr}{5} = \frac{73}{15}gr; u = \sqrt{\frac{73gr}{15}}$$

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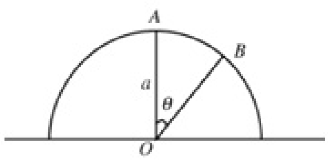
Motion in a circle
Exercise G, Question 14

Question:



A particle is at the highest point A on the outer surface of a fixed smooth hemisphere of radius a and centre O. The hemisphere is fixed to a horizontal surface with the plane face in contact with the surface. The particle is projected horizontally from A with speed u, where $u < \sqrt{ag}$. The particle leaves the sphere at the point B, where OB makes an angle θ with the upward vertical, as shown in the diagram.

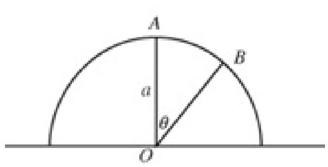
a Find an expression for $\cos\theta$ in terms of u, g and a.

The particle strikes the horizontal surface with speed $\sqrt{\frac{5ag}{2}}$.

b Find the value of θ.

Solution:

a



Equating the K.E. + P.E. at A and B:

$$\frac{1}{2}mu^2 + mga = \frac{1}{2}mv^2 + mga\cos\theta$$

$$\Rightarrow v^2 = u^2 + 2ga(1 - \cos\theta)$$

Resolving towards O: $mg\cos\theta - R = \frac{mv^2}{a}$

$$R = 0 \Rightarrow ag\cos\theta = u^2 + 2ga(1 - \cos\theta)$$

$$3ag\cos\theta = u^2 + 2ag$$

$$\cos\theta = \frac{u^2 + 2ag}{3ag}$$

b Conservation of energy from A to surface:

$$\frac{1}{2}mu^2 + mga = \frac{1}{2}m \times \frac{5ag}{2}, u^2 = \frac{ag}{2}, \cos\theta = \frac{5}{6}, \theta \approx 34°$$

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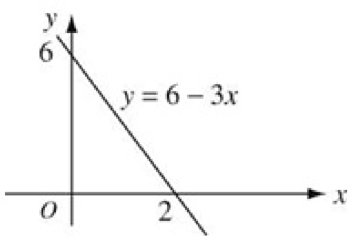
Statics of rigid bodies
Exercise A, Question 1

Question:

Find, by integration, the centre of mass of the uniform triangular lamina enclosed by the lines $y = 6 - 3x, x = 0$ and $y = 0$.

Solution:



Mass $= M = \frac{1}{2}\rho \times 2 \times 6 = 6\rho$, where ρ is the mass per unit area.

$$M\bar{x} = \int_0^2 \rho x(6-3x)\,dx \qquad M\bar{y} = \int_0^2 \rho\frac{1}{2}(6-3x)^2\,dx$$

$$= \rho\int_0^2 6x - 3x^2\,dx \qquad = \frac{1}{2}\rho\int_0^2 36 - 36x + 9x^2\,dx$$

$$= \rho\left[3x^2 - x^3\right]_0^2 \qquad = \frac{1}{2}\rho\left[36x - 18x^2 + 3x^3\right]_0^2$$

$$= \rho\left[4 - 0\right] \qquad = \frac{1}{2}\rho\left[24 - 0\right]$$

$$= 4\rho \qquad = 12\rho$$

$$\therefore \bar{x} = \frac{4\rho}{M} = \frac{4\rho}{6\rho} \qquad \therefore \bar{y} = \frac{12\rho}{M} = \frac{12\rho}{6\rho}$$

$$= \frac{2}{3} \qquad = 2$$

The centre of mass is at the point with coordinates $\left(\frac{2}{3}, 2\right)$.

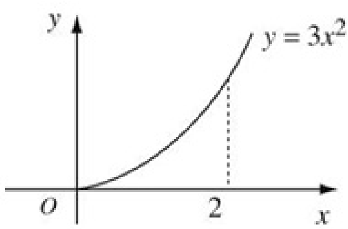
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 2

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = 3x^2$, the x-axis and the line $x = 2$.

Solution:



$$M = \int_0^2 \rho y \, dx$$
$$= \rho \int_0^2 3x^2 \, dx$$
$$= \rho \left[x^3 \right]_0^2$$
$$= 8\rho$$

$$M\bar{x} = \int_0^2 \rho x 3x^2 \, dx$$
$$= \rho \int_0^2 3x^3 \, dx$$
$$= \rho \left[\frac{3}{4} x^4 \right]_0^2$$
$$= 12\rho$$
$$\therefore \bar{x} = \frac{12\rho}{M} = \frac{12\rho}{8\rho}$$
$$= \frac{3}{2} = 1.5$$

$$M\bar{y} = \int_0^2 \frac{1}{2} \rho (3x^2)^2 \, dx$$
$$= \frac{1}{2} \rho \int_0^2 9x^4 \, dx$$
$$= \frac{9}{2} \rho \left[\frac{x^5}{5} \right]_0^2$$
$$= \frac{9 \times 32}{10} \rho$$
$$= \frac{144}{5} \rho$$
$$\therefore \bar{y} = \frac{144\rho}{5M} = \frac{144\rho}{40\rho}$$
$$= 3.6$$

The centre of mass is at the point with coordinates (1.5, 3.6).

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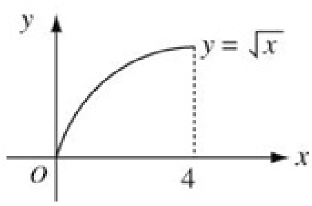
Statics of rigid bodies
Exercise A, Question 3

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y=\sqrt{x}$, the x-axis and the line $x=4$.

Solution:



$$M=\rho\int_0^4 y\,dx$$
$$=\rho\int_0^4 x^{\frac{1}{2}}\,dx$$
$$=\left[\frac{2}{3}\rho x^{\frac{3}{2}}\right]_0^4$$
$$=\frac{16\rho}{3}$$

$$M\bar{x}=\rho\int_0^4 xx^{\frac{1}{2}}dx$$
$$=\rho\int_0^4 x^{\frac{3}{2}}\,dx$$
$$=\left[\frac{2}{5}\rho x^{\frac{5}{2}}\right]_0^4$$
$$=\frac{64\rho}{5}$$
$$\therefore \bar{x}=\frac{64\rho}{5M}$$
$$=\frac{64\rho}{5}\div\frac{16\rho}{3}$$
$$=\frac{64\rho}{5}\times\frac{3}{16\rho}$$
$$=\frac{12}{5}=2.4$$

$$M\bar{y}=\frac{1}{2}\rho\int_0^4\left(x^{\frac{1}{2}}\right)^2\,dx$$
$$=\frac{1}{2}\rho\int_0^4 x\,dx$$
$$=\frac{1}{2}\rho\left[\frac{x^2}{2}\right]_0^4$$
$$=4\rho$$
$$\therefore \bar{y}=\frac{4\rho}{M}$$
$$=4\rho\div\frac{16\rho}{3}$$
$$=4\rho\times\frac{3}{16\rho}$$
$$=\frac{3}{4}=0.75$$

The centre of mass is at the point with coordinates (2.4, 0.75).

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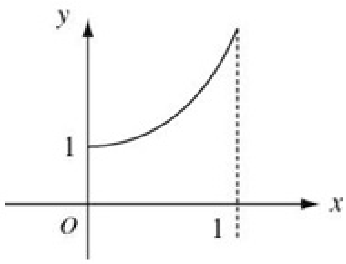
Statics of rigid bodies
Exercise A, Question 4

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = x^3 + 1$, the x-axis and the line $x = 1$.

Solution:



$$M = \rho\int_0^1 (x^3+1)\,dx = \rho\left[\frac{1}{4}x^4 + x\right]_0^1 = \frac{5}{4}\rho$$

$$M\bar{x} = \rho\int_0^1 x(x^3+1)\,dx = \rho\int_0^1 x^4 + x\,dx = \rho\left[\frac{1}{5}x^5 + \frac{1}{2}x^2\right]_0^1 = \frac{7}{10}\rho$$

$$\therefore \bar{x} = \frac{7}{10}\rho \div M = \frac{7}{10}\rho \div \frac{5}{4}\rho = \frac{7\rho}{10}\times\frac{4}{5\rho} = \frac{28}{50} = \frac{14}{25}$$

$$M\bar{y} = \frac{1}{2}\rho\int_0^1 (x^3+1)^2\,dx = \frac{1}{2}\rho\int_0^1 x^6 + 2x^3 + 1\,dx = \frac{1}{2}\rho\left[\frac{1}{7}x^7 + \frac{1}{2}x^4 + x\right]_0^1 = \frac{\rho}{2}\times\frac{23}{14} = \frac{23\rho}{28}$$

$$\therefore \bar{y} = \frac{23\rho}{28} \div M = \frac{23\rho}{28} \div \frac{5}{4}\rho = \frac{23\rho}{28}\times\frac{4}{5\rho} = \frac{23\rho}{35\rho} = \frac{23}{35}$$

The centre of mass is at the point with coordinates $\left(\frac{14}{25}, \frac{23}{35}\right)$.

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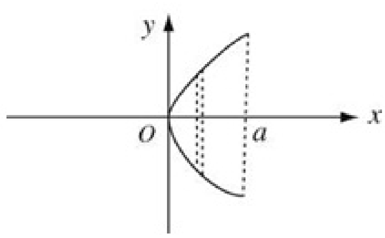
Statics of rigid bodies
Exercise A, Question 5

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y^2 = 4ax$, and the line $x = a$, where a is a positive constant.

Solution:



$$M = \rho\int_0^a 2y\,dx$$
$$= \rho\int_0^a 2\times 2a^{\frac{1}{2}}x^{\frac{1}{2}}\,dx$$
$$= 4\rho a^{\frac{1}{2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$
$$= \frac{8}{3}\rho a^2$$

$$M\,\bar{x} = \rho\int_0^a x\times 4a^{\frac{1}{2}}x^{\frac{1}{2}}\,dx$$
$$= \rho\int_0^a 4a^{\frac{1}{2}}x^{\frac{3}{2}}\,dx$$
$$= \rho\times 4a^{\frac{1}{2}}\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^a$$
$$= \frac{8\rho}{5}a^3$$
$$\therefore \bar{x} = \frac{8\rho a^3}{5}\div M$$
$$= \frac{8\rho a^3}{5}\div\frac{8\rho a^2}{3}$$
$$= \frac{3}{5}a$$

From symmetry $\bar{y} = 0$.

$\therefore$ The centre of mass is at the point with coordinates $\left(\frac{3}{5}a, 0\right)$.

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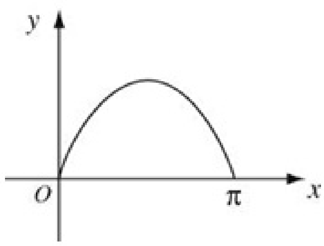
Statics of rigid bodies
Exercise A, Question 6

Question:

Find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = \sin x, (0 \le x \le \pi)$ and the line $y = 0$.

Solution:



$$y = \sin x$$
$$M = \rho\int_0^\pi \sin x \, dx$$
$$= \rho[-\cos x]_0^\pi$$
$$= 2\rho$$

From symmetry $\bar{x} = \frac{\pi}{2}$

$$M\bar{y} = \frac{1}{2}\rho\int_0^\pi \sin^2 x \, dx$$
$$= \frac{1}{2}\rho \times \frac{1}{2}\int_0^\pi 1 - \cos 2x \, dx$$
$$= \frac{1}{4}\rho\left[x - \frac{1}{2}\sin 2x\right]_0^\pi$$
$$= \frac{1}{4}\rho\pi$$
$$\therefore \bar{y} = \frac{1}{4}\rho\pi \div M = \frac{1}{4}\rho\pi \div 2\rho$$
$$= \frac{1}{8}\pi$$

The centre of mass is at the point with coordinates $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$.

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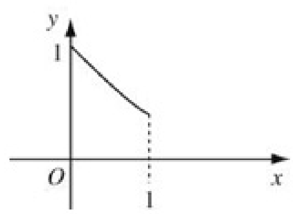
Statics of rigid bodies
Exercise A, Question 7

Question:

Find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = \frac{1}{1+x}, (0 < x < 1)$ and the lines $x = 0, x = 1$ and $y = 0$.

Solution:



$$M = \rho \int_0^1 y \, dx$$
$$= \rho \int_0^1 \frac{1}{1+x} \, dx$$
$$= \rho \left[\ln(1+x) \right]_0^1$$
$$= \rho \ln 2$$

$$M\bar{x} = \rho \int_0^1 \frac{x}{1+x} \, dx$$
$$= \rho \int_0^1 1 - \frac{1}{1+x} \, dx$$
$$= \rho \left[x - \ln(1+x) \right]_0^1$$
$$= \rho \left[1 - \ln 2 \right]$$
$$\therefore \bar{x} = \rho \frac{[1 - \ln 2]}{M}$$
$$= \rho \frac{[1 - \ln 2]}{\rho \ln 2}$$
$$= \frac{1 - \ln 2}{\ln 2}$$

$$M\bar{y} = \frac{1}{2} \rho \int_0^1 \frac{1}{(1+x)^2} \, dx$$
$$= \frac{1}{2} \rho \left[-(1+x)^{-1} \right]_0^1$$
$$= \frac{1}{2} \rho \left[\frac{-1}{2} + 1 \right]$$
$$= \frac{1}{4} \rho$$
$$\therefore \bar{y} = \frac{1}{4} \rho \div M$$
$$= \frac{\frac{1}{4} \rho}{\rho \ln 2}$$
$$= \frac{1}{4 \ln 2}$$

The centre of mass is at the point with coordinates $\left(\frac{1 - \ln 2}{\ln 2}, \frac{1}{4 \ln 2} \right)$.

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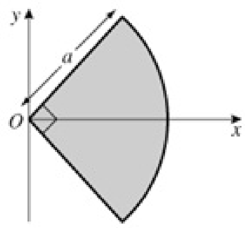
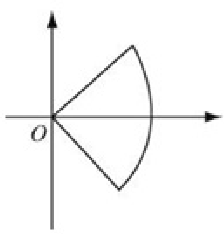
Statics of rigid bodies
Exercise A, Question 8

Question:



Find, by integration, the centre of mass of a uniform lamina in the shape of a quadrant of a circle of radius r as shown.

Solution:



$M = \frac{1}{4}\rho\pi r^2$ as this is a quarter of a circle.

$$M\bar{x} = \rho\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2 \times \frac{2}{3}r\cos\theta \, d\theta$$

$$= \rho \times \frac{1}{3}r^3 \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos\theta \, d\theta$$

$$= \frac{1}{3}\rho r^3 \left[\sin\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{3}\rho r^3 \left[\frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}}\right)\right]$$

$$= \frac{2\rho r^3}{3\sqrt{2}}$$

$$\therefore \bar{x} = \frac{2\rho r^3}{3\sqrt{2}} \div \frac{1}{4}\rho\pi r^2$$

$$= \frac{8r}{3\pi\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{8\sqrt{2}r}{6\pi} = \frac{4\sqrt{2}r}{3\pi}$$

Also $\bar{y} = 0$ from symmetry.

The centre of mass is at the point with coordinates $\left(\frac{4\sqrt{2}r}{3\pi}, 0\right)$.

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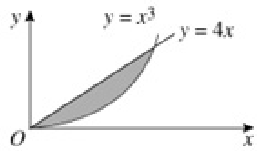
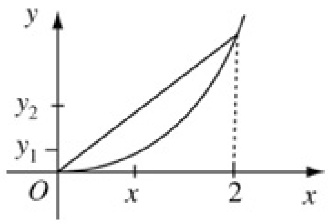
Statics of rigid bodies
Exercise A, Question 9

Question:

The figure shows a uniform lamina bounded by the curve $y = x^3$ and the line with equation $y = 4x$, where $x > 0$. Find the coordinates of the centre of mass of the lamina.



Solution:



$y = x^3$ meets $y = 4x$ when $x^3 = 4x$ i.e. $x = \pm 2$.
$\therefore$ when $x > 0, x = 2$

The small strip shown has dimensions $(y_2 - y_1)$ by δx and centre of mass at $\left(x, \frac{1}{2}(y_1 + y_2)\right)$.

$$M = \rho \int_0^2 (4x - x^3)\,dx = \rho\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 4\rho$$

$$M\bar{x} = \rho \int_0^2 x(4x - x^3)\,dx = \rho \int_0^2 4x^2 - x^4\,dx = \rho\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \rho\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{64\rho}{15}$$

$$\therefore \bar{x} = \frac{64\rho}{15} \div M = \frac{64\rho}{15} \div 4\rho = \frac{16}{15}$$

$$M\bar{y} = \frac{1}{2}\rho \int_0^2 (y_1 + y_2)(y_2 - y_1)dx = \frac{1}{2}\rho \int_0^2 (4x + x^3)(4x - x^3)dx = \frac{1}{2}\rho \int_0^2 16x^2 - x^6\,dx = \frac{\rho}{2}\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2 = \frac{\rho}{2}\left[\frac{128}{3} - \frac{128}{7}\right] = \frac{256}{21}\rho$$

$$\therefore \bar{y} = \frac{256}{21}\rho \div M = \frac{256}{21}\rho \div 4\rho = \frac{64}{21}$$

The centre of mass is at the point with coordinates $\left(\frac{16}{15}, \frac{64}{21}\right)$.

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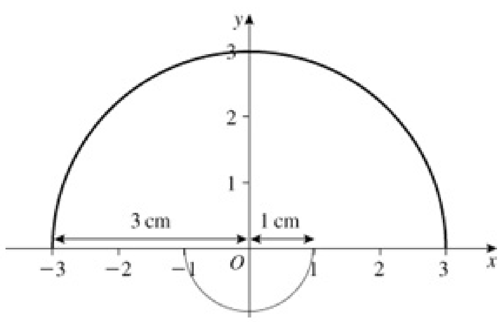
Statics of rigid bodies
Exercise A, Question 10

Question:

The figure shows a badge cut from a uniform sheet of fabric. The badge is formed from one semi-circle of radius 1 cm and a semi-circle of radius 3 cm joined as shown in the figure to make a plane lamina. Both semi-circles have the same centre O. Determine, in terms, of pi, the distance from O of the centre of mass.



Solution:

This question may be answered using M2 techniques. List the shapes with their masses in a table.

Shape	Mass	Position of centre of mass
Large semi-circle	$\frac{9\pi\rho}{2}$	$\left(0, \frac{4}{\pi}\right)$
Small semi-circle	$\frac{\pi\rho}{2}$	$\left(0, \frac{-4}{3\pi}\right)$
Total	$5\pi\rho$	$(0, \bar{y})$

From symmetry, the centre of mass lies on the axis of symmetry, taken as the y-axis. The common diameter is taken as the x-axis.

Then use $\sum m_i y_i = \bar{y} \sum m_i$ to give

$$\frac{9\pi\rho}{2} \times \frac{4}{\pi} + \frac{\pi\rho}{2} \times \frac{-4}{3\pi} = 5\pi\rho\bar{y}$$

i.e. $18 - \frac{2}{3} = 5\pi\bar{y}$

i.e. $\frac{52}{15\pi} = \bar{y}$

The centre of mass is on the axis of symmetry at a distance $\frac{52}{15\pi}$ above the common diameter.

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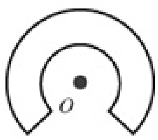
Statics of rigid bodies
Exercise A, Question 11

Question:



The figure shows a uniform lamina made from a sector of a circle with radius 5 cm from which a similar sector of radius 2.5 cm has been removed. The sector is three quarters of the original circle in each case, and both circles have the same centre O. Find the distance of the centre of mass of the lamina from the point O.

Solution:

The centre of mass lies an the axis of symmetry.

Centre of mass of a sector is $\frac{2r\sin\alpha}{3\alpha}$, where the angle of the sector is 2α.

In the given shape $2\alpha = \frac{3\pi}{2} \Rightarrow \alpha = \frac{3\pi}{4}$ and centre of mass of the sectors is at a

distance from the centre O of $\dfrac{2r\times\frac{1}{\sqrt{2}}}{\frac{9\pi}{4}} = \frac{8r}{9\pi\sqrt{2}} = \frac{4r\sqrt{2}}{9\pi}$.

Shape	Mass	Distance from O of centre of mass
Large sector	$\frac{3}{4}\times 25\pi\rho$	$\frac{20\sqrt{2}}{9\pi}$
Small sector	$\frac{3}{4}\times 6.25\pi\rho$	$\frac{10\sqrt{2}}{9\pi}$
Remainder	$\frac{3}{4}\times 18.75\pi\rho$	$\bar{y}$

From the moments equation

$$\frac{3}{4}\times 25\pi\rho\times\frac{20\sqrt{2}}{9\pi} - \frac{3}{4}\times 6.25\pi\rho\times\frac{10\sqrt{2}}{9\pi} = \frac{3}{4}\times 18.75\pi\rho\bar{y}$$

$$\therefore 4\times\frac{20\sqrt{2}}{9\pi} - 1\times\frac{10\sqrt{2}}{9\pi} = 3\bar{y}$$

Divide each term by $\frac{3}{4}\times 6.25\pi\rho$.

$$\therefore \bar{y} = \frac{70\sqrt{2}}{27\pi} \approx 1.2$$

The distance of the centre of mass from O is 1.2 cm (2 s.f.).

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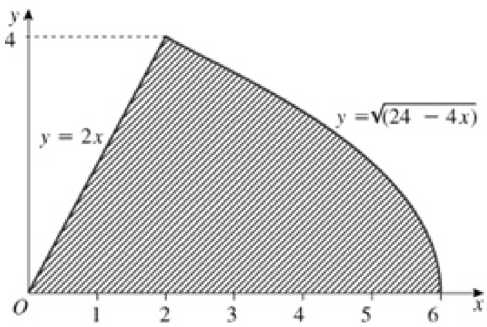
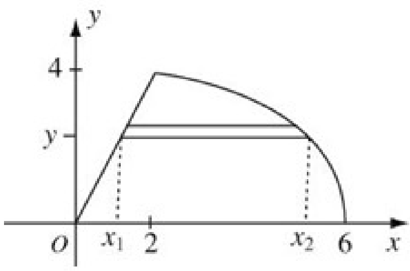
Statics of rigid bodies
Exercise A, Question 12

Question:

The figure shows a uniform lamina occupying the finite region bounded by the x-axis, the curve $y = \sqrt{(24 - 4x)}$, where $2 \le x \le 6$, and the line with equation $y = 2x$, where $0 \le x \le 2$. Find the coordinates of the centre of mass of the lamina.



Solution:



Divide the region into horizontal strips of dimensions $(x_2 - x_1)$ by δy.
The centre of mass of such strips lies at $\left(\frac{x_1 + x_2}{2}, y\right)$
where $x_1 = \frac{y}{2}$ and $x_2 = \frac{24 - y^2}{4}$.

Using $M = \rho\int_0^4 (x_2 - x_1)\,dy$

$$\therefore M = \rho\int_0^4 \left(6 - \frac{1}{4}y^2 - \frac{y}{2}\right)dy$$
$$= \rho\left[6y - \frac{1}{12}y^3 - \frac{1}{4}y^2\right]_0^4$$
$$= \rho\left[24 - \frac{16}{3} - 4\right]$$
$$= \frac{44}{3}\rho$$

Using $M\bar{y} = \rho\int_0^4 y(x_2 - x_1)\,dy$

$$M\bar{y} = \rho\int_0^4 6y - \frac{1}{4}y^3 - \frac{y^2}{2}\,dy$$
$$= \rho\left[3y^2 - \frac{1}{16}y^4 - \frac{1}{6}y^3\right]_0^4$$
$$= \rho\left[48 - 16 - \frac{64}{6}\right]$$
$$= \frac{64}{3}\rho$$
$$\therefore \bar{y} = \frac{64}{3}\rho \div M = \frac{64}{3} \times \frac{3}{44}$$
$$= \frac{16}{11}$$
$$= 1.5 \text{ (2 s.f.)}$$

Using $M\bar{y} = \frac{\rho}{2}\int_0^4 (x_1 + x_2)(x_2 + x_1)\,dy$

$$= \frac{\rho}{2}\int_0^4 (x_2^2 - x_1^2)\,dy$$

$$= \frac{\rho}{2}\int_0^4 \left[\left(6 - \frac{1}{4}y^2\right)^2 - \frac{1}{4}y^2\right]dy$$

$$= \frac{\rho}{2}\int_0^4 36 - 3y^2 + \frac{1}{16}y^4 - \frac{1}{4}y^2\ \,dy$$

$$= \frac{\rho}{2}\int_0^4 36 - \frac{13}{4}y^2 + \frac{1}{16}y^4\ \,dy$$

$$= \frac{\rho}{2}\left[36y - \frac{13}{12}y^3 + \frac{1}{80}y^4\right]_0^4$$

$$= \frac{\rho}{2}\left[144 - \frac{13}{12}\times 64 + \frac{1024}{80}\right]$$

$$= \frac{\rho}{2}\left[87\frac{7}{15}\right]$$

$$\therefore\quad = \frac{1}{2}\times 87\frac{7}{15} \div \frac{44}{3}$$

$$= 2\frac{54}{55}$$

$$= 3.0\ (2\text{ s.f.})$$

The centre of mass is at the point with coordinates $\left(2\frac{54}{55}, 1\frac{5}{11}\right)$.

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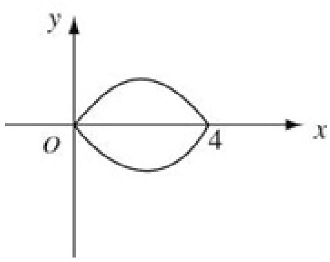
Statics of rigid bodies
Exercise B, Question 1

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.
The finite region bounded by the curve $y = x^2 - 4x$ and the x-axis is rotated through $360°$ about the x-axis to form a solid of revolution. Find the coordinates of its centre of mass.

Solution:



From symmetry the centre of mass lies an the x-axis.

As $y = x^2 - 4x$ meets the x axis when

$$x^2 - 4x = 0$$

i.e. $x(x-4) = 0$

$$\therefore x = 0 \text{ and } x = 4$$

Again from symmetry the centre of mass lies at $x = 2$.

The coordinates of the centre of mass are (2, 0).

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Statics of rigid bodies
Exercise B, Question 2

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.
The finite region bounded by the curve $(x-1)^2+y^2=1$ is rotated through $180°$ about the x-axis to form a solid of revolution. Find the coordinates of its centre of mass.

Solution:

The curve with equation
$(x-1)^2+y^2=1$

is a circle, centre (1, 0) radius 1.

It is rotated about the x-axis to form a sphere – centre (1, 0).

The centre of mass is at (1, 0).

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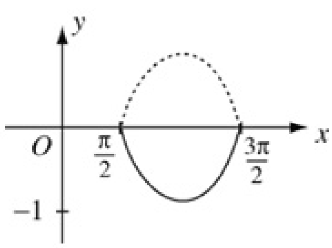
Statics of rigid bodies
Exercise B, Question 3

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.

The finite region bounded by the curve $y = \cos x$ $\frac{\pi}{2} \le x \le \frac{3x}{2}$, and the x-axis, is rotated through 360° about the x-axis to form a solid of revolution. Find the coordinates of its centre of mass.

Solution:



From symmetry the centre of mass of the solid of revolution is at the point with coordinates $(\pi, 0)$.

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Statics of rigid bodies
Exercise B, Question 4

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.
The finite region bounded by the curve $y^2 + 6y = x$ and the y-axis, is rotated through $360°$ about the y-axis to form a solid of revolution. Find the coordinates of its centre of mass.

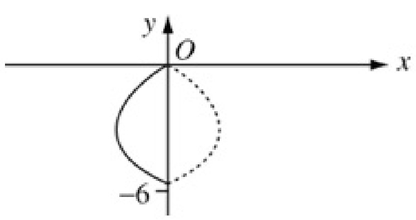
Solution:

The curve with equation $y^2 + 6y = x$ meets the x-axis at $x = 0$, and meets the y-axis when $y^2 + 6y = 0$

i.e. $y(y + 6) = 0$

$\therefore y = 0$ or -6

The curve is shown in the diagram, and is rotated about the y-axis through $360°$.



From symmetry the centre of mass lies at the point $(0, -3)$.

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Statics of rigid bodies
Exercise B, Question 5

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 3x^2$, the line $x = 1$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies on the x-axis, from symmetry.

Using the formula

$$\bar{x} = \frac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}, \text{ as } y = 3x^2$$

$$\bar{x} = \frac{\int_0^1 \pi \times 9x^5 \, dx}{\int_0^1 \pi \times 9x^4 \, dx}$$

$$= \frac{\pi \left[\frac{9}{6}x^6\right]_0^1}{\pi \left[\frac{9}{5}x^5\right]_0^1}$$

$$= \frac{9}{6} \div \frac{9}{5}$$

$$= \frac{5}{6}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{5}{6}, 0\right)$.

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Statics of rigid bodies
Exercise B, Question 6

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y=\sqrt{x}$, the line $x=4$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies an the x-axis, from symmetry.

Using the formula $\bar{x}=\dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y=\sqrt{x}$,

$$\bar{x}=\frac{\int_0^4 \pi x \times x \, dx}{\int_0^4 \pi x \, dx}=\frac{\int_0^4 \pi x^2 \, dx}{\int_0^4 \pi x \, dx}$$

then $$=\frac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4}$$

$$=\frac{64}{3}\div 8$$

$$=\frac{8}{3}=2\frac{2}{3}$$

$\therefore$ The centre of mass lies at the point $\left(2\frac{2}{3},0\right)$.

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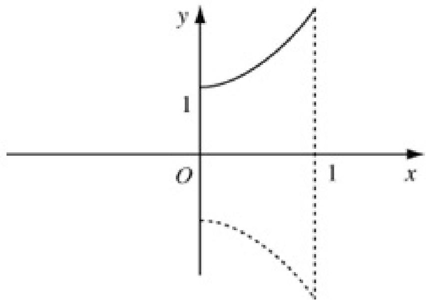
Statics of rigid bodies
Exercise B, Question 7

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 3x^2 + 1$, the lines $x = 0$, $x = 1$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:



The centre of mass lies on the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, as $y = 3x^2 + 1$,

$$\bar{x} = \frac{\int_0^1 \pi (3x^2+1)^2 x \, dx}{\int_0^1 \pi (3x^2+1)^2 \, dx}$$

$$= \frac{\pi \int_0^1 9x^5 + 6x^3 + x \, dx}{\pi \int_0^1 9x^4 + 6x^2 + 1 \, dx}$$

$$= \frac{\left[\frac{9}{6}x^6 + \frac{6}{4}x^4 + \frac{1}{2}x^2\right]_0^1}{\left[\frac{9}{5}x^5 + \frac{6}{3}x^3 + x\right]_0^1}$$

$$= \frac{\frac{9}{6} + \frac{6}{4} + \frac{1}{2}}{\frac{9}{5} + \frac{6}{3} + 1} = \frac{3\frac{1}{2}}{4\frac{4}{5}}$$

$$= \frac{35}{48}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{35}{48}, 0\right)$.

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Statics of rigid bodies
Exercise B, Question 8

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = \frac{3}{x}$, the lines $x = 1$, $x = 3$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies an the x-axis from symmetry.

Using the formula $\bar{x} = \frac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = \frac{3}{x}$, then

$$\bar{x} = \frac{\int_1^3 \pi \left(\frac{9}{x^2}\right) x \, dx}{\int_1^3 \pi \left(\frac{9}{x^2}\right) dx}$$

$$= \frac{\pi \int_1^3 \frac{9}{x} \, dx}{\pi \int_1^3 9x^{-2} \, dx}$$

$$= \frac{[9 \ln x]_1^3}{[-9x^{-1}]_1^3}$$

$$= \frac{9 \ln 3}{9 - 3}$$

$$= \frac{3}{2} \ln 3.$$

$\therefore$ The centre of mass lies at the point $\left(\frac{3}{2} \ln 3, 0\right) = (1.65, 0)$ (3 s.f.)

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Statics of rigid bodies
Exercise B, Question 9

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 2e^x$, the lines $x = 0$, $x = 1$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies on the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = 2e^x$, then

$$\bar{x} = \frac{\int_0^1 \pi \times 4e^{2x} \times x \, dx}{\int_0^1 \pi \times 4e^{2x}}$$

$$= \frac{2\pi \left\{ \left[xe^{2x} \right]_0^1 - \int_0^1 e^{2x} \, dx \right\}}{2\pi \int_0^1 2e^{2x} \, dx}$$

$$= \frac{\left[xe^{2x} - \frac{1}{2} e^{2x} \right]_0^1}{\left[e^{2x} \right]_0^1}$$

$$= \frac{e^2 - \frac{1}{2} e^2 + \frac{1}{2}}{e^2 - 1}$$

$$= \frac{1}{2} \frac{(e^2 + 1)}{(e^2 - 1)}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{1}{2} \frac{(e^2 + 1)}{(e^2 - 1)}, 0 \right)$.

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Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise B, Question 10

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 3e^{-x}$, the lines $x = 0, x = 2$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies an the x-axis from symmetry.

Use the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = 3e^{-x}$.

$$\text{Then } \bar{x} = \frac{\int_0^2 \pi \times 9e^{-2x} x \, dx}{\int_0^2 \pi \times 9e^{-2x} \, dx}$$

$$= \frac{9\pi \int_0^2 xe^{-2x} \, dx}{9\pi \int_0^2 e^{-2x} \, dx}$$

$$= \frac{\left[-\frac{1}{2}xe^{-2x}\right] + \int_0^2 \frac{1}{2}e^{-2x} \, dx}{\left[-\frac{1}{2}e^{-2x}\right]_0^2}$$

$$= \frac{\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^2}{-\frac{1}{2}e^{-4} + \frac{1}{2}}$$

$$= \frac{\left[-e^{-4} - \frac{1}{4}e^{-4}\right] + \frac{1}{4}}{-\frac{1}{2}e^{-4} + \frac{1}{2}}$$

$$= \frac{1 - 5e^{-4}}{2(1 - e^{-4})} \quad \text{or} \quad \frac{e^4 - 5}{2(e^4 - 1)} = 0.46 \text{ (2 s.f.)}$$

$\therefore$ The centre of mass lies at the point with coordinates $\left(\dfrac{e^4 - 5}{2(e^4 - 1)}, 0\right)$.

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Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise B, Question 11

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A uniform solid right circular cone of height 10 cm and base radius 5 cm is joined at its base to the base of a uniform solid hemisphere. The centres of their bases coincide and their axes are collinear. The radius of the hemisphere is also 5 cm. Find the position of the centre of mass of the composite body,
a when both the cone and the hemisphere have the same density,
b when the hemisphere has density twice that of the cone.

Solution:

a Let O be the point at the centre of the plane circular face of the cone and of the hemisphere. Let positive displacement be towards the vertex of the cone

Shape	Mass	Units of mass	Distance from O to centre of mass
Cone	$\frac{1}{3}\pi\rho\times 25\times 10$	1	$\frac{10}{4}$
Hemisphere	$\frac{2}{3}\pi\rho\times 5^3$	1	$-\frac{15}{8}$
Composite body	$\frac{250}{3}\pi\rho+\frac{250}{3}\pi\rho$	2	$\bar{x}$

Take moments about O:

$$1\times\frac{10}{4}+1\times\left(\frac{-15}{8}\right)=2\bar{x}$$

$$\therefore \bar{x}=\frac{5}{16}$$

The centre of mass lies an the axis of symmetry at a point $\frac{5}{16}$ cm from O towards the vertex of the cone.

b If the hemisphere has twice the density of the cone then the ratio of the masses becomes cone 1, hemisphere 2, composite body 3 so the moments equation becomes

$$1\times\frac{10}{4}+2\times\frac{-15}{8}=3\bar{x}$$

$$\therefore \bar{x}=\frac{-15}{12}$$

The centre of mass lies on the axis of symmetry at a point $\frac{5}{12}$ cm from O towards the rim of the hemisphere.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise B, Question 12

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A solid is composed of a uniform solid right circular cylinder of height 10 cm and base radius 6 cm joined at its top plane face to the base of a uniform hemisphere of the same radius. The centres of their adjoining circular faces coincide at point O and their axes are collinear. The radius of the hemisphere is also 6 cm. Find the position of the centre of mass of the composite body,

a if the cylinder and hemisphere are of the same density,

b if the hemisphere has three times the density of the cylinder.

Solution:

Let ρ be the mass per unit volume of the cylinder.

a The centre of mass lies an the axis of symmetry.

Shape	Mass	Units of mass	Distance of centre of mass from O (+ve towards top of hemisphere)
Cylinder	$\rho\pi \times 36 \times 10$	5	-5
Hemisphere	$\rho\frac{2}{3}\pi \times 6^3$	2	$\frac{3}{8}\times 6$
Composite body	$360\pi\rho + 144\pi\rho$	7	$\bar{x}$

Take moments about O.

$$5\times -5 + 2\times\frac{18}{8} = 7\bar{x}$$

$$\text{i.e. } 9 - 50 = 14\bar{x}$$

$$\therefore \bar{x} = -\frac{41}{14}$$

$\therefore$ Centre of mass is an axis of symmetry at a distance $2\frac{13}{14}$ cm away from base of hemisphere.

b Redraw the table noting that the masses of the hemisphere and of the composite body have changed.

Shape	Mass	Units of mass	Distance of centre of mass from O (+ve as before)
Cylinder	$\rho\pi \times 36 \times 10$	5	-5
Hemisphere	$3\rho\times\frac{2}{3}\pi \times 6^3$	6	$\frac{3}{8}\times 6$
Composite body	$360\pi\rho + 432\pi\rho$	11	$\bar{x}$

Take moments about O:

$$-25 + 6\times\frac{18}{8} = 11\bar{x}$$

$$\therefore \bar{x} = \frac{-23}{22}$$

$\therefore$ Centre of mass is on the axis of symmetry at a distance $1\frac{1}{22}$ cm away from the base of the hemisphere.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

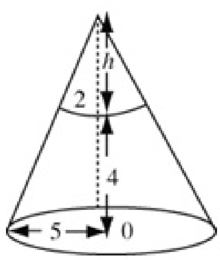
Statics of rigid bodies
Exercise B, Question 13

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
Find the position of the centre of mass of the frustum of a right circular uniform solid cone, where the frustum has end radii 2 cm and 5 cm, and has height 4 cm.

Solution:

A frustum of a cone is obtained by removing a small cone from a large cone.
Draw a diagram, showing the cones and the frustum and let the height of the small cone be h cm.



Using similar triangles:

$$\frac{h}{h+4} = \frac{2}{5}$$

$$\therefore 5h = 2(h+4)$$

$$\therefore 3h = 8$$

$$\text{i.e. } h = \frac{8}{3}$$

The centre of mass lies an the axis of symmetry. Let the centre of the base of the large cone be O.

Shape	Mass	Units of mass	Distance of centre of mass from O
Large cone	$\rho\times\frac{1}{3}\pi\times5^2\times\frac{20}{3}$	500	$\frac{5}{3}$
Small cone	$\rho\times\frac{1}{3}\pi\times2^2\times\frac{8}{3}$	32	$4+\frac{2}{3}$
Frustum	$\frac{500\pi}{9}\rho-\frac{32\pi}{9}\rho$	468	$\bar{x}$

Take moments about O

$$500\times\frac{5}{3}-32\times\left(\frac{14}{3}\right)=468\bar{x}$$

$$\therefore \frac{2052}{3}=468\bar{x}$$

$$\therefore \bar{x}=1\frac{6}{13} \text{ or } 1.46 \quad (3\text{ s.f.})$$

Solutionbank M3

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Statics of rigid bodies
Exercise B, Question 14

Question:

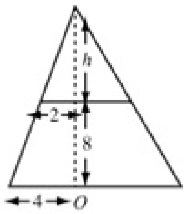
You may quote results for the centres of mass of cones and hemispheres obtained earlier.

a Find the position of the centre of mass of the frustum of a right circular uniform solid cone, where the frustum has end radii 2 cm and 4 cm, and has height 8 cm.

b A cylindrical hole of radius 1 cm with the same axis as that of the frustum is now drilled through the frustum. Find the distance of the new centre of mass from the larger face of the frustum.

Solution:

a



From similar triangles:

$$\frac{h}{h+8} = \frac{2}{4}$$

$$\therefore 4h = 2(h+8)$$

$$\therefore 2h = 16$$

i.e. $h = 8$

Shape	Mass	Units of mass	Distance of centre of mass from O
Large cone	$\rho \times \frac{1}{3}\pi \times 4^2 \times 16$	8	4
Small cone	$\rho \times \frac{1}{3}\pi \times 2^2 \times 8$	1	$8+2=10$
Frustum	$\frac{256\pi}{3}\rho - \frac{32\pi}{3}\rho$	7	$\bar{x}$

Take moments about O:

$8\times 4 - 1\times 10 = 7\bar{x}$

$\therefore \bar{x} = \frac{22}{7} \quad = 3.14$ (3 s.f.).

Required distance is 3.14 cm.

b

Shape	Mass	Units of mass	Distance of centre of mass from O
Frustum	$\frac{224\pi}{3}\rho$	28	$\frac{22}{7}$
Cylindrical hole	$\pi \times 1^2 \times 8\rho$	3	4
Remainder	$\frac{200\pi}{3}\rho$	25	$\bar{x}$

Take moments about O:

$$28\times\frac{22}{7} - 3\times 4 = 25\bar{x}$$

i.e. $88 - 12 = 25\bar{x}$

$$\therefore \bar{x} = \frac{76}{25} = 3.04$$

Required distance is 3.04 cm.

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Statics of rigid bodies

Exercise B, Question 15

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.

A thin uniform hemispherical shell has a circular base of the same material. Find the position of the centre of mass above the base in terms of its radius r.

Solution:

Let the density of the material be ρ and its thickness be t.

Then the mass will be proportional to the surface area.

The centre of mass will be an the axis of symmetry

Shape	Mass	Units of mass	Distance of centre of mass above base
Hemispherical shell	$\rho t \times 2\pi r^2$	2	$\frac{r}{2}$
Circular base	$\rho t \times \pi r^2$	1	0
Composite body	$\rho t \times 3\pi r^2$	3	$\bar{x}$

Take moments about centre of base:

$$2 \times \frac{r}{2} + 0 = 3\bar{x}$$

$$\therefore \bar{x} = \frac{1}{3} r$$

So the centre of mass is at a distance $\frac{r}{3}$ above the base.

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Statics of rigid bodies
Exercise B, Question 16

Question:

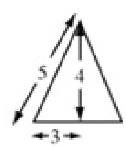
You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A thin uniform hollow cone has a circular base of the same material. Find the position of the centre of mass above the base, given that the radius of the cone is 3 cm and its height is 4 cm.

Solution:

Let the density of the material be ρ and its thickness be t.
Then the mass will be proportional to the surface area.
The centre of mass will be on the axis of symmetry

Shape	Mass	Units of mass	Distance of centre of mass above base
Hollow cone	$\rho t\times\pi\times3\times5$	15	$\frac{1}{3}\times4$
Circular base	$\rho t\times\pi\times3^2$	9	0
Composite body	$15\pi\rho t+9\pi\rho t$	24	$\bar{x}$

[The surface area of a cone is given by the formula πrl where l is the length of the slant side. As $r=3$ and $h=4$ then $l=5$ from Pythagoras-Theorem.]



Take moments about centre of base:

$$\therefore 15\times\frac{4}{3}+0=24\bar{x}$$

$$\therefore \bar{x}=\frac{20}{24}=\frac{5}{6}$$

So the centre of mass is at a distance $\frac{5}{6}$ cm above the base.

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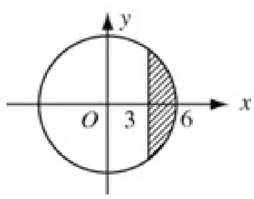
Statics of rigid bodies
Exercise B, Question 17

Question:

Use calculus to obtain your answer.
A cap of height 3 cm is cut from a uniform solid sphere of radius 6 cm. Using calculus, find the position of the centre of mass of the cap, giving the distance from the plane circular surface.

Solution:

This cap is the solid of revolution obtained when the arc of the circle with equation $x^2 + y^2 = 6^2, 3 \le x \le 6$, is rotated through $180°$ about the x-axis.



The centre of mass lies an the x-axis, from symmetry.

Use
$$\bar{x} = \frac{\int_3^6 \pi y^2 x \, dx}{\int_3^6 \pi y^2 \, dx} = \frac{\int_3^6 (36 - x^2) x \, dx}{\int_3^6 (36 - x^2) \, dx}$$

$$= \frac{\int_3^6 36x - x^3 \, dx}{\int_3^6 36 - x^2 \, dx}$$

$$= \frac{\left[18x^2 - \frac{1}{4}x^4\right]_3^6}{\left[36x - \frac{1}{3}x^3\right]_3^6}$$

$$= \frac{\left(18 \times 6^2 - \frac{1}{4} \times 6^4\right) - \left(18 \times 3^2 - \frac{1}{4} \times 3^4\right)}{\left(36 \times 6 - \frac{1}{3} \times 6^3\right) - \left(36 \times 3 - \frac{1}{3} \times 3^3\right)}$$

$$= \frac{324 - 141.75}{144 - 99} = \frac{182.25}{45} = 4.05$$

$\therefore$ Distance from $x = 3$ is 1.05 cm.
$\therefore$ The distance of the centre of mass from the plane circular face is 1.05 cm.

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Statics of rigid bodies
Exercise B, Question 18

Question:

Use calculus to obtain your answer.
Show that the centre of mass of a cap of height h of a sphere of radius a is on its axis of symmetry at a distance $\dfrac{h(4a-h)}{4(3a-h)}$ from the circular base of the cap.

Solution:

The arc of the circle $x^2+y^2=a^2, a-h\le x\le a$ is rotated about the x-axis.

$$\bar{x}=\frac{\pi\int_{a-h}^{a}(a^2-x^2)x\,\mathrm{d}x}{\pi\int_{a-h}^{a}(a^2-x^2)\,\mathrm{d}x}=\frac{\left[\frac{1}{2}a^2x^2-\frac{1}{4}x^4\right]_{a-h}^{a}}{\left[a^2x-\frac{1}{3}x^3\right]_{a-h}^{a}}$$

$$=\frac{\frac{1}{4}a^4-\frac{1}{2}a^2(a-h)^2+\frac{1}{4}(a-h)^4}{\frac{2}{3}a^3-a^2(a-h)+\frac{1}{3}(a-h)^3}$$

$$=\frac{\frac{1}{4}(a^2-(a-h)^2)^2}{\frac{1}{3}(2a^2-3a^2(a-h)+(a-h)^3)}$$

$$=\frac{3}{4}\frac{(2ah-h^2)^2}{3h^2a-h^3}$$

$$=\frac{3}{4}\frac{(2a-h)^2}{(3a-h)}$$

$\therefore$ Distance of centre of mass from base of cap (i.e. $x=a-h$) is

$$\frac{3(2a-h)^2}{4(3a-h)}-(a-h)=\frac{3(2a-h)^2-4(3a-h)(a-h)}{4(3a-h)}$$

$$=\frac{4ah-h^2}{4(3a-h)}$$

i.e. required distance is $\dfrac{h(4a-h)}{4(3a-h)}$

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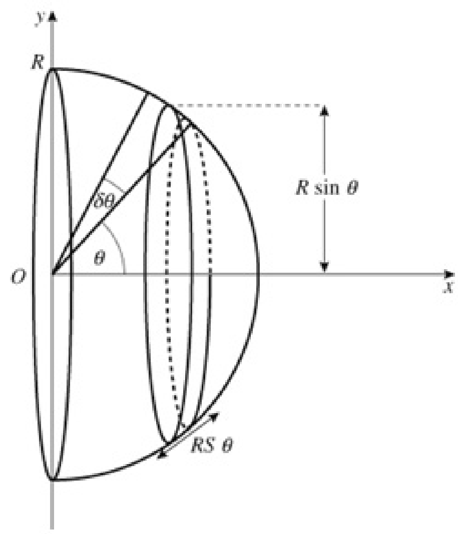
Statics of rigid bodies
Exercise B, Question 19

Question:

Use calculus to obtain your answer.
Using calculus, find the centre of mass of the uniform hemispherical shell with radius R.

Hint: Divide the shell into small elemental cylindrical rings, centred on the x-axis, with radius $R\sin\theta$, and height $R\delta\theta$, where θ is the angle between the radius R and the x-axis.



Solution:

The curved surface area of the elemental disc is $2\pi\ NPR\ \delta\theta = 2\pi R\sin\theta\times\delta\theta$

$\therefore$ its mass is $2\pi\rho R^2\sin\theta\delta\theta$, where ρ is the mass per unit area.

$\therefore$ Since $\sum m_i\times\bar{x}=\sum m_i x_i$, where $x_i = R\cos\theta$

and $m_i = 2\pi\ \rho R^2\sin\theta\ \delta\theta$ then

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} 2\pi\rho R^2\sin\theta\times R\cos\theta\ d\theta}{\int_0^{\frac{\pi}{2}} 2\pi\rho R^2\sin\theta\ d\theta}$$

$$= \frac{R\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\ d\theta}{\int_0^{\frac{\pi}{2}}\sin\theta\ d\theta}$$

$$= \frac{R\times\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2\theta\ d\theta}{\left[-\cos\theta\right]_0^{\frac{\pi}{2}}}$$

$$= \frac{1}{4}R\frac{\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}}}{1}$$

$$= \frac{1}{4}R\left[1-(-1)\right]$$

$$= \frac{R}{2}$$

This proof is **not** expected to be known for the examination.

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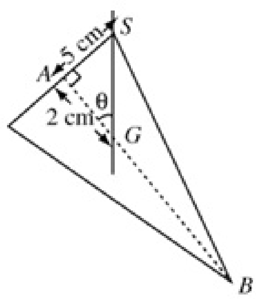
Statics of rigid bodies
Exercise C, Question 1

Question:

A uniform solid right circular cone is suspended by a string attached to a point on the rim of its base. Given that the radius of the base is 5 cm and the height of the cone is 8 cm, find the angle between the vertical and the axis of the cone when it is in equilibrium.

Solution:



The diagram shows the equilibrium position with the centre of mass G, vertically below the point of suspension S.

As $AG = \frac{1}{4}h$ for a cone

$\therefore AG = 2$ cm

Also the radius $AS = 5$ cm .

Let the angle between the vertical and the axis be θ.

Then from $\Delta ASG, \tan\theta = \frac{5}{2}$

$\therefore \theta = 68°$ (to the nearest degree)

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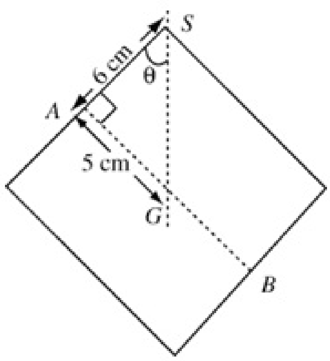
Statics of rigid bodies
Exercise C, Question 2

Question:

A uniform solid right circular cylinder is suspended by a string attached to a point on the rim of its base. Given that the radius of the base is 6 cm and the height of the cylinder is 10 cm, find the angle between the vertical and the circular base of the cylinder when it is in equilibrium.

Solution:



The diagram shows the equilibrium position with the centre of mass G below the point of suspension S.

As $AG = \frac{1}{2}h$ for a uniform cylinder

$\therefore AG = 5$ cm

Also the radius $AS = 6$ cm .

The angle between the vertical and the circular base of the cylinder is θ.

From ΔASG, $\tan\theta = \frac{5}{6}$

$\therefore \theta = 40°$ (to the nearest degree)

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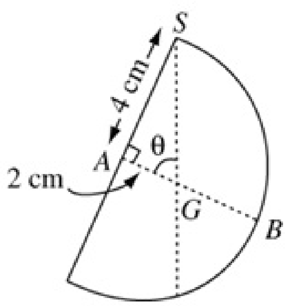
Statics of rigid bodies
Exercise C, Question 3

Question:

A uniform hemispherical shell is suspended by a string attached to a point on the rim of its base. Given that the radius of the base is 4 cm, find the angle between the vertical and the axis of the hemisphere when it is in equilibrium.

Solution:



The diagram shows the equilibrium position, with the centre of mass G below the point of suspension S.

As $AG = \frac{1}{2}r$ for an hemispherical shell

$\therefore AG = 2$ cm

Also the radius $AS = 4$ cm

Let the angle between the vertical and the axis be θ.

Then from $\Delta ASG, \tan\theta = \frac{4}{2}$

$\therefore \theta = 63°$ (nearest degree)

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Statics of rigid bodies
Exercise C, Question 4

Question:

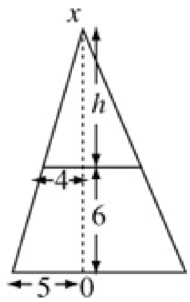
a Find the position of the centre of mass of the frustum of a right circular uniform solid cone, of end radii 4 cm and 5 cm and of height 6 cm. (Give your answer to 3 s.f.)

This frustum is now suspended by a string attached to a point on the rim of its smaller circular face.

b Find the angle between the vertical and the axis of the frustum when it is in equilibrium. (Give your answer to the nearest degree.)

Solution:

a



From similar triangles

$$\frac{h}{h+6} = \frac{4}{5}$$

$$\therefore 5h = 4h + 24$$

$$\text{i.e. } h = 24$$

Centre of mass lies at the axis of symmetry OX.

Shape	Mass	Mass ratios	Position of centre of mass i.e. distance from O
Large cone	$\frac{1}{3}\pi\rho \times 5^2 \times 30$	125	$\frac{30}{4} = 7.5$
Small cone	$\frac{1}{3}\pi\rho \times 4^2 \times 24$	64	$6 + \frac{24}{4} = 12$
Frustum	$\frac{250\pi}{3}\rho - 128\pi\rho$	61	$\bar{x}$

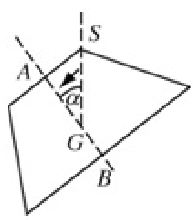
Take moments about O

$$125 \times 7.5 - 64 \times 12 = 61\bar{x}$$

$$\therefore 169.5 = 61\bar{x}$$

$$\therefore \bar{x} = 2.78 \text{ (3 s.f.) (or } \frac{339}{122})$$

b



In equilibrium the centre of mass G lies vertically below the point of suspension S.
Let the required angle be α.
AS is smaller radius = 4 cm

$$AG = 6 - 2.78 = 3.22 \text{ cm (3 s.f.)}$$

$$\tan\alpha = \frac{AS}{AG} = \frac{4}{3.22}$$

$$\therefore \alpha = 51° \text{ (to the nearest degree)}$$

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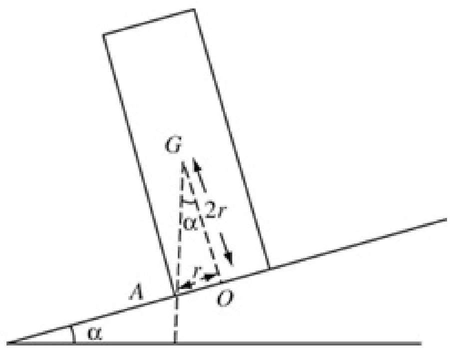
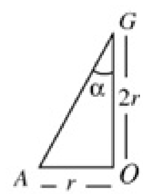
Statics of rigid bodies
Exercise C, Question 5

Question:

A uniform solid cylinder of radius r and height $4r$ rests in equilibrium with its base in contact with a rough inclined plane, which is sufficiently rough to prevent sliding. The plane is inclined at an angle α to the horizontal. Show that equilibrium is maintained provided that $\tan\alpha \leq k$ and find the value of k.

Solution:





The diagram shows the limiting case when the point vertically below the centre of mass G is on the edge of the area of contact.
In this position angle AGO is also equal to α , the angle of inclination of the plane to the horizontal.

From $\Delta AGO, \tan\alpha = \frac{r}{2r} = \frac{1}{2}$
$(\therefore \alpha = 26.6 \text{ (3 s.f.)})$

For any larger angle tilting will occur.

$\therefore$ Equilibrium is maintained provided
$\tan\alpha \leq \frac{1}{2} \quad \left(\text{i.e. } k = \frac{1}{2}\right)$

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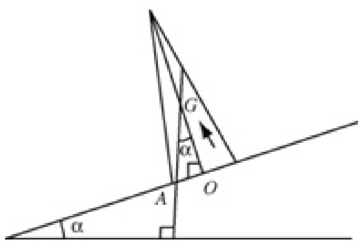
Edexcel AS and A Level Modular Mathematics

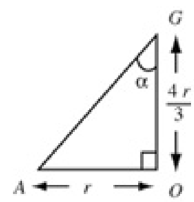
Statics of rigid bodies
Exercise C, Question 6

Question:

A uniform hollow cone of radius r and height $4r$ rests in equilibrium with its base in contact with a rough inclined plane, which is sufficiently rough to prevent sliding. The plane is inclined at an angle α to the horizontal. Show that equilibrium is maintained provided that $\tan\alpha \leq k$ and find the value of k.

Solution:





The diagram shows the limiting case where A, a point on the circumference of the circular base of the cone, is vertically below G – the centre of mass.

$$A\hat{G}O = \alpha$$
$$AO = r$$
$$OG = \frac{1}{3} \times 4r = \frac{4r}{3}$$

From $\Delta AGO, \tan\alpha = \dfrac{r}{\frac{4r}{3}} = \dfrac{3}{4}$ $\quad(\alpha \approx 37°)$

For any larger angle tilting will occur.

Equilibrium is maintained provided

$\tan\alpha \leq \dfrac{3}{4}$ $\quad$ (i.e. $k = \dfrac{3}{4}$).

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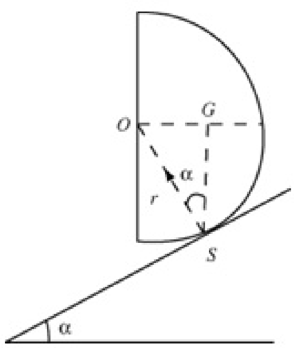
Statics of rigid bodies
Exercise C, Question 7

Question:

A uniform solid hemisphere of radius r rests in equilibrium with its curved surface in contact with a rough inclined plane, which is sufficiently rough to prevent sliding. The plane is inclined at an angle α to the horizontal, and the plane face of the hemisphere is in a vertical position. Find the value of α, giving your answer to the nearest degree.

Solution:



The diagram shows the centre of mass G above the point of contact S.

$G\hat{S}O = \alpha$

$SO = r$

and $OG = \frac{3}{8}r$.

From $\Delta GOS, \sin\alpha = \frac{GO}{OS} = \frac{\frac{3}{8}r}{r} = \frac{3}{8}$

$\therefore \alpha = 22^\circ$ (nearest degree)

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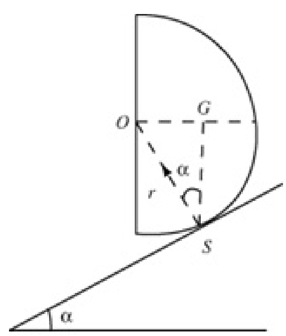
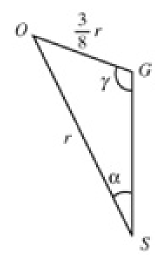
Statics of rigid bodies
Exercise C, Question 8

Question:

A solid uniform hemisphere rests in equilibrium with its curved surface in contact with a rough plane inclined at α to the horizontal where $\sin\alpha = \frac{3}{16}$.
Find the inclination of the axis of symmetry of the hemisphere to the vertical.

Solution:





You need a good diagram and use of the sine rule.

The diagram shows the equilibrium position with the centre of mass G vertically above the point of contact with the plane S. O is the centre of the plane face of the hemisphere.

Let the obtuse angle between the axis of symmetry and the vertical be γ. Let the radius of the hemisphere be r.

In ΔOGS, $O\hat{S}G = \alpha$

$$OS = r$$

$$\text{and } OG = \frac{3}{8}r$$

Using the sine rule

$$\frac{\sin\gamma}{r} = \frac{\sin\alpha}{\frac{3}{8}r} \quad \text{But } \sin\alpha = \frac{3}{16}$$

$$\therefore \sin\gamma = \frac{\frac{3}{16}r}{\frac{3}{8}r} = \frac{1}{2}$$

$\therefore \gamma = 150°$ and the acute angle between the axis of the hemisphere and the vertical is $30°$.

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Statics of rigid bodies
Exercise C, Question 9

Question:

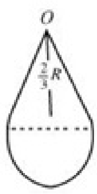
A solid object is made up of a right circular uniform solid cone joined to a uniform solid hemisphere so that the base of the cone coincides with the plane surface of the hemisphere. Their common radius is r and the height of the cone is $\frac{2}{3}r$.

a Find the position of the centre of mass of the composite object giving the distance of this centre of mass from the vertex of the cone.

b Show that the object will remain in equilibrium on a smooth horizontal plane, if it is placed with a curved surface of the cone in contact with the plane.

Solution:

a



Let the vertex of the cone be O and let the mass per unit volume be ρ.

Shape	Mass	Mass ratios	Distance of centre of mass from O
Cone	$\frac{1}{3}\pi\rho r^2 \times \frac{2}{3}r$	2	$\frac{3}{4}\times\frac{2}{3}r = \frac{1}{2}r$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	6	$\frac{3}{8}r + \frac{2}{3}r = \frac{25}{24}r$
Composite body	$\frac{8}{9}\pi\rho r^3$	8	$\bar{x}$

Take moments about O:

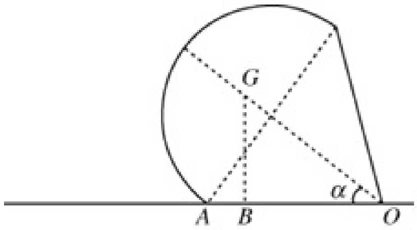
$$2\times\frac{1}{2}r + 6\times\frac{25}{24}r = 8\bar{x}$$

$$\therefore 8\bar{x} = r + \frac{25r}{4}$$

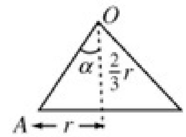
$$8\bar{x} = \frac{29r}{4}$$

$$\therefore \bar{x} = \frac{29r}{32}$$

b



α is the angle between the axis of symmetry and the slant side of the cone



For equilibrium, the point vertically below G, i.e. point B, must lie between O and A, where A is a point on the common face of the cone and hemisphere.

$$OA = \text{slant side of cone} = \sqrt{r^2 + \left(\frac{2}{3}r\right)^2} \quad \text{(from Pythagoras)}$$

$$= \frac{r}{3}\sqrt{9+4} = \frac{r\sqrt{13}}{3}$$

$$OB = OG\cos\alpha$$

$$= \frac{29r}{32} \times \frac{2}{\sqrt{13}}$$

$$= \frac{29r\sqrt{13}}{16 \times 13}$$

$$= \frac{29r\sqrt{13}}{208}$$

As $\frac{1}{3} > \frac{29}{208}$ $\therefore OA > OB$ and equilibrium is maintained.

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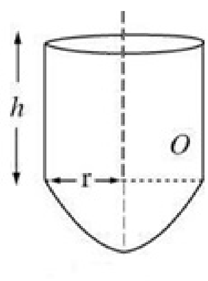
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise C, Question 10

Question:

A uniform solid consists of a hemisphere of radius r and a right circular uniform cylinder of base radius r and height h fixed together so that their circular faces coincide. The solid can rest in equilibrium with any point of the curved surface of the hemisphere in contact with a horizontal plane. Find h in terms of r.

Solution:



Shape	Mass	Distance from O of centre of mass
Cylinder	$\pi\rho r^2 h$	$\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$-\frac{3}{8}r$
Composite body	$\pi\rho r^2\left(h+\frac{2}{3}r\right)$	$\bar{x}$

As the solid can rest in equilibrium with any point of the curved surface in contact with the horizontal plane, the centre of mass must be at the centre of the plane face of the hemisphere.

i.e. $\bar{x}=0$
$\therefore$ Taking moments about O

$$\rho\pi r^2 h\cdot\frac{h}{2}-\frac{2}{3}\pi\rho r^3\cdot\frac{3}{8}r=0$$

$$\frac{r^2h^2}{2}=\frac{1}{4}r^4$$

$$\therefore r^2=2h^2$$

$$\text{i.e. } h=\frac{r}{\sqrt{2}} \text{ or } \frac{r\sqrt{2}}{2}$$

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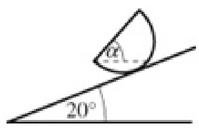
Statics of rigid bodies
Exercise C, Question 11

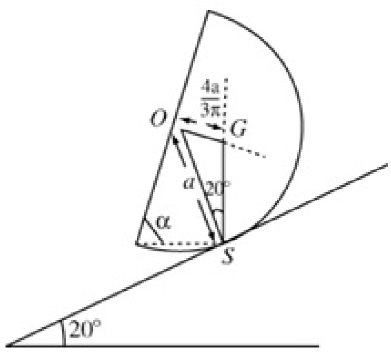
Question:

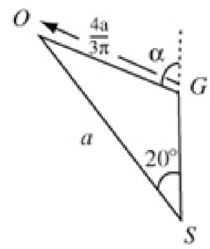
You may assume that the centre of mass of a uniform semi circular lamina of radius a is at a distance $\frac{4a}{3\pi}$ from the centre.



A uniform solid right circular cylinder is cut in half through its axis to form two prisms of semi-circular cross section. One of these is placed with its curved surface in contact with a rough inclined plane as shown in the figure. The inclined plane makes an angle of $20°$ with the horizontal. Show that when the prism is in equilibrium, its rectangular plane face makes an angle α with the horizontal, where α is approximately $54°$.

Solution:





Draw a clear diagram showing equilibrium with the centre of mass G above the point of contact S.

Identify the lengths of the sides and the angles in ΔOGS.

Use the sine rule in ΔOGS:

Then $\dfrac{\sin(180-\alpha)}{a} = \dfrac{\sin 20^\circ}{\frac{4a}{3\pi}}$

$$\frac{\sin(180-\alpha)}{a} = \frac{\sin 20^\circ}{\frac{4a}{3\pi}}$$

$$\therefore \sin\alpha = \frac{3\pi}{4}\times \sin 20$$

$$= 0.806$$

$$\therefore \alpha = 53.6^\circ \text{ (3 s.f.)}$$

$$\therefore \alpha \approx 54^\circ$$

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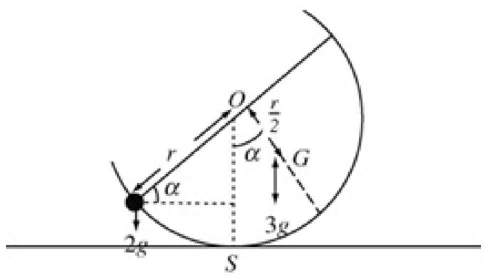
Statics of rigid bodies
Exercise C, Question 12

Question:

A hemispherical bowl, which may be modelled as a uniform hemispherical shell, has mass 3 kg. A weight of 2 kg is placed on the rim and the bowl rests in equilibrium on a smooth horizontal plane. The plane surface of the bowl makes an angle α with the horizontal. Show that $\tan\alpha = \frac{4}{3}$.

Solution:



Let the radius of the bowl be r.

The distance $OG = \frac{r}{2}$.

Take moments about point S — the point of contact with the plane:

$$3g \times \frac{r}{2}\sin\alpha = 2g \times r\cos\alpha$$

$$\therefore 3\sin\alpha = 4\cos\alpha$$

$$\therefore \tan\alpha = \frac{4}{3}, \text{ as required}$$

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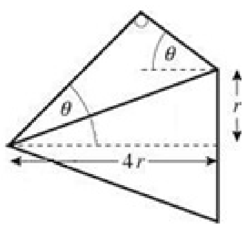
Statics of rigid bodies
Exercise C, Question 13

Question:

A uniform solid right circular cone has base radius r, height $4r$ and mass m. One end of a light inextensible string is attached to the vertex of the cone and the other end is attached to a point on the rim of the base. The string passes over a smooth peg and the cone rests in equilibrium with the axis horizontal, and with the strings equally inclined to the horizontal at an angle θ, as shown in the figure.

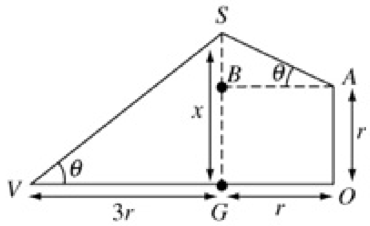


a Show that angle θ satisfies the equation $\tan\theta = \frac{1}{2}$.

b Find the tension in the string, giving your answer as an exact multiple of mg.

Solution:

a



In equilibrium the centre of mass G lies below the point of suspension S. Let distance $SG = x$.
O is the centre of the base of the cone and V is its vertex.
A and B are shown on the diagram.

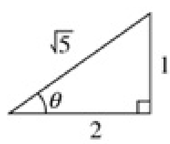
$\tan\theta = \frac{x}{3r}$ (from ΔVSG)

Also $\tan\theta = \frac{x-r}{r}$ (from ΔABS)

$$\therefore \frac{x}{3r} = \frac{x-r}{r}$$
$$\therefore x = 3x - 3r$$
$$\therefore 2x = 3r$$
$$\therefore x = \frac{3r}{2}$$
$$\therefore \tan\theta = \frac{1}{2}$$

b Resolve vertically for the forces acting an the cone:

$$2T\sin\theta = mg$$
$$\therefore T = \frac{mg}{2\sin\theta}$$



As $\tan\theta = \frac{1}{2}$, $\sin\theta = \frac{1}{\sqrt{5}}$ (from Pythagoras)

$$\therefore T = \frac{\sqrt{5}\,mg}{2}$$

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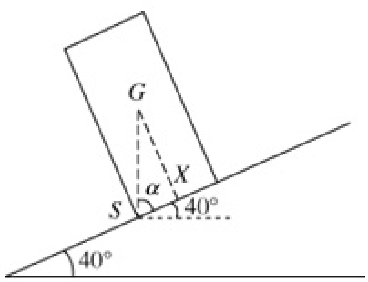
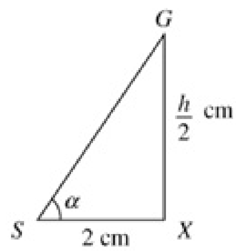
Statics of rigid bodies
Exercise D, Question 1

Question:

A uniform solid right circular cylinder with base diameter 4 cm stands on a rough plane inclined at 40° to the horizontal. What is the maximum height that such a cylinder can have without toppling over?

Solution:





Let the maximum height be h cm. The cylinder is about to topple and so its centre of mass G is directly above the point S on the circumference of the base. X is the mid-point of the base.

As $\alpha + 40^{\circ} = 90^{\circ}, \alpha = 50^{\circ}$.

In $\Delta GSX, SX = 2$ cm (radius)

$$GX = \frac{h}{2} \quad \text{(position of centre of mass)}$$

$$\therefore \tan 50^{\circ} = \frac{\frac{h}{2}}{2}$$

$$\therefore h = 4\tan 50^{\circ}$$

$$\therefore h = 4.77 \text{ cm (3 s.f.)}$$

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Statics of rigid bodies

Exercise D, Question 2

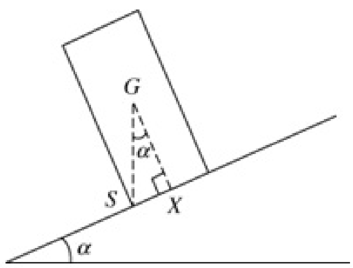
Question:

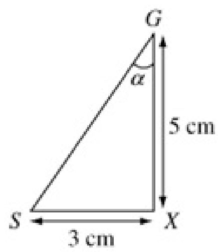
A uniform solid right circular cylinder with base radius 3 cm and height 10 cm is placed with its circular plane base on a rough plane. The plane is gradually tilted.

a Find the angle which the plane makes with the horizontal if the cylinder topples over before it slides.

b What can you deduce about the value of the coefficient of friction?

Solution:





a When the cylinder is about to topple, G is vertically above point S. X is the mid-point of the base.

Let α be the angle which the plane makes with the horizontal.

In triangle GSX, $S\hat{G}X = \alpha$

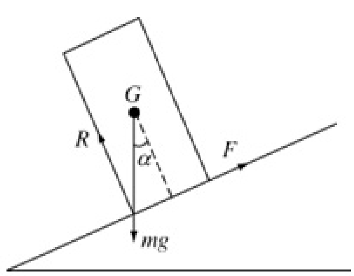
$$GX = \frac{1}{2} \times 10 \text{ cm} = 5 \text{ cm} \quad \text{(position of centre of mass)}$$

$$SX = 3 \text{ cm} \quad \text{(radius)}$$

$$\therefore \tan\alpha = \frac{3}{5}$$

i.e. $\alpha = 31°$ (to the nearest degree)

b



R(↗)

$F - mg\sin\alpha = 0$

$\therefore F = Mg\sin\alpha$

R(↖)

$R - Mg\cos\alpha = 0$

$\therefore R = Mg\cos\alpha$

As $F \le \mu R, Mg\sin\alpha \le Mg\cos\alpha \times \mu$

$\therefore \mu \ge \tan\alpha$

i.e. $\mu \ge \frac{3}{5}$

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Statics of rigid bodies
Exercise D, Question 3

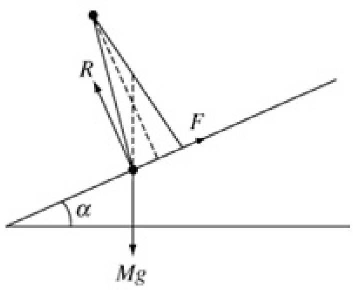
Question:

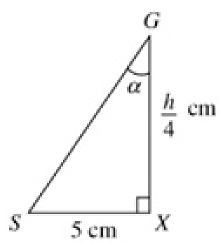
A uniform solid right circular cone with base radius 5 cm and height h cm is placed with its circular plane base on a rough plane. The coefficient of friction is $\frac{\sqrt{3}}{3}$. The plane is gradually tilted.

a Find the angle which the plane makes with the horizontal if the cone is about to slide and topple at the same time.

b Calculate the value of the height of the cone, h cm.

Solution:





a When the cone is about to slide $F = \mu R$

$$\text{i.e.} F = \frac{\sqrt{3}}{3}R \quad ①$$

R(↗)
Then $F - mg\sin\alpha = 0 \quad \therefore F = Mg\sin\alpha \quad ②$
R(↖)
Then $R - Mg\cos\alpha = 0 \quad \therefore R = Mg\cos\alpha \quad ③$

Substituting F and R into equation ①

Then $Mg\sin\alpha = \frac{\sqrt{3}}{3}Mg\cos\alpha$

$$\therefore \tan\alpha = \frac{\sqrt{3}}{3}$$
$$\therefore \alpha = 30°$$

b From ΔGSX, where G is the centre of mass of the cone, X the centre of its base and S a point an the circumference of the base about which topping is about to occur:

$$\tan\alpha = \frac{5}{\frac{h}{4}} = \frac{20}{h}$$

$$\therefore h = \frac{20}{\tan\alpha} = 20 \div \frac{\sqrt{3}}{3} = 20\sqrt{3} = 35 \text{ cm (2 s.f.)}$$

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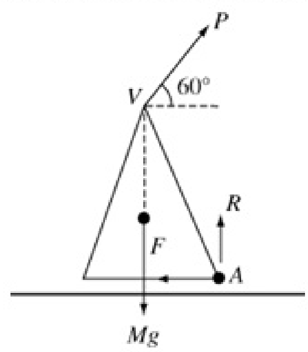
Statics of rigid bodies
Exercise D, Question 4

Question:

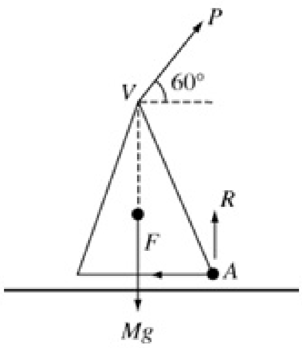
A uniform solid right circular cone of mass M with base radius r and height $2r$ is placed with its circular plane base on a rough horizontal plane. A force P is applied to the vertex V of the cone at an angle of $60°$ above the horizontal as shown in the figure. The cone begins to topple and to slide at the same time.

a Find the magnitude of the force P in terms of M.

b Calculate the value of the coefficient of friction.



Solution:



Let the point about which toppling occurs be A.

Take moments about point A.

When toppling is about to occur, R and F act through point A.

So $P\cos 60\times 2r + P\sin 60\times r = Mg\times r$

$$\therefore Pr + \frac{P\sqrt{3}}{2}r = Mgr$$

$$\therefore P\left(1+\frac{\sqrt{3}}{2}\right) = Mg$$

So $P = \dfrac{2Mg}{2+\sqrt{3}}$

R$(\rightarrow)$

$P\cos 60° - F = 0$

$$\therefore F = \frac{Mg}{2+\sqrt{3}}$$

R$(\uparrow)$

$P\sin 60° + R - Mg = 0$

$\therefore R = Mg - \dfrac{Mg\sqrt{3}}{2+\sqrt{3}} = \dfrac{2Mg}{2+\sqrt{3}}$

As the cone is on the point of slipping, $F = \mu R$

$\therefore \mu = F \div R = \dfrac{1}{2}$

i.e. μ, the coefficient of friction, $= \dfrac{1}{2}$

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Statics of rigid bodies
Exercise D, Question 5

Question:

A frustum of a right circular solid cone has two plane circular end faces with radii r and $2r$ respectively. The distance between the end faces is $2r$.

a Show that the centre of mass of the frustum is at a distance $\frac{11r}{14}$ from the larger circular face.

b Find whether this solid can rest without toppling on a rough plane, inclined to the horizontal at an angle of $40°$, if the face in contact with the inclined plane is

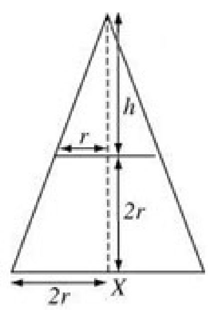
i the large circular end,

ii the small circular end.

c In order to answer part **b** you assumed that slipping did not occur. What does this imply about the coefficient of friction μ?

Solution:

a



Let the height of the small cone shown be h.
Using similar triangles

$$\frac{h}{h+2r} = \frac{r}{2r}$$

$$\therefore 2h = h + 2r$$

$$\therefore h = 2r$$

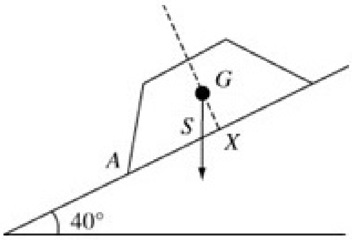
Shape	Mass	Ratio of masses	Distance of centre of mass from X
Large cone	$\rho\frac{1}{3}\pi(2r)^2(4r)$	8	r
Small cone	$\rho\frac{1}{3}\pi r^2 \times 2r$	1	$2r + \frac{2r}{4} = \frac{5r}{2}$
Frustum	$\rho\frac{1}{3}\pi \times 14r^3$	7	$\bar{x}$

Take moments about X:

$$8r - \frac{5r}{2} = 7\bar{x}$$

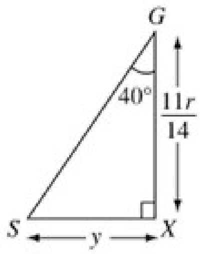
$$\therefore \bar{x} = \frac{11r}{14}$$

b i



Let G be the position of the centre of mass.
Let S be the point an the plane vertically below G.
Let X be the centre of the circular face with radius $2r$ and A be the point about which tilting would occur.

If $SX < AX$ then the solid rests in equilibrium without toppling



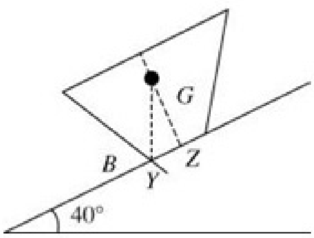
Let $SX = y$.

Then $\tan 40^\circ = \dfrac{y}{\frac{11r}{14}}$

$\therefore y = \dfrac{11r}{14}\tan 40^\circ = 0.66r \quad (2\text{ s.f.})$

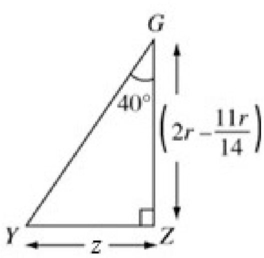
As $SX = 0.66r$ and $AX = 2r$
$SX < AX$ and the solid rests without toppling.

ii



This time Y is vertically below G. Z is the centre of the circular face and B is the point about which toppling would occur.

If $YZ > BZ$ then toppling occurs.



Let $YZ = z$

Then $\tan 40^\circ = \dfrac{z}{\frac{17r}{14}}$

$\therefore z = \dfrac{17r}{14}\tan 40^\circ = 1.02r$

As $YZ = 1.02r$ and $BZ = r$
$YZ > BZ$ and toppling would occur.

c As the angle of slope is 40° limiting friction would imply $\mu = \tan 40^\circ$.
No slipping implies $\mu \geq 0.839$ (3 s.f.)

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Statics of rigid bodies
Exercise D, Question 6

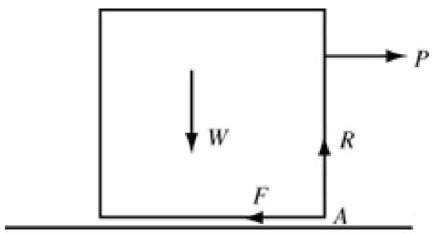
Question:

A uniform cube with edges of length $6a$ and weight W stands on a rough horizontal plane. The coefficient of friction is μ. A gradually increasing force P is applied at right angles to a vertical face of the cube at a point which is a distance a above the centre of that face.

a Show that equilibrium will be broken by sliding or toppling depending on whether $\mu < \frac{3}{4}$ or $\mu > \frac{3}{4}$.

b If $\mu = \frac{1}{4}$, and the cube is about to slip, find the distance from the point where the normal reaction acts, to the nearest vertical face of the cube.

Solution:



a Consider the cube in equilibrium, on the point of toppling, so R acts through the corner A.

$$R(\rightarrow): P - F = 0 \therefore F = P$$

$$R(\uparrow): R - W = 0 \therefore R = W$$

$$\circlearrowleft M(A): P \times 4a = W \times 3a$$

$$\therefore P = \frac{3}{4}W$$

If equilibrium is broken by toppling $P = \frac{3}{4}W$, so $F = \frac{3}{4}W$

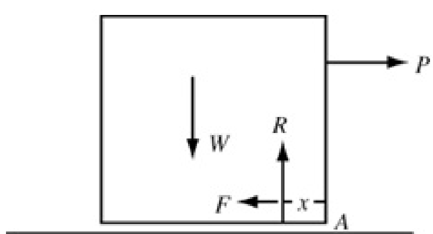
But $F < \mu R$

$\therefore \frac{3}{4}W < \mu W$ so $\mu > \frac{3}{4}$ is the condition for toppling.

If however, $\mu < \frac{3}{4}$ then the cube will be on the point of slipping when $F = \mu R$

i.e. when $P = \mu W$ the cube will start to slip.

b



Let R act at a point x from A.

$$R(\rightarrow) P - F = 0 \therefore P = F$$

$$R(\uparrow) R - W = 0 \therefore R = W$$

When the cube is about to slip: $F = \mu R$

$$\therefore P = \frac{1}{4}W$$

$$\circlearrowleft M(A): P \times 4a + Rx = W \times 3a \text{ (substitute for } P\text{)}$$

$$\therefore \frac{1}{4}W \times 4a + Rx = W \times 3a \text{ (substitute for } R\text{)}$$

$$\therefore Wx = W \times 2a$$

$$\text{i.e. } x = 2a$$

The required distance is $2a$.

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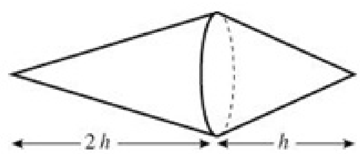
Statics of rigid bodies
Exercise D, Question 7

Question:

A spindle is formed by joining two solid right circular cones so that their circular bases coincide. The cones have the same base radius and have the same uniform density. The heights of the two cones are h and $2h$ as shown in the figure.



a Find the distance of the centre of mass of the spindle from the vertex of the larger cone.

The spindle is placed on horizontal ground with the sloping surface of the smaller cone in contact with the ground. It rests in equilibrium but is on the point of toppling.

b Show that the radius of the common base of the two cones is $\frac{1}{2}h$.

Solution:

a Let the point V be at the vertex of the large cone. The centre of mass lies an the axis of symmetry. Let the radius of the bases of the cones be r.

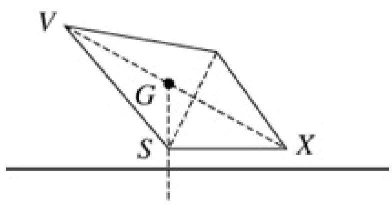
Shape	Mass	Mass ratios	Distance of centre of mass from V
Large cone	$\frac{1}{3}\pi\rho r^2 2h$	2	$\frac{3}{4}\times 2h$
Small cone	$\frac{1}{3}\pi\rho r^2 h$	1	$2h+\frac{1}{4}h$
Spindle	$\frac{1}{3}\pi\rho r^2 3h$	3	$\bar{x}$

$$\circlearrowleft \text{M}(V):\ 2\times\frac{3}{4}\times 2h + 1\times\left(2h+\frac{1}{4}h\right) = 3\bar{x}$$

$$\therefore 3h + \frac{9h}{4} = 3\bar{x}$$

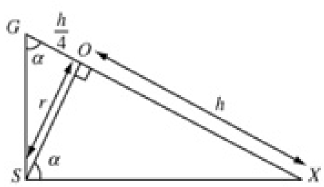
$$\text{i.e. } \bar{x} = \frac{7}{4}h$$

b



When the spindle is on the point of toppling the centre of mass G is vertically above point S, on the rim of the common face.

Let the vertex of the small cone be X and the centre of the bases of the cones be O.



In the figure let $X\hat{S}O = \alpha$,

then $S\hat{G}O = \alpha$ also

$$GO = 2h - \left(\frac{7h}{4}\right)$$

$$= \frac{h}{4}$$

$OS = r$ and $OX = h$

From ΔGOS, $\tan\alpha = \dfrac{r}{\frac{1}{4}h}$

and from ΔSOX, $\tan\alpha = \dfrac{h}{r}$

$$\therefore \frac{h}{r} = \frac{4r}{h}$$

$$\text{i.e. } h^2 = 4r^2$$

$$\text{or } r = \frac{1}{2}h$$

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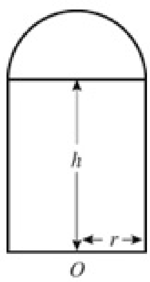
Statics of rigid bodies
Exercise D, Question 8

Question:

A uniform solid cylinder of base radius r and height h has the same density as a uniform solid hemisphere of radius r. The plane face of the hemisphere is joined to a plane face of the cylinder to form the composite solid S shown. The point O is the centre of the plane base of S.

a Show that the distance from O to the centre of mass of S is $\dfrac{6h^2+8hr+3r^2}{4(3h+2r)}$



The solid is placed on a rough plane which is inclined at an angle α to the horizontal. The plane base of S is in contact with the inclined plane.

b Given that $h=3r$ and that S is on the point of toppling, find α to the nearest degree.

c Given that the solid did not slip before it toppled, find the range of possible values for the coefficient of friction. **[E] [adapted]**

Solution:

a

Shape	Mass	Mass ratios	Distance of centre of mass from O
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$h+\frac{3}{8}r$
Cylinder	$\pi\rho r^2 h$	$3h$	$\frac{h}{2}$
Composite solid	$\pi\rho r^2\left(\frac{2}{3}r+h\right)$	$2r+3h$	$\bar{x}$

$$\circlearrowleft \text{M}: 2r\left(h+\frac{3}{8}r\right)+3h\times\frac{h}{2} = (2r+3h)\bar{x}$$

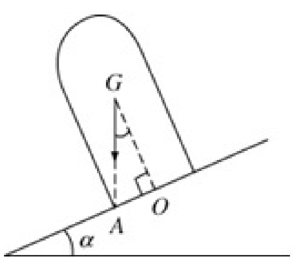
$$\therefore 2rh+\frac{3}{4}r^2+\frac{3}{2}h^2 = (2r+3h)\bar{x}$$

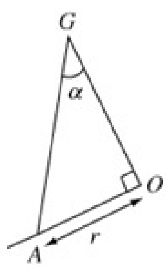
Multiply both sides by 4

$$8rh+3r^2+6h^2 = 4(2r+3h)\bar{x}$$

$$\therefore \bar{x} = \frac{6h^2+8hr+3r^2}{4(3h+2r)}$$

b





When the solid is on the point of toppling the centre of mass G is vertically above point A as shown.

In ΔGOA,

$\angle AGO = \alpha$

$OA = r$

and $OG = \frac{6(3r)^2+8(3r^2)+3r^2}{4(9r+2r)}$

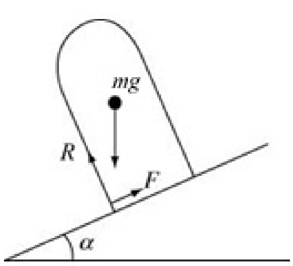
(i.e. $\bar{x}$ with $h = 3r$)

$$\therefore OG = \frac{81r^2}{44r} = \frac{81r}{44}$$

$$\therefore \tan\alpha = \frac{r}{\frac{81}{44}r} = \frac{44}{81}$$

$\therefore \alpha = 29^\circ$ (nearest degree)

c



$\text{R}(\nearrow) F - mg\sin\alpha = 0 \therefore F = mg\sin\alpha$

$\text{R}(\nwarrow) R - mg\cos\alpha = 0 \therefore R = mg\cos\alpha$

The solid does not slip

$\therefore F \le \mu R$

i.e., $mg\sin\alpha \le \mu mg\cos\alpha$

$\therefore \mu \ge \tan\alpha$

i.e: $\mu > \frac{44}{81}$ if the solid did not slip before it toppled.

[If $\mu = \frac{44}{81}$ it slips and topples at the same time.]

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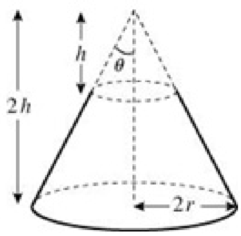
Statics of rigid bodies
Exercise D, Question 9

Question:

A uniform solid paperweight is in the shape of a frustum of a cone. It is formed by removing a right circular cone of height h from a right circular cone of height $2h$ and base radius $2r$.



a Show that the centre of mass of the paperweight lies at a height of $\frac{11}{28}h$ from its base.

When placed with its curved surface on a horizontal plane, the paperweight is on the point of toppling.

b Find θ, the semi-vertical angle of the cone, to the nearest degree.

Solution:

a Let the mass per unit volume be ρ.

Shape	Mass	Mass ratio	Position of centre of mass – distance from O
Large cone	$\frac{1}{3}\pi\rho(2r^2)2h$	8	$\frac{2h}{4}$
Small cone	$\frac{1}{3}\pi\rho r^2 h$	1	$h+\frac{h}{4}$
Frustum	$\frac{1}{3}\pi\rho(8r^2 h - r^2 h)$	7	$\bar{x}$

The centre of the base is the point O.
The radius of the small cone is obtained by sinular triangles.

$$\circlearrowleft MO: 8\times\frac{2h}{4} - 1\times\frac{5h}{4} = 7\bar{x}$$

$$\therefore \frac{11h}{4} = 7\bar{x}$$

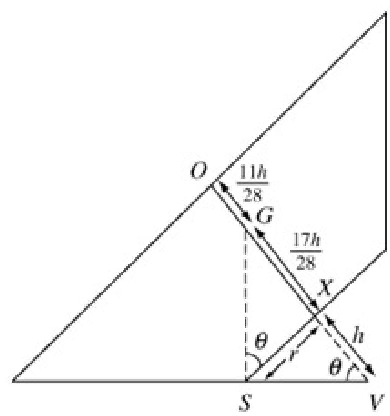
$$\text{i.e. } \bar{x} = \frac{11}{28}h$$

b As $OG = \frac{11h}{28}, GX = h - \frac{11h}{28}$

$$= \frac{17h}{28}$$

From ΔS GXS and VXS shown:

$$\tan\theta = \frac{\frac{17h}{28}}{r} \text{ and } \tan\theta = \frac{r}{h}$$



Eliminating r, $h\tan\theta = \dfrac{\frac{17h}{28}}{\tan\theta}$

$$h\tan\theta = \frac{\frac{17h}{28}}{\tan\theta}$$

$$\therefore \tan^2\theta = \frac{17}{28}$$

$$\therefore \theta = 38^\circ \text{ (nearest degree)}$$

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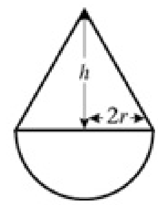
Statics of rigid bodies
Exercise D, Question 10

Question:

A child's toy is made from joining a right circular uniform solid cone, radius r and height h, to a uniform solid hemisphere of the same material and radius r. They are joined so that their plane faces coincide as shown in the figure.



a Show that the distance of the centre of mass of the toy from the base of the cone is $\left|\frac{h^2 - 3r^2}{4(2r + h)}\right|$.

The toy is placed with its hemisphere in contact with a horizontal plane and with its axis vertical. It is slightly displaced and released from rest.

b Given that the plane is sufficiently rough to prevent slipping, explain clearly, with reasons, what will happen in each of the following cases:

i $h > r\sqrt{3}$

ii $h < r\sqrt{3}$

iii $h = r\sqrt{3}$. **[E]**

Solution:

a Let the mass per unit volume of the solids be ρ. Let O be the centre of the plane circular faces which coincide.

Shape	Mass	Ratio of masses	Distance of centres of mass from O
Cone	$\frac{1}{3}\pi\rho r^2 h$	h	$\frac{h}{4}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$\frac{-3r}{8}$
Toy	$\frac{1}{3}\pi\rho(r^2h+2r^3)$	$h+2r$	$\bar{x}$

$$\circlearrowleft O\ (h+2r)\bar{x} = h\times\frac{h}{4}+2r\left(\frac{-3r}{8}\right)$$

$$= \frac{h^2}{4}-\frac{3r^2}{4}$$

$$\therefore \bar{x} = \frac{(h^2-3r^2)}{4(h+2r)}$$

b **i** If $h > r\sqrt{3}$ then $\bar{x} > 0$ so the centre of mass is in the cone – the cone will fall over.

ii If $h < r\sqrt{3}$ then $\bar{x} < 0$ so the centre of mass is in the hemisphere, the toy will return to vertical position

iii If $h = r\sqrt{3}$, then $\bar{x} = 0$ so the centre of mass is on the join at point O. The toy will stay still in equilibrium.

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Statics of rigid bodies
Exercise E, Question 1

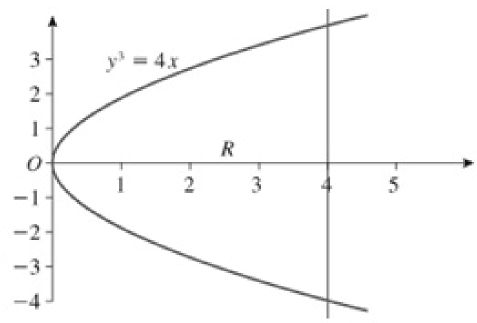
Question:

The curve shows a sketch of the region R bounded by the curve with equation $y^2 = 4x$ and the line with equation $x = 4$. The unit of length on both the axes is the centimetre. The region R is rotated through π radians about the x-axis to form a solid S.

a Show that the volume of the solid S is $32\pi \text{ cm}^3$.

b Given that the solid is uniform, find the distance of the centre of mass of S from O.

[E]



Solution:

a $V = \int \pi y^2 \, dx = \pi \int_0^4 4x \, dx$

$= \pi \left[2x^2 \right]_0^4$

$= 32\pi$

b $M \bar{x} = \rho \int x\pi y^2 \, dx = \rho\pi \int_0^4 4x^2 \, dx$

$= \rho\pi \left[\frac{4}{3} x^3 \right]_0^4$

$= \frac{256}{3} \rho\pi$

$\therefore 32\pi\rho\bar{x} = \frac{256}{3} \pi\rho$

$\therefore \bar{x} = \frac{8}{3}$

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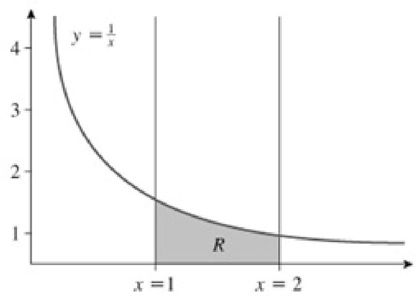
Statics of rigid bodies
Exercise E, Question 2

Question:



The region R is bounded by the curve with equation $y = \frac{1}{x}$, the lines $x = 1, x = 2$ and the x-axis, as shown in the figure. The unit of length on both the axes is 1 m. A solid plinth is made rotating R through 2π radians about the x-axis.

a Show that the volume of the plinth is $\frac{\pi}{2}\text{m}^3$.

b Find the distance of the centre of mass of the plinth from its larger plane face, giving your answer in cm to the nearest cm.

Solution:

a $V = \int \pi y^2 \, dx = \pi \int_1^2 \frac{1}{x^2} \, dx$

$= \pi \left[\frac{-1}{x} \right]_1^2$

$= \pi \left[\frac{-1}{2} + 1 \right]$

Volume $= \frac{\pi}{2}\text{m}^3$

b $M\bar{x} = \rho \int x\pi y^2 \, dx = \rho\pi \int_1^2 x \times \frac{1}{x^2} \, dx$

$= \rho\pi \int_1^2 \frac{1}{x} \, dx$

$= \rho\pi [\ln x]_1^2$

$= \rho\pi \ln 2$

$\therefore \frac{\pi}{2} \rho \bar{x} = \rho\pi \ln 2$

$\therefore \bar{x} = 2\ln 2$

So the distance of the centre of mass from the plane face $x = 1$ is $2\ln 2 - 1 = 0.386\text{ m}$ (3 s.f.)

i.e. 39 cm to the nearest cm.

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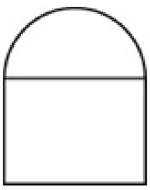
Statics of rigid bodies
Exercise E, Question 3

Question:

The figure shows a uniform solid standing on horizontal ground. The solid consists of a uniform solid right circular cylinder, of diameter 80 cm and height 40 cm, joined to a uniform solid hemisphere of the same density. The circular base of the hemisphere coincides with the upper circular end of the cylinder and has the same diameter as that of the cylinder. Find the distance of the centre of mass of the solid from the ground.



Solution:

Let the density of the solids be ρ . Let O be the centre of the circular base of the solid.

Shape	Mass	Ratio of masses	Distance of centre of mass from O
Cylinder	$\pi \times 40^2 \times 40\rho$	1	20 cm
Hemisphere	$\frac{2}{3}\pi\rho \times 40^3$	$\frac{2}{3}$	$\left(40 + \frac{3}{8} \times 40\right)$ cm
Solid	$\pi\rho \times 40^3\left(1 + \frac{2}{3}\right)$	$\frac{5}{3}$	$\bar{x}$

$$\circlearrowleft \mathrm{M}(O) \quad \frac{5}{3}\bar{x} = 1 \times 20 + \frac{2}{3} \times \left(40 + \frac{3}{8} \times 40\right)$$
$$= 20 + \frac{110}{3}$$
$$\therefore \bar{x} = \frac{170}{5}$$
$$= 34$$

$\therefore$ The centre of mass of the solid is at a height of 34 cm above the ground.

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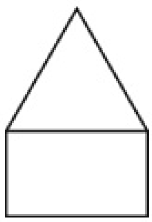
Statics of rigid bodies
Exercise E, Question 4

Question:

A simple wooden model of a rocket is made by taking a uniform cylinder, of radius r and height $3r$, and carving away part of the top two thirds to form a uniform cone of height $2r$ as shown in the figure. Find the distance of the centre of mass of the model from its plane face.



Solution:

Let the mass per unit volume be ρ .

Shape	Mass	Mass ratios	Distance of centre of mass from plane face
Cylinder	$\pi\rho r^2 \times r$	1	$\frac{r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times 2r$	$\frac{2}{3}$	$r+\frac{2r}{4}$
Model	$\pi\rho r^2 \times 1\frac{2}{3}r$	$1\frac{2}{3}$	$\bar{x}$

Note that the cylindrical base of this rocket has height r.

$$\circlearrowleft \text{M (plane face)}: 1\frac{2}{3}\bar{x} = 1\times\frac{r}{2}+\frac{2}{3}\times\left(r+\frac{2r}{4}\right)$$

$$\text{i.e.} \frac{5}{3}\bar{x} = \frac{r}{2}+\frac{2r}{3}+\frac{1}{3}r$$

$$\text{i.e.} \frac{5}{3}\bar{x} = \frac{3r}{2}$$

$$\therefore \bar{x} = \frac{9r}{10}$$

$\therefore$ The centre of mass is at a distance $\frac{9r}{10}$ from the plane face.

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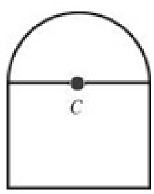
Statics of rigid bodies
Exercise E, Question 5

Question:

The figure shows a cross section containing the axis of symmetry of a uniform body consisting of a solid right circular cylinder of base radius r and height kr surmounted by a solid hemisphere of radius r. Given that the centre of mass of the body is at the centre C of the common face of the cylinder and the hemisphere, find the value of k, giving your answer to 2 significant figures.
Explain briefly why the body remains at rest when it is placed with any part of its hemispherical surface in contact with a horizontal plane.

[E]



Solution:

Let the density of the solid be ρ.

Shape	Mass	Mass ratio	Distance of C of mass from C
Cylinder	$\pi\rho r^2 \times kr$	k	$-\frac{kr}{2}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$\frac{2}{3}$	$\frac{3}{8}r$
Composite body	$\pi\rho r^3\left(k+\frac{2}{3}\right)$	$k+\frac{2}{3}$	0

$$\circlearrowleft \text{M(about } C\text{)}: k\times\left(-\frac{kr}{2}\right)+\frac{2}{3}\times\frac{3}{8}r=0$$

$$\therefore \frac{k^2 r}{2} = \frac{r}{4}$$

$$\therefore k^2 = \frac{1}{2} \Rightarrow k=\frac{1}{\sqrt{2}}=0.71\ (2\text{ s.f.})$$

The centre of mass of the body is at C which is always directly above the contact point.

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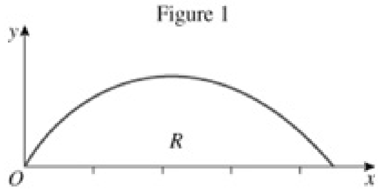
Statics of rigid bodies
Exercise E, Question 6

Question:

A uniform lamina occupies the region R bounded by the x-axis and the curve with equation $y=\frac{1}{4}x(4-x)0\leq x\leq 4$, as shown in Figure 1.

a Show by integration that the y-coordinate of the centre of mass of the lamina is $\frac{2}{5}$.

Figure 1



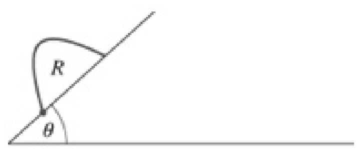


Figure 2

A uniform prism P has cross section R. The prism is placed with its rectangular face on a slope inclined at an angle θ to the horizontal. The cross section R lies in a vertical plane as shown in Figure 2. The surfaces are sufficiently rough to prevent P from sliding.

b Find the angle θ, for which P is about to topple.

Solution:

a $$\bar{y} = \frac{\rho\int \frac{1}{2}y^2\ \mathrm{d}x}{\rho\int y\ \mathrm{d}x} = \frac{\frac{1}{2}\int_0^4 \frac{x^2}{16}(16-8x+x^2)\ \mathrm{d}x}{\frac{1}{4}\int_0^4 4x - x^2\ \mathrm{d}x}$$

$$= \frac{\frac{1}{2}\int_0^4 x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4\ \mathrm{d}x}{\frac{1}{4}\left[2x^2 - \frac{1}{3}x^3\right]_0^4}$$

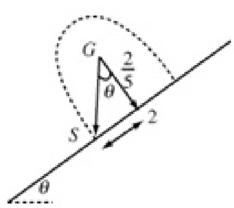
$$= 2\frac{\left[\frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{80}x^5\right]_0^4}{32 - \frac{64}{3}}$$

$$= \frac{6}{32}\left[\frac{64}{3} - 32 + \frac{64}{5}\right]$$

$$= \frac{6}{32} \times \frac{32}{15}$$

$$= \frac{6}{15} = \frac{2}{5}$$

b From symmetry the x-coordinate of the centre of mass is 2.
When P is about to topple the centre of mass G is directly above the lower edge of the prism S.



$$\therefore \tan\theta = \frac{2}{\frac{2}{5}} = 5$$

$$\therefore \theta = 79^\circ \quad \text{(nearest degree)}$$

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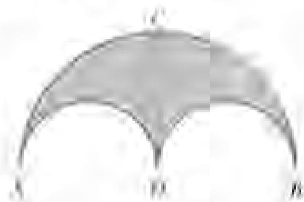
Statics of rigid bodies
Exercise E, Question 7

Question:

A uniform semi-circular lamina has radius $2a$ and the mid-point of the bounding diameter AB is O.

a Using integration, show that the centre of mass of the lamina is at a distance $\frac{8a}{3\pi}$ from O.



The two semi-circular laminas, each of radius a and with AO and OB as diameters, are cut away from the original lamina to leave the lamina $AOBC$ shown in the diagram, where OC is perpendicular to AB.

b Show that the centre of mass of the lamina $AOBC$ is at a distance $\frac{4a}{\pi}$ from O.

The lamina $AOBC$ is of mass M and a particle of mass M is attached to the lamina at B to form a composite body.

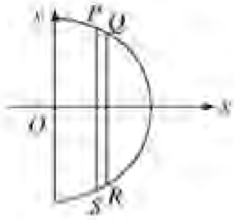
c State the distance of the centre of mass of the body from OC and from OB.

The body is smoothly hinged at A to a fixed point and rests in equilibrium in a vertical plane.

d Calculate, to the nearest degree, the acute angle between AB and the horizontal.

Solution:

a



Take the diameter as the y-axis and the mid-point of the diameter as the origin.

Then $M\bar{x} = \rho\int 2yx \, dx$ where

$M = \frac{1}{2}\rho\pi(2a)^2$ and where $x^2 + y^2 = (2a)^2$

$$\therefore 2\rho\pi a^2\bar{x} = \rho\int_0^{2a} 2x\sqrt{4a^2 - x^2}\, dx$$

$$= \frac{-2\rho}{3}\left[(4a^2 - x^2)^{\frac{3}{2}}\right]_0^{2a}$$

$$\therefore 2\rho\pi a^2\bar{x} = \frac{2\rho}{3}\times 8a^3$$

$$\therefore \bar{x} = \frac{16}{3}a^3 \div 2\pi a^2$$

$$= \frac{8a}{3\pi}$$

b

Shape	Mass	Mass ratios	Centre of mass (distance from AB)
Large semi-circle	$2\pi\rho a^2$	4	$\frac{8a}{3\pi}$
Semi-circle diameter AD	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Semi-circle diameter OB	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Remainder	$\pi\rho a^2$	2	$\bar{x}$

$$\circlearrowleft MO: 4\times\frac{8a}{3\pi} - 1\times\frac{4a}{3\pi} - 1\times\frac{4a}{3\pi} = 2\bar{x}$$

$$\therefore \frac{24a}{3\pi} = 2\bar{x}$$

$$\therefore \bar{x} = \frac{4a}{\pi}$$

c The distance from OC is a

The distance from OB is $\frac{2a}{\pi}$

d

Let N be the foot of the perpendicular from G onto AB.
In the diagram θ is the angle between AB and the vertical.
From ΔANG

$$\tan\theta = \frac{NG}{AN} = \frac{\frac{2a}{\pi}}{2a + a}$$

$$= \frac{2}{3\pi}$$

$\therefore \theta = 12°$ (to the nearest degree)

$\therefore$ The angle between AB and the horizontal is $90 - 12 = 78°$ (to the nearest degree)

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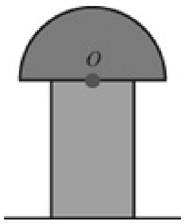
Statics of rigid bodies
Exercise E, Question 8

Question:



A uniform wooden 'mushroom', used in a game, is made by joining a solid cylinder to a solid hemisphere. They are joined symmetrically, such that the centre O of the plane face of the hemisphere coincides with the centre of one of the ends of the cylinder. The diagram shows the cross section through a plane of symmetry of the mushroom, as it stands on a horizontal table.
The radius of the cylinder is r, the radius of the hemisphere is $3r$, and the centre of mass of the mushroom is at the point O.

a Show that the height of the cylinder is $r\sqrt{\frac{81}{2}}$.

The table top, which is rough enough to prevent the mushroom from sliding, is slowly tilted until the mushroom is about to topple.

b Find, to the nearest degree, the angle with the horizontal through which the table top has been tilted.

Solution:

a

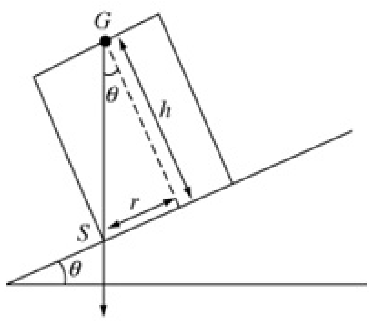
Shape	Mass	Mass ratio	Distance of centre of mass from O
Cylinder	$\pi\rho r^2 h$	h	$-\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi\rho(3r)^3$	$18r$	$\frac{3}{8}(3r)$
Mushroom	$\pi\rho r^2(h+18r)$	$h+18r$	0

$$\circlearrowleft \text{M}(O)\ -h\times\frac{h}{2}+18r\times\frac{3}{8}\times 3r=0$$

$$\therefore \frac{h^2}{2}=\frac{81r^2}{4}$$

$$\therefore h=r\sqrt{\frac{81}{2}}$$

b



When the mushroom is about to topple GS is vertical.

From the diagram $\tan\theta=\frac{r}{h}$

$$=\sqrt{\frac{2}{81}}$$

$\therefore \theta=9^\circ$ (nearest degree)

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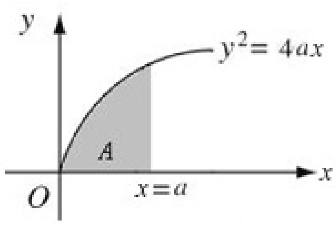
Statics of rigid bodies
Exercise E, Question 9

Question:

Figure 1 shows a finite region A which is bounded by the curve with equation $y^2 = 4ax$, the line $x = a$ and the x-axis.
A uniform solid S_1 is formed by rotating A through 2π radians about the x-axis.



a Show that the volume of S_1 is $2\pi a^3$.

b Show that the centre of mass of S_1 is a distance $\frac{2a}{3}$ from the origin O.

Figure 2

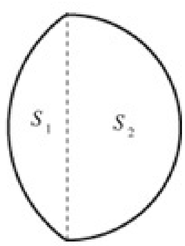


Figure 2 shows a cross section of a uniform solid S which has been obtained by attaching the plane base of solid S_1 to the plane base of a uniform hemisphere S_2 of the base radius $2a$.

c Given that the densities of solids S_1 and S_2 are ρ_1 and ρ_2 respectively, find the ratio $\rho_1 : \rho_2$ which ensures that the centre of mass of S lies in the common plane face of S_1 and S_2.

d Given that $\rho_1 : \rho_2 = 6$, explain why the solid S may rest in equilibrium with any point of the curved surface of the hemisphere in contact with a horizontal plane.

Solution:

a $V = \pi \int y^2 \, dx$

$= \pi \int_0^a 4ax \, dx$

$= \pi \left[2ax^2 \right]_0^a$

$= 2\pi a^3$

b $\bar{x} = \dfrac{\pi \int xy^2 \, dx}{\pi \int y^2 \, dx}$

$= \dfrac{\pi \int_0^a 4ax^2 \, dx}{2\pi a^3}$

$= \pi \dfrac{\left[\frac{4ax^3}{3} \right]_0^a}{2\pi a^3}$

$= \dfrac{\frac{4}{3}\pi a^4}{2\pi a^3}$

$= \dfrac{2}{3} a$

c

Shape	Mass	Mass ratios	Distance of centre of mass from X
S_1	$2\pi\rho a^3$	ρ_1	$-\dfrac{a}{3}$
S_2	$\dfrac{2}{3}\pi\rho_2(2a)^3$	$\dfrac{8}{3}\rho_2$	$\dfrac{3}{8}(2a)$
Combined solid	$2\pi a^3(\rho_1 + \dfrac{8}{3}\rho_2)$	$\rho_1 + \dfrac{8}{3}\rho_2$	0

The centre of mass of S_1 is at a distance $\left(a - \dfrac{2a}{3} \right)$ from its plane face.

X is the centre of the common plane base.

↺ M(X):

$-\rho_1 \times \dfrac{a}{3} + \dfrac{8}{3}\rho_2 \times \dfrac{6a}{8} = 0$

$\therefore \dfrac{1}{3}\rho_1 = 2\rho_2$

$\therefore \rho_1 = 6\rho_2$

$\rho_1 : \rho_2 = 6:1$

d Given that $\rho_1 : \rho_2 = 6:1$, then as centre of mass is at centre of hemisphere this will always be above the point of contact with the plane when a point of the curved surface area of the hemisphere is in contact with a horizontal plane.
(Tangent – radius property)

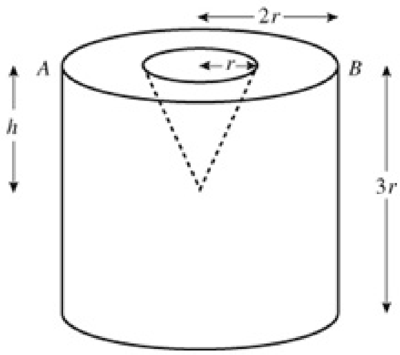
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise E, Question 10

Question:

A mould for a right circular cone, base radius r and height h, is produced by making a conical hole in a uniform cylindrical block, base radius $2r$ and height $3r$. The axis of symmetry of the conical hole coincides with that of the cylinder, and AB is a diameter of the top of the cylinder, as shown in the figure.

a Show that the distance from AB of the centre of mass of the mould is $\dfrac{216r^2 - h^2}{4(36r - h)}$



The mould is suspended from the point A, and hangs freely in equilibrium.

b In the case $h = 2r$, calculate, to the nearest degree, the angle between AB and the downward vertical.

Solution:

a

Shape	Mass	Mass ratio	Distance of centre of mass from AB
Cylinder	$\pi\rho(2r)^2 \times 3r$	$12r$	$\frac{3r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times h$	$\frac{1}{3}h$	$\frac{1}{4}h$
Remainder	$\pi\rho(12r^3 - \frac{1}{3}r^3 h)$	$12r - \frac{1}{3}h$	$\bar{x}$

$$\circlearrowleft: \left(12r - \frac{1}{3}h\right)\bar{x} = 12r \times \frac{3r}{2} - \frac{1}{3}h \times \frac{1}{4}h$$

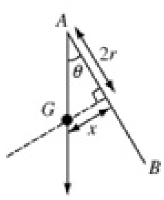
$$\therefore \left(12r - \frac{1}{3}h\right)\bar{x} = 18r^2 - \frac{1}{12}h^2$$

$$\therefore \bar{x} = \frac{18r^2 - \frac{1}{12}h^2}{12r - \frac{1}{3}h}$$

Multiply numerator and denominator by 12

$$\therefore \bar{x} = \frac{216r^2 - h^2}{4(36r - h)}$$

b



From the diagram

$$\tan\theta = \frac{\bar{x}}{2r}$$

$$\text{As } h = 2r, \bar{x} = \frac{216r^2 - (2r)^2}{4(36r - 2r)} = \frac{212r^2}{136r} = \frac{53}{34}r$$

$$\therefore \tan\theta = \frac{53}{68}$$

$$\therefore \theta = 38^\circ \quad \text{(nearest degree)}$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 1

Question:

A particle P moves in a straight line. At time t seconds, the acceleration of P is e^{2t} m s^{-2}, where $t \geq 0$. When $t = 0$, P is at rest. Show that the speed, v m s^{-1}, of P at time t seconds is given by

$v = \frac{1}{2}(e^{2t} - 1)$ **[E]**

Solution:

$a = \frac{dv}{dt} = e^{2t}$

$v = \int e^{2t}\, dt = \frac{1}{2}e^{2t} + A$

To find the acceleration, you integrate the velocity with respect to time. Remember to include a constant of integration.

When $t = 0, v = 0$

$0 = \frac{1}{2} + A \Rightarrow A = -\frac{1}{2}$

Hence $v = \frac{1}{2}(e^{2t} - 1)$, as required

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Review Exercise 1
Exercise A, Question 2

Question:

A particle P moves along the x-axis in such a way that when its displacement from the origin O is x m, its velocity is v m s^{-1} and its acceleration is $4x$ m s^{-2}. When $x = 2$, $v = 4$.
Show that $v^2 = 4x^2$. **[E]**

Solution:

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x$$

When the acceleration is a function of the displacement, x metres, you write $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and integrate both sides of the equation with respect to x.

$$\frac{1}{2}v^2 = \int 4x \, dx = 2x^2 + A$$

$$v^2 = 4x^2 + B, \text{ where } B = 2A$$

At $x = 2, v = 4$

$$16 = 16 + B \Rightarrow B = 0$$

Even when, as here, the constant of integration is 0, it is essential for you to show how this follows from the information given in the question to gain full marks.

Hence
$$v^2 = 4x^2$$

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Review Exercise 1
Exercise A, Question 3

Question:

A particle P moves along the x-axis in the positive direction. At time t seconds, the velocity of P is v m s^{-1} and its acceleration is $\frac{1}{2}e^{-\frac{1}{6}t}$ m s^{-1}. When $t = 0$ the speed of P is $10\,\text{m s}^{-1}$.

a Express v in terms of t.

b Find, to 3 significant figures, the speed of P when $t = 3$.

c Find the limiting value of v. **[E]**

Solution:

a $a = \frac{dv}{dt} = \frac{1}{2}e^{-\frac{1}{6}t}$

$v = \int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = -3e^{-\frac{1}{6}t} + A$

When $t = 0, v = 10$

$10 = -3 + A \Rightarrow A = 13$

Hence $v = 13 - 3e^{-\frac{1}{6}t}$

> Using $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + A$, then
>
> $\int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = \frac{1}{2\times(-\frac{1}{6})}e^{-\frac{1}{6}t} + A = -\frac{1}{\frac{1}{3}}e^{-\frac{1}{6}t} + A$
>
> $= -3e^{-\frac{1}{6}t} + A.$

b When $t = 3$

$v = 13 - 3e^{-\frac{1}{2}} = 11.180\ldots$

The speed of P when $t = 3$ is $11.2\,\text{m s}^{-1}$ (3 s.f.).

c As $t \to \infty, e^{-\frac{1}{6}t} \to 0$ and $v \to 13$.

The limiting value of v is 13.

> As t gets large, $e^{-\frac{1}{6}t}$ gets very small. For example, if $t = 120$, then $e^{-\frac{1}{6}t} \approx 2.06 \times 10^{-9}$. In this question, as t gets larger, v gets closer and closer to 13 and so 13 is the limiting value of v.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 4

Question:

A particle P moves on the positive x-axis. When $OP = x$ metres, where O is the origin, the acceleration of P is directed away from O and has magnitude $\left(1 - \frac{4}{x^2}\right)\text{ms}^{-2}$. When $OP = x$ metres, the velocity of P is $v\text{ ms}^{-1}$. Given that when $x = 1$, $v = 3\sqrt{2}$ show that when $x = \frac{3}{2}, v^2 = \frac{49}{3}$. **[E]**

Solution:

$$a = \frac{\text{d}}{\text{d}x}\left(\frac{1}{2}v^2\right) = 1 - \frac{4}{x^2} = 1 - 4x^{-2}$$

$$\frac{1}{2}v^2 = \int (1 - 4x^{-2})\,\text{d}x$$

$$= x - \frac{4x^{-1}}{-1} + A = x + \frac{4}{x} + A$$

$$v^2 = 2x + \frac{8}{x} + B, \text{ where } B = 2A$$

Multiplying the equation $\frac{1}{2}v^2 = 2x + \frac{4}{x} + A$ throughout by 2. Twice one arbitrary constant is another arbitrary constant.

At $x = 1, v = 3\sqrt{2}$

$$18 = 2 + 8 + B \Rightarrow B = 8$$

Hence $v^2 = 2x + \frac{8}{x} + 8$

At $x = \frac{3}{2}$

You use the information that at $x = 1, v = 3\sqrt{2}$ to evaluate the constant of integration B. You then substitute $x = \frac{3}{2}$ into the resulting equation and show that $v^2 = \frac{49}{3}$.

$$v^2 = 2 \times \frac{3}{2} + 8 \times \frac{2}{3} + 8 = 11 + \frac{16}{3} = \frac{49}{3}, \text{ as required.}$$

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Review Exercise 1
Exercise A, Question 5

Question:

A particle P is moving in a straight line. When P is at a distance x metres from a fixed point O on the line, the acceleration of P is $(5+3\sin 3x)\text{m s}^{-2}$ in the direction OP.
Given that P passes through O with speed 4 m s^{-1}, find the speed of P at $x=6$
Give your answer to 3 significant figures.

Solution:

$a=\frac{\text{d}}{\text{d}x}\left(\frac{1}{2}v^2\right)=5+3\sin 3x$

$\frac{1}{2}v^2=\int(5+3\sin 3x)\text{d}x=5x-\cos 3x+A$

$v^2=10x-2\cos 3x+B$, where $B=2A$

At $x=0, v=4$

$16=0-2+B\Rightarrow B=18$

Hence $v^2=10x-2\cos 3x+18$

At $x=6$

$v^2=60-2\cos 18+18=76.679\ldots$

$v=\sqrt{76.679\ldots}=8.756\ldots$

The speed of P at $x=6$ is 8.76 m s^{-1} (3 s.f.)

You use the information that at $x=0, v=4$ to evaluate the constant of integration B. You then substitute $x=6$ into the resulting equation and use your calculator to find v.

When calculus has been used, it is assumed that all angles are measured in radians and you must make sure that your calculator is in the correct mode.

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Review Exercise 1
Exercise A, Question 6

Question:

A particle P is moving along the positive x-axis in the direction of x increasing. When $OP = x$ metres, the velocity of P is v m s^{-1} and the acceleration of P is $\frac{4k^2}{(x+1)^2}$ m s^{-2}, where k is a positive constant. At $x = 1$, $v = 0$.

a Find v^2 in terms of x and k.

b Deduce that v cannot exceed $2k$.

[E]

Solution:

a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{4k^2}{(x+1)^2} = 4k^2(x+1)^{-2}$

$\frac{1}{2}v^2 = \int 4k^2(x+1)^{-2}\,\mathrm{d}x = \frac{4k^2(x+1)^{-1}}{-1} + A$

$v^2 = B - \frac{8k^2}{x+1}$, where $B = 2A$

At $x = 1, v = 0$

$0 = B - \frac{8k^2}{2} \Rightarrow B = 4k^2$

Hence $v^2 = 4k^2 - \frac{8k^2}{x+1} = 4k^2\left(1 - \frac{2}{x+1}\right)$

b $v = 2k\sqrt{\left(1 - \frac{2}{x+1}\right)}$ ← As P is moving on the positive x-axis in the direction of x increasing, you need not consider the possibility of a negative square root.

As x is positive, $1 - \frac{1}{x+1} < 1$ ← As x is positive, $\frac{1}{1+x}$ is positive and one minus a positive number must be less than one.

Hence $v < 2k$ and v cannot exceed $2k$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 7

Question:

A particle P moves along the x-axis. At time $t = 0$, P passes through the origin O, moving in the positive x-direction. At time t seconds, the velocity of P is v m s^{-1} and $OP = x$ metres. The acceleration of P is $\frac{1}{12}(30 - x)$m s^{-2}, measured in the positive x- direction.

a Give a reason why the maximum speed of P occurs when $x = 30$.

Given that the maximum speed of P is 10 m s^{-1},

b find an expression for v^2 in terms of x. **[E]**

Solution:

a At the maximum value of v, $\frac{dv}{dt} = 0$.

As $a = \frac{dv}{dt}$, the maximum speed of P occurs when $a = \frac{1}{12}(30 - x) = 0 \Rightarrow x = 30$.

b $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{12}(30 - x)$

$\frac{1}{2}v^2 = \int \frac{1}{12}(30 - x)dx = \int\left(\frac{5}{2} - \frac{x}{12}\right)dx$

$= \frac{5x}{2} - \frac{x^2}{24} + A$

$v^2 = 5x - \frac{x^2}{12} + B$, where $B = 2A$

Multiplying the equation $\frac{1}{2}v^2 = \frac{5x}{2} - \frac{x^2}{24} + A$ throughout by 2. Twice one arbitrary constant is another arbitrary constant.

At $x = 30, v = 10$

$100 = 5 \times 30 - \frac{900}{12} + B$

$B = 100 + \frac{900}{12} - 150 = 25$

Hence $v^2 = 5x - \frac{x^2}{12} + 25$

An alternative form of this answer, completing the square, is $v^2 = 100 - \frac{1}{12}(30 - x)^2$. This confirms that the speed has a maximum at $x = 30$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 8

Question:

A particle P moves along the x-axis. At time t seconds the velocity of P is v m s^{-1} and its acceleration is $2\sin\frac{1}{2}t$ m s^{-2}, both measured in the direction Ox. Given that $v = 4$ when $t = 0$,

a find v in terms of t,

b calculate the distance travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$. **[E]**

Solution:

a $a = \frac{dv}{dt} = 2\sin\frac{1}{2}t$

$v = \int 2\sin\frac{1}{2}t \, dt = -4\cos\frac{1}{2}t + A$

When $t = 0, v = 4$

$4 = -4 + A \Rightarrow A = 8$

Hence $v = 8 - 4\cos\frac{1}{2}t$

Using the formula $\int \sin at \, dt = -\frac{1}{a}\cos at + A$,
$\int 2\sin\frac{1}{2}t \, dt = -\frac{2}{\frac{1}{2}}\cos\frac{1}{2}t + A = -4\cos\frac{1}{2}t + A$

b The distance, s metres, travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is given by

$s = \int_0^{\frac{\pi}{2}} \left(8 - 4\cos\frac{1}{2}t\right) dt$

$= \left[8t - 8\sin\frac{1}{2}t\right]_0^{\frac{\pi}{2}}$

$= 4\pi - 8\sin\frac{\pi}{4} = 4\pi - \frac{8}{\sqrt{2}}$

$= 4\pi - 4\sqrt{2} = 4(\pi - \sqrt{2})$

The change in the displacement of P between any two times, say t_1 and t_2, can be found by calculating the definite integral of the velocity between the limits t_1 and t_2. If P has not turned round, this will also give the distance travelled by P. The particle in this question does turn round when $\frac{dv}{dt} = a = 0$ but that does not happen until $t = 2\pi$, so P does not turn round in the interval $0 \le t \le \frac{\pi}{2}$.

The distance travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is $4(\pi - \sqrt{2})$m .

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 9

Question:

A particle P moves along the x-axis. At time t seconds its acceleration is $(-4e^{-2t})$m s^{-2} in the direction of x increasing. When $t = 0$, P is at the origin O and is moving with speed 1 m s^{-1} in the direction of x increasing.

a Find an expression for the velocity of P at time t.
b Find the distance of P from O when P comes to instantaneous rest. **[E]**

Solution:

a $a = \frac{dv}{dt} = -4e^{-2t}$

$v = -\int 4e^{-2t}\, dt = 2e^{-2t} + A$

At $t = 0, v = 1$

$1 = 2 + A \Rightarrow A = -1$

Hence $v = 2e^{-2t} - 1$

b P is instantaneously at rest when $v = 0$.

$0 = 2e^{-2t} - 1$

To find the speed when P is instantaneously at rest, you will need to know the value of t when $v = 0$.

$e^{-2t} = \frac{1}{2} \Rightarrow e^{2t} = 2$

Take natural logarithms of both sides of this equation and use the property that, for any x, $\ln(e^x) = x$

$2t = \ln 2 \Rightarrow t = \frac{1}{2}\ln 2$

$x = \int v\, dt = \int (2e^{-2t} - 1)\, dt$

$= -e^{-2t} - t + B$

When $t = 0, x = 0$

$0 = -1 - 0 + B \Rightarrow B = 1$

Using $e^0 = 1$. It is a common error to obtain $B = 0$ by carelessly writing $e^0 = 0$.

Hence $x = 1 - e^{-2t} - t$

When $t = \frac{1}{2}\ln 2$

$x = 1 - e^{-2(\frac{1}{2}\ln 2)} - \frac{1}{2}\ln 2 = 1 - e^{-\ln 2} - \frac{1}{2}\ln 2$

Using the law of logarithms, $\ln a^n = n\ln a$ with $n = -1$, $-\ln 2 = (-1)\ln 2 = \ln 2^{-1} = \ln\frac{1}{2}$. Then as for any x, $e^{\ln x} = x, e^{\ln\frac{1}{2}} = \frac{1}{2}$.

$= 1 - e^{\ln\frac{1}{2}} - \frac{1}{2}\ln 2 = 1 - \frac{1}{2} - \frac{1}{2}\ln 2$

$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$

The distance of P from O when P comes to instantaneous rest is $\frac{1}{2}(1 - \ln 2)$m.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 10

Question:

At time $t = 0$, a particle P is at the origin O moving with speed 18 m s^{-1} along the x-axis in the positive x-direction. At time t seconds $(t > 0)$ the acceleration of P has magnitude $\frac{3}{\sqrt{(t+4)}} \text{ m s}^{-2}$ and is directed towards O.

a Show that, at time t seconds, the velocity of P is $\left[30 - 6\sqrt{(t+4)}\right] \text{m s}^{-1}$. **[E]**

b Find the distance of P from O when P comes to instantaneous rest.

Solution:

a $a = \frac{dv}{dt} = -\frac{3}{\sqrt{(t+4)}} = -3(t+4)^{-\frac{1}{2}}$

As the acceleration is towards O, $\frac{dv}{dt}$, which is always measured in the direction of x increasing, is negative.

$v = -3\int (t+4)^{-\frac{1}{2}}\, dt = \frac{-3(t+4)^{\frac{1}{2}}}{\frac{1}{2}} + A = A - 6(t+4)^{\frac{1}{2}}$

When $t = 0, v = 18$

$18 = A - 6 \times 2 \Rightarrow A = 30$

Hence

$v = 30 - 6(t+4)^{\frac{1}{2}}$

The velocity of P is $\left[30 - 6\sqrt{(t+4)}\right] \text{m s}^{-1}$, as required.

b $0 = 30 - 6(t+4)^{\frac{1}{2}}$

$(t+4)^{\frac{1}{2}} = 5 \Rightarrow t + 4 = 25 \Rightarrow t = 21$

There are three steps needed to solve part **b**. First you must find the value of t for which P is instantaneously at rest; that is when $v = 0$. You must also find x in terms of t by integrating the expression you proved in part **a**. Finally you substitute your value of t into your expression for x. It is a characteristic of harder questions at this level that you often have to construct for yourself the steps needed to solve a problem.

$v = \frac{dx}{dt} = 30 - 6(t+4)^{\frac{1}{2}}$

$x = \int \left(30 - 6(t+4)^{\frac{1}{2}}\right) dt = 30t - \frac{6(t+4)^{\frac{3}{2}}}{\frac{3}{2}} + B$

$= 30t - 4(t+4)^{\frac{3}{2}} + B$

When $t = 0, x = 0$

$0 = 0 - 4 \times 4^{\frac{3}{2}} + B$

$B = 4 \times 4^{\frac{3}{2}} = 4 \times 8 = 32$

Hence $x = 30t - 4(t+4)^{\frac{3}{2}} + 32$

When $t = 21$

$x = 30 \times 21 - 4(25)^{\frac{3}{2}} + 32 = 630 - 500 + 32 = 162$

The distance of P from O when P comes to instantaneous rest is 162 m.

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Review Exercise 1
Exercise A, Question 11

Question:

A particle P starts at rest and moves in a straight line. The acceleration of P initially has magnitude 20 m s^{-2} and, in a first model of the motion of P, it is assumed that this acceleration remains constant.

a For this model, find the distance moved by P while accelerating from rest to a speed of 6 m s^{-1}.

The acceleration of P when it is x metres from its initial position is $a \text{ m s}^{-2}$ and it is then established that $a = 12$ when $x = 2$. A refined model is proposed in which $a = p - qx$, where p and q are constants.

b Show that, under the refined model, $p = 20$ and $q = 4$.

c Hence find, for this model, the distance moved by P in first attaining a speed of 6 m s^{-1}. **[E]**

Solution:

a $u = 0, a = 20, v = 6, s = ?$

$v^2 = u^2 + 2as$

$36 = 0 + 2 \times 20 \times s$

$s = \frac{36}{40} = 0.9$

For the first model, the distance moved by P while accelerating from rest to 6 m s^{-1} is 0.9 m.

> The model in part **a** is that of constant acceleration, which you studied in module M1. The specification for M3 includes 'a knowledge of the specifications for M1 and M2 and their prerequisites and associated formulae … is assumed and may be tested'.

b $a = p - qx$

At $x = 0, a = 20$

$20 = p - 0 \Rightarrow p = 20$

Hence $a = 20 - qx$

At $x = 2, a = 12$

$12 = 20 - 2q \Rightarrow q = \frac{20-12}{2} = 4$

$p = 20, q = 4$, as required.

> The initial acceleration is 20 m s^{-2}. This applies to all parts of the question. Additionally in part **b**, you are given that $a = 12$ when $x = 2$. The two conditions enable you to find the two unknowns p and q.

c $a = \frac{\text{d}}{\text{d}x}\left(\frac{1}{2}v^2\right) = 20 - 4x$

$\frac{1}{2}v^2 = \int (20 - 4x)\,\text{d}x = 20x - 2x^2 + A$

$v^2 = 40x - 4x^2 + B$, where $B = 2A$

At $x = 0, v = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

Hence $v^2 = 40x - 4x^2$

When $v = 6$

$36 = 40x - 4x^2 \Rightarrow 4x^2 - 40x + 36 = 0$

$x^2 - 10x + 9 = (x-1)(x-9) = 0$

$x = 1, 9$

The distance moved by P in first attaining a speed of 6 m s^{-1} is 1 m.

> Divide this equation throughout by 4 and factorise.

> Comparing this with result in part **a**, the revised model predicts that P moves a little further before reaching the speed of 6 m s^{-1}.

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Review Exercise 1
Exercise A, Question 12

Question:

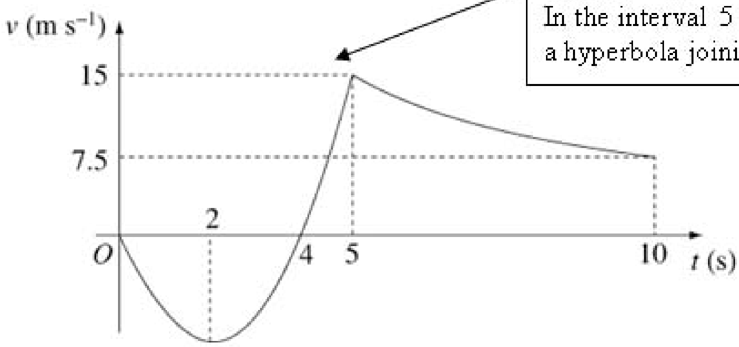
A particle moving in a straight line starts from rest at a point O at time $t = 0$.
At time t seconds, the velocity v m s^{-1} is given by

$$v = \begin{cases} 3t(t-4), & 0 \le t \le 5 \\ 75t^{-1}, & 5 < t \le 10 \end{cases}$$

a Sketch a velocity - time graph for the particle for $0 \le t \le 10$.
b Find the set of values of t for which the acceleration of the particle is positive.
c Show that the total distance travelled by the particle in the interval $0 \le t \le 5$ is 39 m.
d Find, to 3 significant figures, the value of t at which the particle returns to O. **[E]**

Solution:

a



In the interval $0 \le t \le 5$, the graph is part of a parabola which meets the t-axis at the origin and where $t = 4$.
In the interval $5 < t \le 10$, the graph is a segment of a hyperbola joining (5, 15) to (10, 7.5).

b The set of values of t for which the acceleration is positive is $2 < t < 5$.

The acceleration is positive when the velocity–time graph has a positive gradient. By the symmetry of a parabola, the graph has a minimum when $t = 2$ and the set of values of t for which the gradient is positive can be written down by inspecting the graph.

c $\int_0^4 3t(t-4)\,dt = \int_0^4 (3t^2 - 12t)\,dt$

$= \left[t^3 - 6t^2\right]_0^4$

$= (64 - 96) - 0 = -32$

$\int_4^5 3t(t-4)\,dt = \int_4^5 (3t^2 - 12t)\,dt$

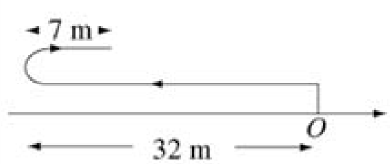
$= \left[t^3 - 6t^2\right]_4^5$

$= (125 - 150) - (64 - 96)$

$= 7$

The distance travelled by P in the interval $0 \le t \le 5$ is $(32 + 7)\text{m} = 39$ m.

Taking the direction of v increasing as positive, for the first 4 seconds the particle travels 32 m in the negative direction. In the next second, it travels 7 m in the positive direction. So in 5 seconds, it travels a total of $(32+7)$m ending at a point which is $(32-7)$m from O in the negative direction.



d For $t > 5$

$x = \int v\,dt = \int 75t^{-1}\,dt$

$= 75\ln t + A$

At time $t = 5$, the particle is $(32 - 7)\text{m} = 25$ m from O in the negative direction.

So when $t = 5, x = -25$

$-25 = 75\ln 5 + A \Rightarrow A = -75\ln 5 - 25$

Hence

$x = 75\ln t - 75\ln 5 - 25 = 75\ln\left(\frac{t}{5}\right) - 25$

Using the law of logarithms $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$,

$75\ln t - 75\ln 5 = 75(\ln t - \ln 5) = 75\ln\left(\frac{t}{5}\right)$.

At $x = 0$

$0 = 75\ln\left(\frac{t}{5}\right) - 25 \Rightarrow \ln\left(\frac{t}{5}\right) = \frac{1}{3}$

$\frac{t}{5} = e^{\frac{1}{3}} \Rightarrow t = 5e^{\frac{1}{3}} = 6.98\ (3\text{ s.f.})$

You solve this equation for t by taking exponentials of both sides of the equation and using $e^{\ln\left(\frac{t}{5}\right)} = \frac{t}{5}$.

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Review Exercise 1
Exercise A, Question 13

Question:

A particle P moves along the positive x-axis. When $OP = x$ metres, the velocity of P is v m s^{-1} and its acceleration is $\frac{72}{(2x+1)^2}$ m s^{-2} in the direction of x increasing.

Initially $x = 1$ and P is moving toward O with speed 6 m s^{-1}. Find

a v in terms of x,

b the minimum distance of P from O.

Solution:

a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{72}{(2x+1)^2} = 72(2x+1)^{-2}$

$\frac{1}{2}v^2 = \int 72(2x+1)^{-2}\,\mathrm{d}x = \frac{72(2x+1)^{-1}}{2\times(-1)} + A$

$= A - \frac{36}{2x+1}$

$v^2 = B - \frac{72}{2x+1}$, where $B = 2A$

At $x = 1, v = -6$

$36 = B - \frac{72}{3} \Rightarrow B = 60$

Hence $v^2 = 60 - \frac{72}{2x+1}$

To integrate $(2x+1)^{-2}$, you can use the formula $\int (ax+b)^n\,\mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + A$.

b At a minimum value of x, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

and so $v = 0$.

Substituting $v = 0$ into the result of part **a**

$0 = 60 - \frac{72}{2x+1} \quad \Rightarrow 60 = \frac{72}{2x+1}$

$2x+1 = \frac{72}{60} = 1.2 \Rightarrow x = \frac{1.2-1}{2} = 0.1$

The minimum distance of P from O is 0.1 m.

In this question, it is not practical to find x in terms of t. However, to find the minimum value of x, this is not necessary. The minimum is a stationary value and at a stationary value $\frac{\mathrm{d}x}{\mathrm{d}t}$, which is v, is zero

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Review Exercise 1
Exercise A, Question 14

Question:

A particle moves on the positive x-axis. The particle is moving towards the origin O when it passes through the point A, where $x = 2a$, with speed $\sqrt{\left(\frac{k}{a}\right)}$, where k is a constant. Given that the particle experiences an acceleration $\frac{k}{2x^2} + \frac{k}{4a^2}$ in a direction away from O,

a show that it comes instantaneously to rest at a point B, where $x = a$.

As soon as the particle reaches B the acceleration changes to $\frac{k}{2x^2} - \frac{k}{4a^2}$ in a direction away from O.

b Show that the particle next comes instantaneously to rest at A. **[E]**

Solution:

a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{k}{2x^2} + \frac{k}{4a^2} = \frac{k}{2}x^{-2} + \frac{k}{4a^2}$

$\frac{1}{2}v^2 = \int\left(\frac{k}{2}x^{-2} + \frac{k}{4a^2}\right)\mathrm{d}x = \frac{k}{2}\times\frac{x^{-1}}{-1} + \frac{kx}{4a^2} + A$

$= -\frac{k}{2x} + \frac{kx}{4a^2} + A$

$v^2 = -\frac{k}{x} + \frac{kx}{2a^2} + B$, where $B = 2A$

At $x = 2a, v = -\sqrt{\left(\frac{k}{a}\right)}$

$\frac{k}{a} = -\frac{k}{2a} + \frac{2ka}{2a^2} + B$

$B = \frac{k}{a} + \frac{k}{2a} - \frac{k}{a} = \frac{k}{2a}$

Hence $v^2 = -\frac{k}{x} + \frac{kx}{2a^2} + \frac{k}{2a}$

At $x = a$

$v^2 = -\frac{k}{a} + \frac{ka}{2a^2} + \frac{k}{2a} = -\frac{k}{a} + \frac{k}{2a} + \frac{k}{2a} = 0$

The particle comes to instantaneous rest where $x = a$.

> To show that the particle comes to rest at A, you use integration to obtain v^2 in terms of x, and then substitute $x = a$ into your expression and show that $v = 0$.

b $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{k}{2x^2} - \frac{k}{4a^2} = \frac{k}{2}x^{-2} - \frac{k}{4a^2}$

$\frac{1}{2}v^2 = \int\left(\frac{k}{2}x^{-2} - \frac{k}{4a^2}\right)\mathrm{d}x = \frac{k}{2}\times\frac{x^{-1}}{-1} - \frac{kx}{4a^2} + C$

$= -\frac{k}{2x} - \frac{kx}{4a^2} + C$

$v^2 = -\frac{k}{x} - \frac{kx}{2a^2} + D$, where $D = 2C$

At $x = a, v = 0$

> Although the acceleration changes at B you can assume that the velocity is continuous and that the final velocity, 0, in part **a** is the initial velocity in part **b**.

$0 = -\frac{k}{a} - \frac{k}{2a} + D \Rightarrow D = \frac{3k}{2a}$

Hence

$v^2 = -\frac{k}{x} - \frac{kx}{2a^2} + \frac{3k}{2a} = \frac{-2ka^2 - kx^2 + 3kax}{2a^2x}$

$= -\frac{k}{2a^2x}(x^2 - 3ax + 2a^2) = -\frac{k}{2a^2x}(x-a)(x-2a)$

> There are a number of different ways of completing this question. The solution shown here puts all of the terms on the right of the equation over a common denominator and factorises the resulting expression.

When $v = 0, (x-a)(x-2a) = 0$

$x = a, 2a$

> $x = a$ corresponds to the point B and $x = 2a$ corresponds to the point A.

After leaving B, the particle next comes to rest at A, where $x = 2a$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 15

Question:

A car is travelling along a straight horizontal road. As it passes a point O on the road, the engine is switched off. At time t seconds after the car has passed O, it is at a point P, where $OP = x$ metres, and its velocity is v m s^{-1}. The motion of the car is modelled by

$$v = \frac{1}{p + qt}$$

where p and q are positive constants.

a Show that, with this model, the retardation of the car is proportional to the square of the speed.

When $t = 0$, the retardation of the car is 0.75 m s^{-2} and $v = 20$.

Using the model, find

b the value of p and the value of q,

c x in terms of t.

Solution:

a $v = \dfrac{1}{p+qt} = (p+qt)^{-1}$

$a = \dfrac{dv}{dt} = (-1)q(p+qt)^{-2}$

$= -\dfrac{q}{(p+qt)^2} = -qv^2$

So the deceleration is proportional to the square of the speed.

The deceleration is the negative of the acceleration. If $d = kv^n$, for any constant k, then d is proportional to v^n.

The square of the speed and the square of the velocity are identical because, for example, $(-20)^2 = 20^2$.

b When $t = 0, a = -0.75$ and $v = 20$

$a = -qv^2$

$-0.75 = -q \times 20^2$

$q = \dfrac{3}{4} \times \dfrac{1}{20^2} = \dfrac{3}{1600}$

The exact decimal answers $p = 0.05$ and $q = 0.001\,875$ are also acceptable.

$v = \dfrac{1}{p+qt}$

When $t = 0, v = 20$

$20 = \dfrac{1}{p} \Rightarrow p = \dfrac{1}{20}$

$p = \dfrac{1}{20}, q = \dfrac{3}{1600}$

c $v = \dfrac{dx}{dt} = \dfrac{1}{p+qt}$

$x = \displaystyle\int \frac{1}{p+qt}\,dt = \frac{1}{q}\ln(p+qt) + A$

$= \dfrac{1600}{3}\ln\left(\dfrac{1}{20} + \dfrac{3}{1600}t\right) + A$

When $t = 0$, $x = 0$

$0 = \dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right) + A \Rightarrow A = -\dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right)$

Hence $x = \dfrac{1600}{3}\ln\left(\dfrac{1}{20} + \dfrac{3}{1600}t\right) - \dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right)$

This expression can be simplified using the law of logarithms $\ln a - \ln b = \ln\dfrac{a}{b}$. However, as the question specifies no particular form for the answer, an unsimplified answer or an answer with decimals would be accepted.

$= \dfrac{1600}{3}\ln\left(\dfrac{\frac{1}{20} + \frac{3}{1600}t}{\frac{1}{20}}\right)$

$x = \dfrac{1600}{3}\ln\left(1 + \dfrac{3}{80}t\right)$

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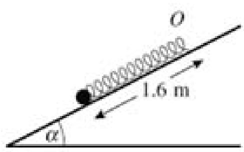
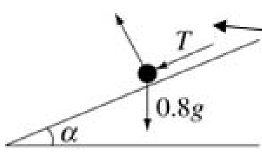
Review Exercise 1
Exercise A, Question 16

Question:



A particle of mass 0.8 kg is attached to one end of a light elastic spring, of natural length 2 m and modulus of elasticity 20 N.
The other end of the spring is attached to a fixed point O on a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$.
The particle is held at a point which is 1.6 m down the line of greatest slope of the plane from O, as shown in the figure. The particle is then released from rest. Find the initial acceleration of the particle. **[E]**

Solution:



Initially the spring is in compression and the force of the spring on the particle is acting down the plane.

Let the thrust in the spring be T newtons.

Hooke's law $T = \frac{\lambda x}{l}$

$= \frac{20 \times 0.4}{2} = 4$

The compression is $(2 - 1.6)\text{m} = 0.4\text{ m}$.

R(↙) $\mathbf{F} = m\mathbf{a}$

$mg\sin\alpha + T = ma$

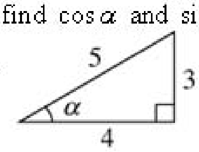
$0.8 \times 9.8 \times \frac{3}{5} + 4 = 0.8a$

$0.8a = 8.704$

$a = 10.88$

When you know $\tan\alpha$ you can draw a triangle to find $\cos\alpha$ and $\sin\alpha$.

$\tan\alpha = \frac{3}{4}$



$\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

The initial acceleration of the particle is 11m s^{-1} (2 s.f.)

As you have used an approximate value of g, you should round your answer to a sensible accuracy. Either 2 or 3 significant figures is acceptable.

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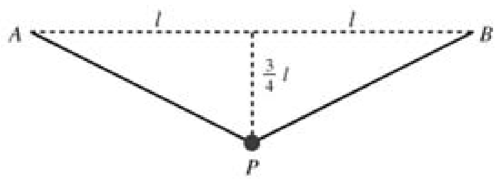
Review Exercise 1
Exercise A, Question 17

Question:

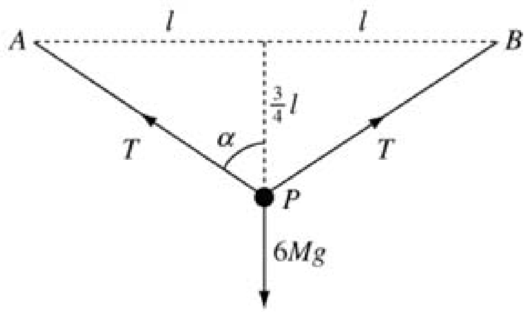


The figure shows a particle P, of mass $6M$, suspended by two light elastic strings from points A and B which are fixed and at a horizontal distance $2l$ apart. Each string has natural length l and P rests in equilibrium at a vertical distance $\frac{3}{4}l$ below the level of AB. Determine

a the tension in either string,

b the modulus of elasticity of either string. **[E]**

Solution:



a Let the angle between PA and the vertical be α.

$\tan\alpha = \frac{l}{\frac{3}{4}l} = \frac{4}{3} \Rightarrow \sin\alpha = \frac{4}{5}, \cos\alpha = \frac{3}{5}$

Let the tensions in AP and BP be T newtons.

By symmetry, the tension in both strings is the same.

$R(\uparrow)\ 2T\cos\alpha = 6Mg$

$2T \times \frac{3}{5} = 6Mg$

$T = 6Mg \times \frac{5}{6} = 5Mg$

The tension in both strings is $5Mg$.

b $AP^2 = l^2 + \left(\frac{3}{4}l\right)^2 = \frac{25}{16}l^2 \Rightarrow AP = \frac{5}{4}l$

You use Pythagoras' theorem to find the length of one of the strings and use this length to find the extension of the string.

The extension of AP is $\left(\frac{5}{4}l - l\right) = \frac{1}{4}l$

Hooke's law $T = \frac{\lambda x}{l}$

$5Mg = \frac{\lambda \times \frac{1}{4}l}{l} = \frac{\lambda}{4}$

As you know the tension in the string from part **a**, you can use Hooke's law to find the modulus of elasticity, λ.

The modulus of elasticity in either string is $20Mg$.

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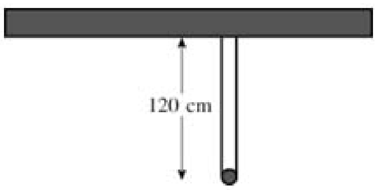
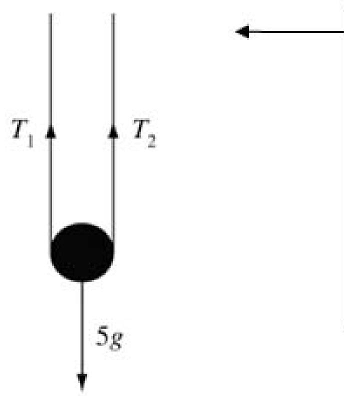
Review Exercise 1
Exercise A, Question 18

Question:



A particle of mass 5 kg is attached to one end of two light elastic strings. The other ends of the string are attached to a hook on a beam. The particle hangs in equilibrium at a distance 120 cm below the hook with both strings vertical, as shown in the figure. One string has natural length 100 cm and modulus of elasticity 175 N. The other string has natural length 90 cm and modulus of elasticity λ newtons.
Find the value of λ. **[E]**

Solution:



You must consider the two strings separately. Here the string of natural length 100 cm is drawn on the left. The extension of this string is $(120-100)\text{cm} = 20\text{ cm}$ and the tension in this string is taken as T_1 newtons. The string of natural length 90 cm is drawn on the right. The extension of this string is $(120-90)\text{cm} = 30\text{ cm}$ and the tension in this string is taken as T_2 newtons.

For the string of natural length 100 cm

Hooke's law $T = \frac{\lambda x}{l}$

$$T_1 = \frac{175 \times 20}{100} = 35$$

For the string of natural length 90 cm

Hooke's law $T = \frac{\lambda x}{l}$

$$T_2 = \frac{\lambda \times 30}{90} = \frac{\lambda}{3}$$

R($\uparrow$) $T_1 + T_2 = 5g$

$$35 + \frac{\lambda}{3} = 5 \times 9.8$$

$$\lambda = 3(5 \times 9.8 - 35) = 42$$

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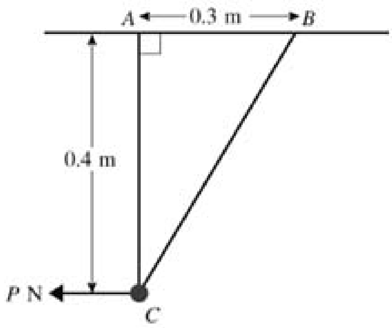
Review Exercise 1
Exercise A, Question 19

Question:



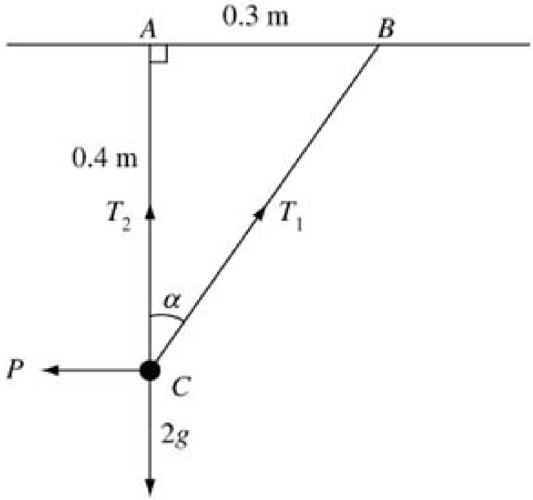
The figure shows a particle C of mass 2 kg suspended by two strings. The strings are fixed to two points A and B on a horizontal ceiling, where $AB = 0.3$ m. The string AC is light and inextensible, with length 0.4 m, while the string BC is light and elastic with natural length 0.4 m and modulus of elasticity 32 N. A horizontal force of magnitude P N holds the system in equilibrium with AC vertical.

a Show that the tension in BC is 8 N.

b Find the value of P.

c Find the tension in AC. **[E]**

Solution:



a Let $BC = x$ m

$$x^2 = 0.3^2 + 0.4^2 = 0.25 \Rightarrow x = 0.5$$

The extension of BC is $(0.5 - 0.4)\text{m} = 0.1\text{ m}$.

Let the tension in BC be T_1 newtons

Hooke's Law $T = \dfrac{\lambda x}{l}$

$$T_1 = \frac{32 \times 0.1}{0.4} = 8$$

The tension in BC is 8 N, as required.

> BC is an elastic string and, to find its tension, you need to know its extension. You use Pythagoras' theorem to find the length of BC and subtract the natural length to find the extension.

b Let $\angle ACB = \alpha$

$$\tan\alpha = \frac{0.3}{0.4} = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$$

$\text{R}(\rightarrow)$ $P = T_1 \sin\alpha$

$$= 8 \times \frac{3}{5} = 4.8$$

> To resolve, you need to know the sines and cosines of an appropriate angle. The dimensions of the triangle ABC enable you to find these.

c Let the tension in AC be T_2 newtons

$\text{R}(\uparrow)$ $T_2 + T_1 \cos\alpha = 2g$

$$T_2 = 2 \times 9.8 - 8 \times \frac{4}{5} = 13.2$$

The tension is AC is 13.2 N.

> When a numerical value of g is used, 2 or 3 significant figures is acceptable.

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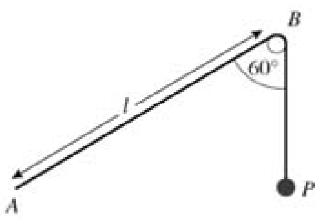
Review Exercise 1
Exercise A, Question 20

Question:



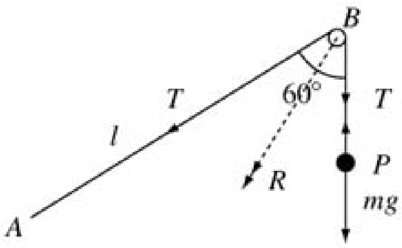
A light elastic string, of natural length l and modulus of elasticity $4mg$, has one end tied to a fixed point A. The string passes over a fixed small smooth peg B and at the other end a particle P, of mass m, is attached. The particle hangs in equilibrium. The distance between A and B is l and AB is inclined at $60°$ to the vertical, as shown in the figure.

a Find, in terms of l, the length of the vertical portion BP of the string.

b Show that the magnitude of the force exerted by the string on the peg is $mg\sqrt{3}$.

[E]

Solution:



As the string is light and the peg is smooth, the tension, T say, is the same throughout the string.

a Let $BP = x$
For the particle P
R(↑) $T = mg$

Hooke's law

$$T = \frac{\lambda x}{l}$$

$$mg = \frac{4mg \times x}{l}$$

$$x = \frac{l}{4}$$

$$BP = \frac{l}{4}$$

As the natural length of the string is l and $AB = l$, the length BP is the extension of the string.

b Resolving along the bisector of the angle between the two portions of the string

R(↙) $R = 2T\cos 30°$

$$= 2mg \times \frac{\sqrt{3}}{2}$$

$$= mg\sqrt{3}, \text{ as required}$$

As both parts of the string exert the same force T on the peg, by symmetry, the resultant force on the peg acts along the angle bisector of the angle between the two portions of the string. You find the magnitude of the resultant force, R say, by resolving along this angle bisector.

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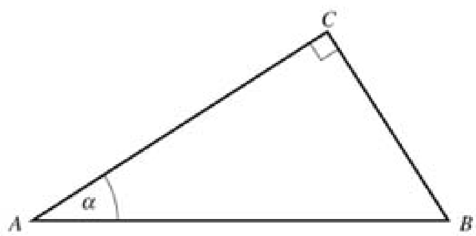
Review Exercise 1
Exercise A, Question 21

Question:



A rod AB, of mass $2m$ and length $2a$, is suspended from a fixed point C by two light strings AC and BC. The rod rests horizontally in equilibrium with AC making an angle α with the rod, where $\tan \alpha = \frac{3}{4}$, and with AC perpendicular to BC, as shown in the figure.

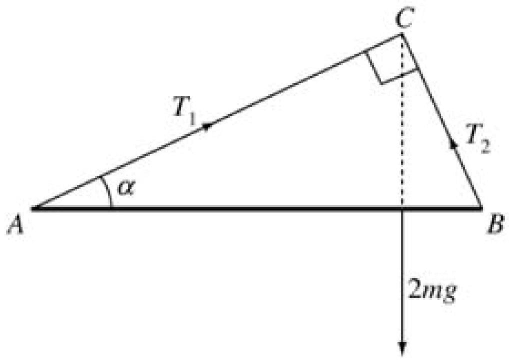
a Give a reason why the rod cannot be uniform.

b Show that the tension in BC is $\frac{8}{5}mg$ and find the tension in AC.

The string BC is elastic, with natural length a and modulus of elasticity kmg, where k is a constant.

c Find the value of k. **[E]**

Solution:



a The line of action of the weight must pass through C which is not above the centre of the rod.

For three forces to be in equilibrium the lines of action of all three forces must pass through the same point. As the lines of action of both tensions pass through C, the line of action of the weight has to pass through C as well and so the rod cannot be uniform.

b $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

Let the tension in AC be T_1 newtons and the tension in BC be T_2 newtons.

$\text{R}(\rightarrow)\quad T_1\cos\alpha = T_2\sin\alpha$

$\frac{4}{5}T_1 = \frac{3}{5}T_2 \Rightarrow T_1 = \frac{3}{4}T_2$

$\text{R}(\uparrow)\quad T_1\sin\alpha + T_2\cos\alpha = 2mg$

You substitute $T_1 = \frac{3}{4}T_2$ and the values of $\sin\alpha$ and $\cos\alpha$ into this equation and solve for T_2.

$$\frac{3}{4}T_2 \times \frac{3}{5} + T_2 \times \frac{4}{5} = 2mg$$

$$\left(\frac{9}{20} + \frac{4}{5}\right)T_2 = \frac{5}{4}T_2 = 2mg$$

$$T_2 = \frac{8}{5}mg$$

The tension in BC is $\frac{8}{5}mg$, as required.

$T_1 = \frac{3}{4}T_2 = \frac{3}{4} \times \frac{8}{5}mg = \frac{6}{5}mg$

The tension in AC is $\frac{6}{5}mg$.

c $BC = AB\sin\alpha = 2a \times \frac{3}{5} = \frac{6}{5}a$

For BC

Hooke's law $\quad T_2 = \frac{\lambda x}{l}$

You find the length of BC by trigonometry. Then the extension of the elastic string BC is $\frac{6}{5}a - a = \frac{1}{5}a$.

$$\frac{8}{5}mg = \frac{kmg \times \frac{1}{5}a}{a} \Rightarrow k = 8$$

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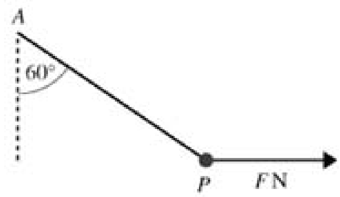
Review Exercise 1
Exercise A, Question 22

Question:



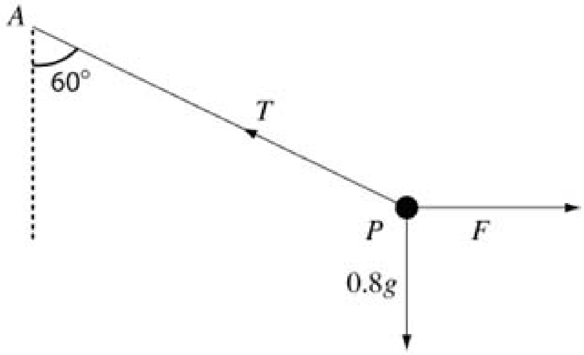
A particle of mass 0.8 kg is attached to one end of a light elastic string, of natural length 1.2 m and modulus of elasticity 24 N. The other end of the string is attached to a fixed point A. A horizontal force of magnitude F newtons is applied to P. The particle is in equilibrium with the string making an angle $60°$ with the downward vertical as shown in the figure. Calculate

a the value of F,

b the extension of the string,

c the elastic energy stored in the string. **[E]**

Solution:



$R(\uparrow)\quad T\cos 60^\circ = 0.8\,g$

$\frac{1}{2}T = 0.8\,g \Rightarrow T = 1.6\,g$

Resolving vertically gives you the tension in the string.

$R(\leftarrow)\quad F = T\cos 30^\circ = 1.6\,g \times \frac{\sqrt{3}}{2}$

$= 13.579\ldots$

$= 14$ (2 s.f.)

Substituting for the tension into the equation obtained by resolving horizontally gives the value of F.

b Hooke's law $\quad T = \frac{\lambda x}{l}$

$1.6\,g = \frac{24x}{1.2}$

$x = \frac{1.6\,g \times 1.2}{24} = 0.784$

The extension of the string is 0.78 m (2 s.f.).

Substituting for the tension into Hooke's Law gives you an equation for the extension.

c The elastic energy stored in the string is given by

$E = \frac{\lambda x^2}{2l}$

$= \frac{24 \times (0.784)^2}{2 \times 1.2} = 6.146\,56$

The elastic energy stored in the string is 6.1 J (2 s.f.)

You need to remember the formula for the energy stored in an elastic string.

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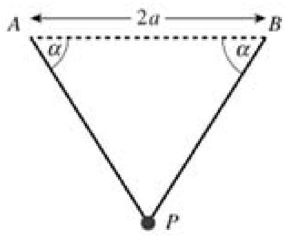
Edexcel AS and A Level Modular Mathematics

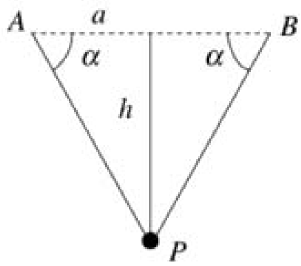
Review Exercise 1
Exercise A, Question 23

Question:



Two light elastic strings each have natural length a and modulus of elasticity λ. A particle P of mass m is attached to one end of each string. The other ends of the string are attached to points A and B, where AB is horizontal and $AB = 2a$. The particle is held at the mid-point of AB and released from rest. It comes to rest for the first time in its subsequent motion when PA and PB make angles α with AB, where $\tan \alpha = \frac{4}{3}$, as shown in the figure. Find λ in terms of m and g. **[E]**

Solution:



When P comes instantaneously to rest, it is not in equilibrium and so the question cannot easily be solved by resolving. It is a common error to attempt the solution of this, and similar questions, by resolving.

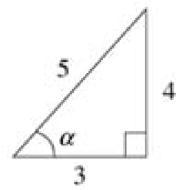
$\tan\alpha = \frac{4}{3} \Rightarrow \cos\alpha = \frac{3}{5}$

Let the distance fallen by P be h.

When you know $\tan\alpha$ you can draw a triangle to find $\cos\alpha$.

$\tan\alpha = \frac{4}{3}$

$\cos\alpha = \frac{3}{5}$



$$h = a\tan\alpha = \frac{4a}{3}$$

$$AP^2 = h^2 + a^2 = \left(\frac{4a}{3}\right)^2 + a^2 = \frac{25a^2}{9}$$

$$AP = \frac{5a}{3}$$

When P first comes to rest the energy stored in one string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{\lambda\left(\frac{2a}{3}\right)^2}{2a} = \frac{2\lambda a}{9}$$

The extension in one string is

$$\text{AP} - \text{natural length} = \frac{5a}{3} - a$$

$$= \frac{2a}{3}$$

When P first comes to rest the potential energy lost is given by

$$mgh = mg \times \frac{4}{3}a$$

Conservation of energy

Elastic energy gained = potential energy lost

$$\frac{4\lambda a}{9} = \frac{4mga}{3}$$

$$\lambda = \frac{4mga}{3} \times \frac{9}{4a} = 3mg$$

Initially P is at rest and, when it has fallen $\frac{5a}{3}$, it is at rest again. So there is no change in kinetic energy. Elastic energy is gained by both strings and potential energy is lost by the particle.

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Review Exercise 1
Exercise A, Question 24

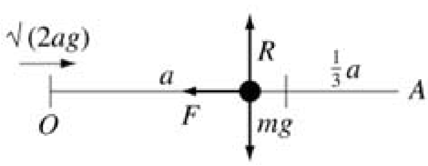
Question:

A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $3.6mg$. The other end of the string is fixed at a point O on a rough horizontal table. The particle is projected along the surface of the table from O with speed $\sqrt{(2ag)}$. At its furthest point from O, the particle is at the point A, where $OA = \frac{4}{3}a$.

a Find, in terms of m, g and a, the elastic energy stored in the string when P is at A.

b Using the work–energy principle, or otherwise, find the coefficient of friction between P and the table. **[E]**

Solution:



a At A, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$
$$= \frac{3.6mg \times (\frac{1}{3}a)^2}{2a}$$
$$= 0.2mga$$

> At A, the extension of the string is $\frac{4}{3}a - a = \frac{1}{3}a$.

b The total energy lost is

$$\frac{1}{2}mu^2 - 0.2mga = \frac{1}{2} \times 2ag - 0.2mga$$
$$= 0.8mga$$

> As P is at rest at A, the net loss of energy is the loss in kinetic energy minus the gain in elastic energy.

At any point in the motion

R(↑) $R = mg$

The friction is given by

$F = \mu R = \mu mg$

By the work–energy principle

$$0.8mg = \mu mg \times \frac{4}{3}a$$
$$\mu = 0.8 \times \frac{3}{4} = 0.6$$

> By the work–energy principle, the net loss in energy is equal to the work done by friction. You find the work done by friction by multiplying the magnitude of the friction, μmg, by the distance the particle moves, $\frac{4}{3}a$. This gives you an equation in μ, which you solve.

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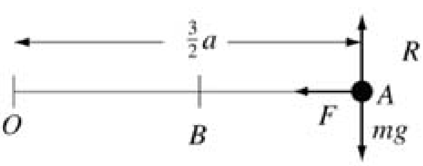
Review Exercise 1
Exercise A, Question 25

Question:

A particle P of mass m is held at a point A on a rough horizontal plane. The coefficient of friction between P and the plane is $\frac{2}{3}$. The particle is attached to one end of a light elastic string, of natural length a and modulus of elasticity $4mg$. The other end of the string is attached to a fixed point O on the plane, where $OA = \frac{3}{2}a$. The particle P is released from rest and comes to rest at a point B, where $OB < a$. Using the work–energy principle, or otherwise, calculate the distance AB. **[E]**

Solution:



At any point in the motion

R(↑) $R = mg$

The friction is given by

$F = \mu R = \frac{2}{3}mg$

At A, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{4mg \times (\frac{1}{2}a)^2}{2a}$$

$$= \frac{1}{2}mga$$

At A, the extension of the string is $\frac{3}{2}a - a = \frac{1}{2}a$.

By the work–energy principle

$$\frac{1}{2}mga = \frac{2}{3}mg \times AB$$

$$AB = \frac{3}{4}a$$

When P comes to rest, as $OB < a$, the string is slack so all of the elastic energy has been lost. This lost energy must equal the work done by friction, which is the magnitude of the friction, $\frac{2}{3}mg$, multiplied by the distance moved by P, which is AB.

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Review Exercise 1
Exercise A, Question 26

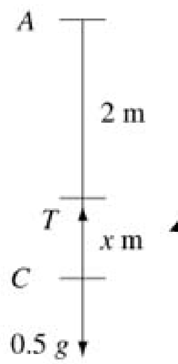
Question:

One end of an light elastic string, of natural length 2 m and modulus of elasticity 19.6 N, is attached to a fixed point A. A small ball B of mass 0.5 kg is attached to the other end of the string. The ball is released from rest at A and first comes to instantaneous rest at the point C, vertically below A.

a Find the distance AC.

b Find the instantaneous acceleration of B at C. **[E]**

Solution:



In solving nearly all questions involving elastic strings and springs you need to find the value of, or an expression for, the extension. If no symbol is given in the question, you should introduce a symbol, here x m, yourself.

a Let the extension of string when B is at C be x m.
Conservation of energy
elastic energy gained = potential energy lost

$$\frac{\lambda x^2}{2l} = mgh$$

$$\frac{19.6x^2}{4} = 0.5 \times 9.8\,(2+x)$$

In falling from A to C, the ball moves a distance of $(2+x)$m and so the potential energy lost is, in Joules, $mg(2+x)$.

$4.9x^2 = 4.9(2+x)$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$

$AC = 4$ m

For there to be elastic energy in a string, the extension must be positive, so you can discard the solution $x = -1$.

b At C
Hooke's law

$$T = \frac{\lambda x}{l} = \frac{19.6 \times 2}{2} = 19.6$$

R(↓) $\qquad \mathbf{F} = m\mathbf{a}$

$$mg - T = m\mathbf{a}$$

$$0.5 \times 9.8 - 19.6 = 0.5\mathbf{a}$$

$$\mathbf{a} = \frac{0.5 \times 9.8 - 19.6}{0.5} = -29.4$$

The instantaneous acceleration of B at C is 29.4 m s^{-2} directed towards A.

The negative acceleration shows you that the acceleration is in the direction of x decreasing, that is towards A.

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Review Exercise 1
Exercise A, Question 27

Question:

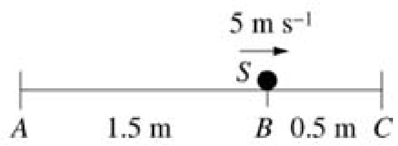
A light elastic string AB of natural length 1.5 m has modulus of elasticity 20 N. The end A is fixed to a point on a smooth horizontal table. A small ball S of mass 0.2 kg is attached to the end B. Initially S is at rest on the table with $AB = 1.5$ m. The ball S is then projected horizontally directly away from A with a speed of 5 m s^{-1}. By modelling S as a particle

a Find the speed of S when $AS = 2$ m.

When the speed of S is 1.5 m s^{-1}, the string breaks.

b Find the tension in the string immediately before the string breaks. **[E]**

Solution:



a Let $AC = 2\text{ m}$. When S is at C, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{20 \times (0.5)^2}{2 \times 1.5} = \frac{5}{3}$$

Let the speed of S at C be $v\text{ m s}^{-1}$.

Conservation of energy

Kinetic energy lost = elastic energy gained

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{5}{3}$$

$$\frac{1}{2} \times 0.2 \times 5^2 - \frac{1}{2} \times 0.2v^2 = \frac{5}{3}$$

$$0.1v^2 = 0.1 \times 25 - \frac{5}{3} = \frac{5}{6}$$

$$v^2 = \frac{25}{3} \Rightarrow v = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3} \approx 2.886\ldots$$

The speed of S when $AS = 2\text{ m}$ is 2.89 m s^{-1} (3 s.f.)

The exact answer $\frac{5\sqrt{3}}{3}\text{ m s}^{-1}$ is also accepted.

b Let the extension of the string immediately before the string breaks be x m.

To find the tension in the string when the speed of S is 1.5 m s^{-1}, you first need to find the extension of the string at this speed. The extension is found using conservation of energy.

When the extension in the string is x m, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l} = \frac{20x^2}{3}$$

Conservation of energy

Kinetic energy lost = elastic energy gained

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{20x^2}{3}$$

$$\frac{1}{2} \times 0.2 \times 5^2 - \frac{1}{2} \times 0.2 \times 1.5^2 = \frac{20x^2}{3}$$

$$\frac{20x^2}{3} = 2.275 \Rightarrow x^2 = 0.34125$$

$$x = \sqrt{(0.34125)}$$

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{20\sqrt{(0.34125)}}{1.5} = 7.788\ldots$$

The tension in the string immediately before the string breaks is 7.79 N (3 s.f.).

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Review Exercise 1
Exercise A, Question 28

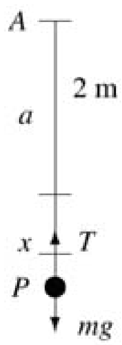
Question:

One end of a light elastic string of natural length a and modulus of elasticity $3\ mg$, is fixed at a point A and the other end carries a particle P of mass m. The particle is held at A and then projected vertically downwards with speed $\sqrt{(3ga)}$.

a Find the distance AP when the acceleration of the particle is instantaneously zero.

b Find the maximum speed attained by the particle during its motion. **[E]**

Solution:



a Let the extension of the string when the acceleration of P is zero be x m.
Hooke's law

$$T = \frac{\lambda x}{l} = \frac{3mgx}{a}$$

R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m \times 0$$

$$mg - \frac{3mgx}{a} = 0 \Rightarrow x = \frac{a}{3}$$

$$AP = a + \frac{a}{3} = \frac{4a}{3}$$

It is usually easier, when using Hooke's law, to work in terms of the extension of the string than to work with the total length of the string. Here, first find the extension when the acceleration is zero and, then, find the total length by adding the extension to the natural length.

b When $AP = \frac{4a}{3}$, the energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l} = \frac{3mg\left(\frac{a}{3}\right)^2}{2a} = \frac{1}{6}mga$$

Let the maximum speed of P be v m s^{-1}

Using conservation of energy

The maximum speed occurs when the acceleration is zero. This is the equivalent of the calculus property that at a maximum value of v, $\frac{dv}{dt} = 0$. In part **a**, you showed that $a = 0$ when $AP = \frac{4a}{3}$. Using conservation of energy, you can find the speed of P when $AP = \frac{4a}{3}$.

kinetic energy gained + elastic energy gained = potential energy lost

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 + \frac{\lambda x^2}{2l} = mgh$$

$$\frac{1}{2}mv^2 - \frac{1}{2}m \times 3ga + \frac{1}{6}mga = mg \times \frac{4a}{3}$$

$$\frac{1}{2}mv^2 = \frac{4mga}{3} + \frac{3mga}{2} - \frac{mga}{6} = \frac{8mga}{3}$$

$$v^2 = \frac{16mga}{3m} \Rightarrow v = 4\left(\frac{ga}{3}\right)^{\frac{1}{2}}$$

The maximum speed attained by P during its motion is $4\left(\frac{ga}{3}\right)^{\frac{1}{2}}$.

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Review Exercise 1
Exercise A, Question 29

Question:

A light elastic string of natural length 30 cm is placed on a smooth horizontal table with one end attached to a fixed point P on the table. The other end of the string is attached to a fixed point Q on the table such that $PQ = 80$ cm .

a Given that the tension in the string is 175 N, find, in newtons, the modulus of elasticity of the string.

The mid-point of the string is pulled a distance 30 cm along the perpendicular bisector of PQ in the plane of the table.

b Find the increase of tension in the string.

c Find, in joules, the corresponding increase in the elastic energy of the string. **[E]**

Solution:

a Hooke's law $T = \frac{\lambda x}{l}$

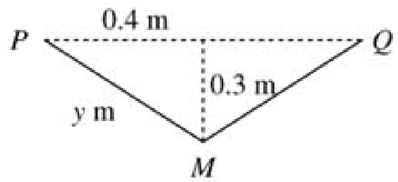
$175 = \frac{\lambda \times 0.5}{0.3}$

$\lambda = 175 \times \frac{0.3}{0.5} = 105$

The modulus of elasticity of the string is 105 N

It is always a good idea to convert all units to base SI units. Here $30 \text{ cm} = 0.3 \text{ m}$, and the extension $50 \text{ cm} = 0.5 \text{ m}$. In this question, using centimetres could be very misleading as you will get the correct answers in parts **a** and **b** but the incorrect answer in part **c**, where the extension is squared.

b



Let the mid-point of the string be M and $PM = y$ m.

$y^2 = 0.4^2 + 0.3^2 = 0.25 \Rightarrow y = 0.5$

The new tension in the string is given by Hooke's law

$T = \frac{\lambda x}{l}$

$= \frac{105(0.5 - 0.15)}{0.15} = 245$

Here, the tension is calculated using half the string PM. You could use the whole of the string when the working is $T = \frac{105(1 - 0.3)}{0.3} = 245$.

The increase in the tension in the string is $(245 - 105)\text{ N} = 140\text{ N}$

c The initial energy, E_1 Joules, say, is given by

$E_1 = \frac{\lambda x^2}{2l} = \frac{105 \times (0.5)^2}{0.6} = 43.75$

The final energy, E_2 Joules, say, is given by

$E_2 = 2 \times \frac{\lambda x^2}{2l} = 2 \times \frac{105 \times (0.35)^2}{0.3} = 85.75$

The final energy has been calculated as twice the energy in half of the string PM.

The increase in the elastic energy of the string is $(85.75 - 43.75)\text{ J} = 42\text{ J}$

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Review Exercise 1
Exercise A, Question 30

Question:

A particle P of mass m is attached to one end of a light elastic string of length a and modulus of elasticity $\frac{1}{2}mg$. The other end of the string is fixed at a point A which is at a height $2a$ above a smooth horizontal table. The particle is held on the table with the string making an angle β with the table, where $\tan\beta = \frac{3}{4}$.

a Find the elastic energy stored in the string.

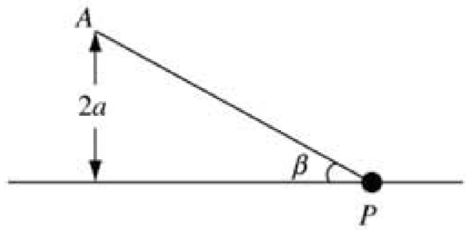
The particle is now released from rest. Assuming that P remains on the table.

b Find the speed of P when the string is vertical.

By finding the vertical component of the tension in the string when P is on the table and AP makes an angle θ with the horizontal,

c show that the assumption that P remains in contact with the table is justified. **[E]**

Solution:



a $\tan\beta = \frac{3}{4} \Rightarrow \sin\beta = \frac{3}{5}$

$$\frac{2a}{AP} = \sin\beta$$

$$AP = \frac{2a}{\sin\beta} = \frac{2a}{\frac{3}{5}} = \frac{10a}{3}$$

The extension of the string is $\frac{10a}{3} - a = \frac{7a}{3}$

You find the extension of the string by calculating the length of AP using trigonometry and subtracting the natural length of the string a.

The energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l} = \frac{\frac{1}{2}mg\left(\frac{7a}{3}\right)^2}{2a} = \frac{49}{36}mga$$

b When the particle is vertically below A, the energy stored in the string, E' say, is given by

$$E' = \frac{\lambda x^2}{2l} = \frac{\frac{1}{2}mg(a)^2}{2a} = \frac{1}{4}mga$$

When P is vertically below A, $AP = 2a$ and the extension of the string is $2a - a = a$.

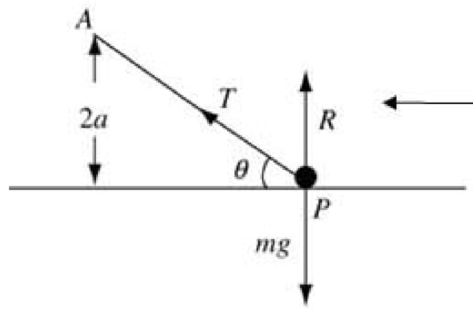
Let v m s^{-1} be the velocity of P when it is vertically below A.
Conservation of energy
kinetic energy gained = elastic energy lost

$$\frac{1}{2}mv^2 = \frac{49}{36}mga - \frac{1}{4}mga = \frac{10}{9}mga$$

$$v^2 = \frac{20ga}{9} \Rightarrow v = \frac{2}{3}\sqrt{(5ga)}$$

The speed of P when the string is vertical is $\frac{2}{3}\sqrt{(5ga)}$.

c



For P to remain in contact with the table, the normal reaction between P and the table, here called R, must remain positive throughout the motion. If $R = 0$, the particle loses contact with the table

$$\frac{2a}{AP} = \sin\theta \Rightarrow AP = \frac{2a}{\sin\theta}$$

The extension in the string is $\frac{2a}{\sin\theta} - a$

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{\frac{1}{2}mg\left(\frac{2a}{\sin\theta} - a\right)}{a}$$

$$= \frac{mg}{\sin\theta} - \frac{1}{2}mg$$

You must consider the forces on P at a general point in its motion. As P moves from its starting position to the point below A, $\sin\theta$ varies from $\frac{3}{5}$ to 1.

The vertical component of the tension, T_V say, is given by

$$T_V = T\sin\theta = \left(\frac{mg}{\sin\theta} - \frac{1}{2}mg\right)\sin\theta = mg - \frac{1}{2}mg\sin\theta$$

The question requires you to consider the vertical component of the tension in the string. You find an expression for the vertical component and then resolve vertically to obtain the normal reaction.

R(↑) $T_V + R = mg$

$$R = mg - T_v$$

$$= mg - \left(mg - \frac{1}{2}mg\sin\theta\right)$$

$$= \frac{1}{2}mg\sin\theta > 0$$

So P remains in contact with the table.

As $\sin\theta$ varies from $\frac{3}{5}$ to 1, the normal reaction R is always positive and so the assumption that P remains in contact with the table is justified.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 31

Question:

A light elastic string has natural length 4 m and modulus of elasticity 58.8 N. A particle P of mass 0.5 kg is attached to one end of the string. The other end of the string is attached to a fixed point A. The particle is released from rest at A and falls vertically.

a Find the distance travelled by P before it comes to instantaneous rest for the first time.

The particle is now held at a point 7 m vertically below A and released from rest.

b Find the speed of the particle when the string first becomes slack. **[E]**

Solution:

a When P comes to rest for the first time, let the extension of the string be x m

Conservation of energy

elastic energy gained = potential energy lost

$$\frac{\lambda x^2}{2l} = mgh$$

$$\frac{58.8x^2}{8} = 0.5 \times 9.8 \times (4+x)$$

$$7.35x^2 = 19.6 + 4.9x$$

$$3x^2 - 2x - 8 = 0$$

$$(x-2)(3x+4) = 0$$

$$x = 2$$

The distance fallen by P is $(4+2)\text{m} = 6\text{ m}$.

Divide this equation throughout by 2.45 and rearrange the terms. If you cannot see this simplification, you can use the quadratic formula but you would be expected to obtain an exact answer.

For the string to have elastic energy, it has to be stretched so you can ignore the negative solution $-\frac{4}{3}$.

b P will first become slack when it has moved 3 m vertically.

Let the velocity at this point be v m s^{-1}.

Conservation of energy

kinetic energy gained + potential energy gained = elastic energy lost

$$\frac{1}{2}mv^2 + mgh = \frac{\lambda x^2}{2l}$$

$$\frac{1}{2}0.5v^2 + 0.5 \times 9.8 \times 3 = \frac{58.8 \times 3^2}{8}$$

$$0.25v^2 + 14.7 = 66.15$$

$$v^2 = \frac{66.15 - 14.7}{0.25} = 205.8$$

$$v = \sqrt{(205.8)} = 14.345\ldots$$

Initially P is at rest and then rises 3 m. So both kinetic and potential energy are gained. Initially the string is stretched but, after rising 3 m, it is slack. So elastic energy is lost. By conservation of energy, the net gain of kinetic and potential energies must equal the elastic energy lost.

The speed of the particle when the string first becomes slack is 14 m s^{-1} (2 s.f.).

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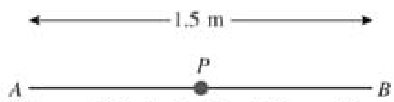
Review Exercise 1
Exercise A, Question 32

Question:

Two light elastic strings each have natural length 0.75 m and modulus of elasticity 49 N. A particle P of mass 2 kg is attached to one end of each string. The other ends of the strings are attached to fixed points A and B, where AB is horizontal and $AB = 1.5\text{ m}$.



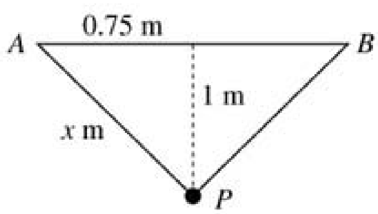
The particle is held at the mid-point of AB. The particle is released from rest, as shown in the figure.

a Find the speed of P when it has fallen a distance of 1 m.

b Given instead that P hangs in equilibrium vertically below the mid-point of AB with $\angle APB = 2\alpha$ show that $\tan\alpha + 5\sin\alpha = 5$. **[E]**

Solution:

a



When P has fallen 1 m, let $AP = x$ m.

$x^2 = 0.75^2 + 1^2 = 1.5625 \Rightarrow x = 1.25$

At this point the extension of the string AP is $(1.25 - 0.75)\text{m} = 0.5\text{ m}$

To find the elastic energy in the string, you need to calculate the extension in the string. First find AP (you could just use the 3, 4, 5 triangle) and then subtract the natural length, 0.75 m.

Let the velocity of P when it has fallen 1 m be $v\text{ m s}^{-1}$.

Conservation of energy

kinetic energy gained + elastic energy gained = potential energy lost

$$\frac{1}{2}mv^2 + 2\times\frac{\lambda x^2}{2l} = mgh$$

$$\frac{1}{2}2\times v^2 + 2\times\frac{49\times 0.5^2}{2\times 0.75} = 2\times 9.8\times 1$$

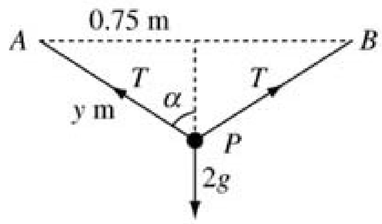
$$v^2 + \frac{49}{3} = 19.6 \Rightarrow v^2 = 3.26\dot{6}$$

$$v = 1.807\ldots$$

Both strings have elastic energy stored in them. By symmetry, the energy in both strings is the same.

The speed of P when it has fallen 1 m is 1.8 m s^{-1} (2 s.f.).

b



Let $AP = y$ m and the angle AP makes with the vertical be α.

By trigonometry

$$\sin\alpha = \frac{0.75}{y} \Rightarrow y = \frac{0.75}{\sin\alpha}$$

R(↑) $2T\cos\alpha = 2g$

$$T = \frac{g}{\cos\alpha} = \frac{9.8}{\cos\alpha}$$

Hooke's law

$$T = \frac{\lambda x}{l}$$

$$= \frac{49}{0.75}(y - 0.75) = \frac{49}{0.75}\left(\frac{0.75}{\sin\alpha} - 0.75\right)$$

$$= 49\left(\frac{1}{\sin\alpha} - 1\right) = 49\left(\frac{1-\sin\alpha}{\sin\alpha}\right)$$

You find two separate expression for T, one by resolving vertically and the other from Hooke's law. Equating the two expressions gives you an equation in α.

$$\frac{9.8}{\cos\alpha} = 49\left(\frac{1-\sin\alpha}{\sin\alpha}\right)$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{49}{9.8}(1-\sin\alpha) = 5(1-\sin\alpha)$$

$$\tan\alpha = 5 - 5\sin\alpha$$

$\tan\alpha + 5\sin\alpha = 5$, as required.

Eliminating T gives an equation in α. You have to manipulate this equation to obtain the printed answer.

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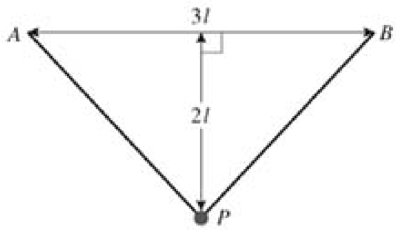
Review Exercise 1
Exercise A, Question 33

Question:



A light elastic string, of natural length $3l$ and modulus of elasticity λ, has ends attached to two points A and B where $AB = 3l$ and AB is horizontal. A particle P of mass m is attached to the mid-point of the string. Given that P rests in equilibrium at a distance $2l$ below AB, as shown in the figure,

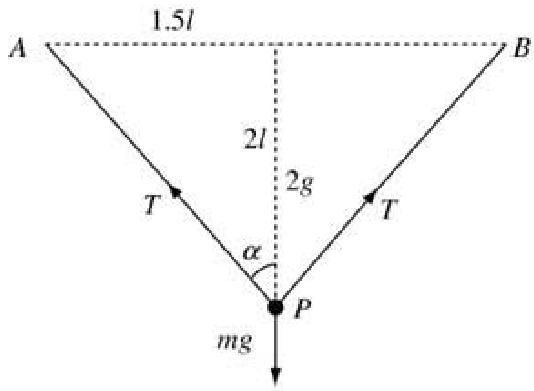
a show that $\lambda = \frac{15mg}{16}$.

The particle is pulled vertically downwards from its equilibrium position until the total length of the elastic string is $7.8l$. The particle is released from rest.

b Show that P comes to instantaneous rest on the line AB. **[E]**

Solution:

a



$AP^2 = (1.5l)^2 + (2l)^2 = 6.25l^2 \Rightarrow AP = 2.5l$

It is acceptable just to write down $AP = 2.5l$, using the 3, 4, 5 triangle.

Let α be the angle between AP and the vertical.

$\cos\alpha = \frac{2l}{2.5l} = \frac{4}{5}$

The extension of half of the string, AP, is $2.5l - 1.5l = l$

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}}$$

$$= \frac{\lambda l}{1.5l} = \frac{2\lambda}{3} \quad ①$$

You find two equations in T and λ by resolving vertically and using Hooke's law. Eliminating T between the two equations gives λ.

R(↑) $2T\cos\alpha = mg$

$$2T \times \frac{4}{5} = mg$$

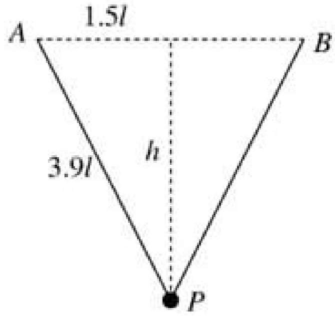
$$T = \frac{5mg}{8} \quad ②$$

Eliminating T between ① and ②

$$\frac{2\lambda}{3} = \frac{5mg}{8}$$

$\lambda = \frac{5mg}{8} \times \frac{3}{2} = \frac{15mg}{16}$, as required.

b



Let the perpendicular distance from the original position of P to AB be h.

$h^2 = (3.9l)^2 - (1.5l)^2 = 12.96l^2 \Rightarrow h = 3.6l$

Let the speed of P as it reaches AB be v m s^{-1}

> To prove that the speed of P at AB is zero, the speed of P at AB is taken as v m s^{-1}. You then use conservation of energy to obtain an equation for v. You complete the proof by solving the equation for v and showing the solution is zero.

Conservation of energy

kinetic energy gained + potential energy gained = elastic energy lost

$$\frac{1}{2}mv^2 + mgh = 2 \times \frac{\lambda x^2}{2 \times \text{natural length}}$$

> Each of the two halves of the string have natural length $1.5l$ and extension $2.4l$.

$$\frac{1}{2}mv^2 + mg \times 3.6l = 2 \times \frac{\left(\frac{15mg}{16}\right)(3.9l - 1.5l)^2}{2 \times 1.5l}$$

$$\frac{1}{2}mv^2 + 3.6mgl = \frac{5mg}{8l} \times (2.4l)^2 = 3.6mgl$$

Hence $\frac{1}{2}mv^2 = 0 \Rightarrow v = 0$

P comes to instantaneous rest on the line AB, as required.

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Review Exercise 1
Exercise A, Question 34

Question:

A particle P, of mass m, is attached to one end of a light elastic string, of natural length l and modulus of elasticity $8mg$. The other end of the string is attached at a point A to a horizontal ceiling which is at a height $2l$ above a horizontal floor. The particle P is held at A and projected vertically downwards with speed u.

a Find the least possible value of u for P to reach the floor.

Given that $u^2 = 16gl$ and that when P strikes the floor its speed is halved, find

b the speed of P when it hits the ceiling after striking the floor once,

c the maximum speed of P during its motion. **[E]**

Solution:

a Conservation of energy

elastic energy gained = potential energy lost + kinetic energy lost

$$\frac{\lambda x^2}{2l} = mgh + \frac{1}{2}mu^2$$

$$\frac{8mg \times l^2}{2l} = mg \times 2l + \frac{1}{2}mu^2$$

$$4mgl = 2mgl + \frac{1}{2}mu^2$$

$$\frac{1}{2}mu^2 = 2mgl \Rightarrow u^2 = 4gl$$

$$u = 2\sqrt{(gl)}$$

The least value of u occurs when P just reaches the floor. That is when the speed of P becomes zero just as it reaches the floor. At that point, P has lost all its kinetic energy $\left(\frac{1}{2}mu^2\right)$, P has fallen a distance $2l$ and the extension of the string is l.

The least possible value of u to reach the floor is $2\sqrt{(gl)}$.

b Let the speed of P as it strikes the floor be v m s^{-1}

Conservation of energy

elastic energy gained = potential energy lost + kinetic energy lost

$$\frac{\lambda x^2}{2l} = mgh + \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$4mgl = 2mgl + \frac{1}{2}m(16gl) - \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = 2mgl + 8mgl - 4mgl = 6mgl$$

$$v^2 = 12gl \Rightarrow v = 2\sqrt{(3gl)}$$

In part **b**, you first find the speed of P immediately before it strikes the ground. In the downward motion, the gain in elastic energy and the loss in potential energy are the same as in part **a**.

P rebounds from the floor with speed, v', say, where $v' = \sqrt{(3gl)}$

As P strikes the ground, its velocity is halved.

Let the speed with which P returns to the ceiling be w m s^{-1}.

Conservation of energy

elastic energy lost + kinetic energy lost = potential energy gained

$$\frac{\lambda x^2}{2l} + \frac{1}{2}mv'^2 - \frac{1}{2}mw^2 = mgh$$

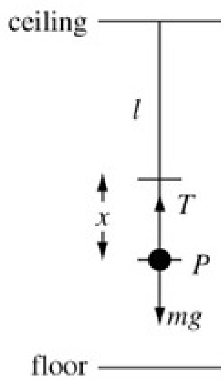
$$4mgl + \frac{1}{2}m(3gl) - \frac{1}{2}mw^2 = 2mgl$$

$$\frac{1}{2}mw^2 = 4mgl + \frac{3}{2}mgl - 2mgl = \frac{7}{2}mgl$$

$$w^2 = 7gl \Rightarrow w = \sqrt{(7gl)}$$

Now you use conservation of energy for the motion of P as it moves from the floor to the ceiling. As P rises, elastic and kinetic energy are lost; potential energy is gained.

The speed of P when it hits the ceiling after striking the floor once is $\sqrt{(7gl)}$.

c 

The maximum speed occurs when the acceleration is zero. Let the extension of the string at this point be x.

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{8mgx}{l}$$

$$R(\downarrow)\ mg - T = m \times 0$$

$$mg - \frac{8mgx}{l} = 0$$

$$x = \frac{l}{8}$$

The maximum value of v is when $a = \frac{dv}{dt} = 0$. You use Newton's second law and Hooke's law to find the extension of the string when the maximum occurs.

Let the maximum speed of P be V m s^{-1}.

Considering the motion of P from the ceiling to the point where $x = \frac{l}{8}$.

kinetic energy gained + elastic energy gained = potential energy lost

$$\frac{1}{2}mV^2 - \frac{1}{2}mu^2 + \frac{\lambda x^2}{2l} = mg(l + x)$$

$$\frac{1}{2}mV^2 - \frac{1}{2}m(16gl) + \frac{8mg\left(\frac{l}{8}\right)^2}{2l} = mg \times \frac{9l}{8}$$

$$\frac{1}{2}mV^2 - 8mgl + \frac{mgl}{16} = \frac{9mgl}{8}$$

$$\frac{1}{2}mV^2 = 8mgl - \frac{mgl}{16} + \frac{9mgl}{8} = \frac{145mgl}{16}$$

$$V^2 = \frac{145gl}{8} \Rightarrow V = \left(\frac{145gl}{8}\right)^{\frac{1}{2}}$$

You consider the fall of P from the point where it first leaves the ceiling to the point where the maximum speed is reached. In that motion kinetic energy and elastic energy are gained, and potential energy is lost.

The maximum speed of P during its motion is $\left(\frac{145gl}{8}\right)^{\frac{1}{2}}$.

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Review Exercise 1
Exercise A, Question 35

Question:

A particle P of mass 0.2 kg moves away from the origin along the positive x-axis. It moves under the action of a force directed away from the origin O of magnitude $\frac{5}{x+1}$ N, where $OP = x$ m. Given that the speed of P is 5 m s^{-1} when $x = 0$, find the value of x, to 3 significant figures, when the speed of P is 15 m s^{-1}.

Solution:

$$\mathbf{F} = m\mathbf{a}$$
$$\frac{5}{x+1} = 0.2a$$
$$a = \frac{5}{0.2(x+1)} = \frac{25}{x+1}$$

You need to remember that $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{25}{x+1}$$

You integrate both sides of this equation with respect to x. It is important to include a constant of integration. Most questions include information which enables you to find the constant.

$$\frac{1}{2}v^2 = \int \frac{25}{x+1}\,\mathrm{d}x = 25\ln(x+1) + A$$
$$v^2 = 50\ln(x+1) + B, \text{ where } B = 2A$$

At $x = 0, v = 5$

You use the information that $v = 5$ when $x = 0$ to evaluate the constant.

$$25 = 50\ln 1 + B \Rightarrow B = 25$$

Hence $v^2 = 50\ln(x+1) + 25$

When $v = 15$

$$15^2 = 50\ln(x+1) + 25$$
$$\ln(x+1) = \frac{225-25}{50} = 4$$

Take exponentials of both sides of this equation and use $e^{\ln(x+1)} = x+1$.

$$x + 1 = e^4$$
$$x = e^4 - 1 = 53.6 \text{ (3 s.f.)}$$

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Review Exercise 1
Exercise A, Question 36

Question:

A particle P of mass 2.5 kg moves along the positive x-axis. It moves away from a fixed origin O, under the action of a force directed away from O. When $OP = x$ metres, the magnitude of the force is $2e^{-0.1x}$ N and the speed of P is v ms^{-1}. When $x = 0, v = 2$, Find

a v^2 in terms of x,

b the value of x when $v = 4$.

c Give a reason why the speed of P does not exceed $\sqrt{20}$ m s^{-1}.

Solution:

a $\mathbf{F} = ma$

$2e^{-0.1x} = 2.5a$

$a = \frac{2}{2.5}e^{-0.1x} = 0.8e^{-0.1x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.8e^{-0.1x}$

Using $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

$\frac{1}{2}v^2 = \int 0.8e^{-0.1x}\,dx = \frac{0.8e^{-0.1x}}{-0.1} + A$

$= -8e^{-0.1x} + A$

$v^2 = B - 16e^{-0.1x}$, where $B = 2A$

Twice one arbitrary constant is another arbitrary constant.

When $x = 0, v = 2$

At $x = 0, e^{-0.1x} = e^0 = 1$.

$4 = B - 16 \Rightarrow B = 20$

Hence $v^2 = 20 - 16e^{-0.1x}$

b When $v = 4$

$16 = 20 - 16e^{-0.1x}$

$e^{-0.1x} = \frac{20-16}{16} = \frac{1}{4}$

Take logarithms of both sides of this equation and use $\ln(e^{-0.1x}) = -0.1x$.

$-0.1x = \ln\left(\frac{1}{4}\right) = -\ln 4$

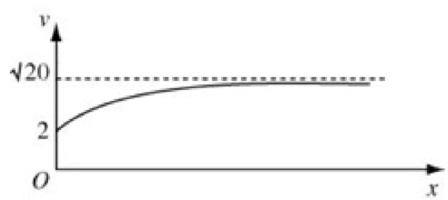
$x = 10\ln 4$

c For all x, $e^{-0.1x} > 0$

So $v^2 = 20 - 16e^{-0.1x} < 20$

and hence $v < \sqrt{20}$

The speed of P does not exceed $\sqrt{20}\text{ m s}^{-1}$.



This sketch, of v against x, shows that as x increases, the velocity of P gets closer to $\sqrt{20}\text{ m s}^{-1}$ but never reaches it. $\sqrt{20}\text{ m s}^{-1}$ is the **terminal** or **limiting speed** of P.

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Review Exercise 1
Exercise A, Question 37

Question:

A toy car of mass 0.2 kg is travelling in a straight line on a horizontal floor. The car is modelled as a particle. At time $t = 0$ the car passes through a fixed point O. After t seconds the speed of the car is v m s^{-1} and the car is at a point P with $OP = x$ metres. The resultant force on the car is modelled as $\frac{1}{10}x(4-3x)$N in the direction OP. The car comes to instantaneous rest when $x = 6$. Find

a an expression for v^2 in terms of x,

b the initial speed of the car. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$$\frac{1}{10}x(4-3x) = 0.2a$$

$$a = \frac{1}{0.2\times 10}x(4-3x) = \frac{1}{2}x(4-3x) = 2x - \frac{3x^2}{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 2x - \frac{3x^2}{2}$$

Integrate both sides of this equation with respect to x. Remember to include a constant of integration.

$$\frac{1}{2}v^2 = \int\left(2x - \frac{3x^2}{2}\right)\mathrm{d}x = x^2 - \frac{x^3}{2} + A$$

$$v^2 = 2x^2 - x^3 + B, \text{ where } B = 2A$$

At $x = 6, v = 0$

$0 = 2\times 36 - 216 + B \Rightarrow B = 144$

Hence $v^2 = 2x^2 - x^3 + 144$

The car comes to instantaneous rest when $x = 6$. So $v = 0$ at $x = 6$.

b When $x = 0$

$v^2 = 144 \Rightarrow v = \pm 12$

The initial speed of the car is 12 m s^{-1}.

Both signs are possible as the car could pass through O in either direction when $t = 0$. However, in either case, the speed of the car, which is the magnitude of the velocity, is 12 m s^{-1}.

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Review Exercise 1
Exercise A, Question 38

Question:

A particle P of mass 0.6 kg is moving along the positive x-axis under the action of a force which is directed away from the origin O. At time t seconds, the force has magnitude $3e^{-0.5t}$ N. When $t = 0$, the particle P is at O and moving with speed 2 m s^{-1} in the direction of x increasing. Find

a the value of t when the speed is 8 m s^{-1},

b the distance of P from O when $t = 2$. **[E]**

Solution:

a $F = ma$

$$3e^{-0.5t} = 0.6a$$

$$a = \frac{3e^{-0.5t}}{0.6} = 5e^{-0.5t}$$

$$\frac{dv}{dt} = 5e^{-0.5t}$$

$$v = \int 5e^{-0.5t}\,dt = \frac{5e^{-0.5t}}{-0.5} + A = A - 10e^{-0.5t}$$

When the acceleration is a function of time, you use $a = \frac{dv}{dt}$. When the acceleration is a function of distance, you can use $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

When $t = 0, v = 2$

$2 = A - 10 \Rightarrow A = 12$

Hence $v = 12 - 10e^{-0.5t}$

When $v = 8$

$$8 = 12 - 10e^{-0.5t} \Rightarrow e^{-0.5t} = \frac{12-8}{10} = \frac{2}{5}$$

$$-0.5t = \ln\left(\frac{2}{5}\right)$$

$$t = -2\ln\left(\frac{2}{5}\right) = 2\ln\left(\frac{5}{2}\right)$$

No particular form of the answer is asked for in the question and an approximate answer, such as $t = 1.83$, would be accepted.

b $v = \frac{dx}{dt} = 12 - 10e^{-0.5t}$

$$x = \int (12 - 10e^{-0.5t})\,dt = 12t + 20e^{-0.5t} + B$$

When $t = 0, x = 0$

$$0 = 0 + 20 + B \Rightarrow B = -20$$

$$x = 12t + 20e^{-0.5t} - 20$$

Using $e^0 = 1$. Carelessly writing $e^0 = 0$ is a common error.

When $t = 2$

$$x = 24 + 20e^{-1} - 20 = 4 + 20e^{-1}$$

The distance of P from O when $t = 2$ is $(4 + 20e^{-1})$ m.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 39

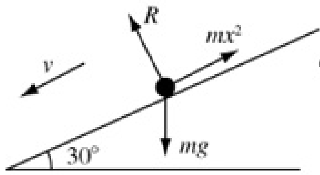
Question:

A particle P of mass m kg slides from rest down a smooth plane inclined at $30°$ to the horizontal. When P has moved a distance x metres down the plane, the resistance to motion from non-gravitational forces has magnitude mx^2 N. Find

a the speed of P when $x = 2$,
b the distance P has moved when it comes to rest for the first time. **[E]**

Solution:

a



Resistance opposes motion, so as P moves down the plane, resistance acts up the plane.

$$R(\swarrow) \quad \mathbf{F} = ma$$

$$mg\sin 30^\circ - mx^2 = ma$$

$$a = g\sin 30^\circ - x^2$$

Divide this equation throughout by m.

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}g - x^2$$

Using $\sin 30^\circ = \frac{1}{2}$.

$$\frac{1}{2}v^2 = \int\left(\frac{1}{2}g - x^2\right)dx = \frac{1}{2}gx - \frac{x^3}{3} + A$$

$$v^2 = gx - \frac{2x^3}{3} + B, \text{ where } B = 2A$$

At $x = 0, v = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

P starts from rest and so at $x = 0, v = 0$.

Hence $v^2 = gx - \frac{2x^3}{3}$

When $x = 2$

$$v^2 = 9.8 \times 2 - \frac{16}{3} = 14.2\dot{6}$$

$$v = \sqrt{14.2\dot{6}} = 3.777\ldots$$

As a numerical value of g has been used, you should round your answer to 2 or 3 significant figures.

The speed of P when $x = 2$ is 3.78 m s^{-1} (3 s.f.)

b $v^2 = gx - \frac{2x^3}{3}$

When $v = 0$

$$0 = gx - \frac{2x^3}{3} = x\left(g - \frac{2x^2}{3}\right)$$

$x = 0$ is the starting position and this root of the equation can be ignored.

P comes to rest when

$$g - \frac{2x^2}{3} = 0 \Rightarrow x^2 = \frac{3g}{2} = 14.7$$

$$x = \sqrt{14.7} = 3.834\ldots$$

The distance moved before P first comes to rest is 3.83 m (3 s.f.).

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Review Exercise 1
Exercise A, Question 40

Question:

Above the Earth's surface, the magnitude of the force on a particle due to the Earth's gravity is inversely proportional to the square of the distance of the particle from the centre of the Earth. Assuming that the Earth is a sphere of radius R, and taking g as the acceleration due to gravity at the surface of the Earth,

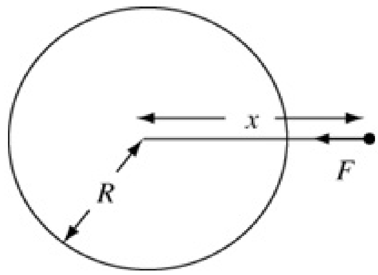
a prove that the magnitude of the gravitational force on a particle of mass m when it is a distance x (where $x \geq R$) from the centre of the Earth is $\frac{mgR^2}{x^2}$.

A particle is fired vertically upwards from the surface of the Earth with initial speed u, where $u^2 = \frac{3}{2}gR$. Ignoring air resistance.

b Find, in terms of g and R, the speed of the particle when it is at a height $2R$ above the surface of the Earth. **[E]**

Solution:

a



The gravitational force is directed towards the centre of the Earth and so is in the direction of x decreasing.

As $F \propto \frac{1}{x^2}$, $F = -\frac{k}{x^2}$

At $x = R, F = -mg$

$-mg = -\frac{k}{R^2} \Rightarrow k = mgR^2$

Hence $F = -\frac{mgR^2}{x^2}$

You introduce a constant of proportionality k and use the fact, that the force due to gravity at the surface of the Earth is known to have magnitude mg, to find k.

The magnitude of the force is $\frac{mgR^2}{x^2}$, as required.

b $\mathbf{F} = m\mathbf{a}$

$$-\frac{mgR^2}{x^2} = ma$$

$$a = -\frac{gR^2}{x^2}$$

In this equation, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -gR^2x^{-2}$$

$$\frac{1}{2}v^2 = -\int gR^2x^{-2}\,\mathrm{d}x = -\frac{gR^2x^{-1}}{-1} + A$$

$$v^2 = \frac{2gR^2}{x} + B, \text{ where } B = 2A$$

At $x = R, v^2 = \frac{3}{2}gR$

$$\frac{3}{2}gR = \frac{2gR^2}{R} + B \Rightarrow B = \frac{3}{2}gR - 2gR = -\frac{1}{2}gR$$

The question gives the velocity of the particle as it is fired from the surface of the Earth. That is the velocity when $x = R$, the radius of the Earth.

Hence $v^2 = \frac{2gR^2}{x} - \frac{1}{2}gR$

When $x = 3R$

$$v^2 = \frac{2gR^2}{3R} - \frac{1}{2}gR = \frac{gR}{6} \Rightarrow v = \sqrt{\left(\frac{gR}{6}\right)}$$

When the particle is at a height of $2R$ above the surface of the Earth, it is $2R + R = 3R$ from the centre of the Earth.

The speed of the particle when it is $2R$ above the surface of the Earth is $\sqrt{\left(\frac{gR}{6}\right)}$.

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Review Exercise 1
Exercise A, Question 41

Question:

A rocket is fired vertically upwards with speed U from a point on the Earth's surface. The rocket is modelled as a particle P of constant mass m, and the Earth as a fixed sphere of radius R. At a distance x from the centre of the Earth, the speed of P is v. The only force acting on P is directed towards the centre of the Earth and has magnitude $\frac{cm}{x^2}$, where c is a constant.

a Show that $v^2 = U^2 + 2c\left(\frac{1}{x} - \frac{1}{R}\right)$.

The kinetic energy of P at $x = R$ is half of the kinetic energy at $x = R$.

b Find c in terms of U and R. **[E]**

Solution:

a $\mathbf{F} = ma$

$$-\frac{cm}{x^2} = ma$$

In the equation of motion, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$$a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -\frac{c}{x^2} = -cx^{-2}$$

$$\frac{1}{2}v^2 = -\int cx^{-2}\,\mathrm{d}x = \frac{c}{x} + A$$

Using $\int x^{-2}\,\mathrm{d}x = \frac{x^{-1}}{-1} + k = -\frac{1}{x} + k$.

$$v^2 = \frac{2c}{x} + B, \text{ where } B = 2A$$

When $x = R, v = U$

$$U^2 = \frac{2c}{R} + B \Rightarrow B = U^2 - \frac{2c}{R}$$

Hence $v^2 = \frac{2c}{x} + U^2 - \frac{2c}{R}$

$$= U^2 + 2c\left(\frac{1}{x} - \frac{1}{R}\right), \text{ as required}$$

b At $x = R, v = U$ and the kinetic energy of P is $\frac{1}{2}mU^2$

At $x = 2R$, using the result of part **a**

$$v^2 = U^2 + 2c\left(\frac{1}{2R} - \frac{1}{R}\right) = U^2 + 2c\left(-\frac{1}{2R}\right)$$

$$v^2 = U^2 - \frac{c}{R}$$

and the kinetic energy of P is $\frac{1}{2}mv^2 = \frac{1}{2}m\left(U^2 - \frac{c}{R}\right)$

(kinetic energy at $x = 2R$) $= \frac{1}{2}$ (kinetic energy at $x = R$)

$$\frac{1}{2}m\left(U^2 - \frac{c}{R}\right) = \frac{1}{2}\left(\frac{1}{2}mU^2\right)$$

Divide this equation throughout by $\frac{1}{2}m$ and then make c the subject of the formula.

$$U^2 - \frac{c}{R} = \frac{1}{2}U^2$$

$$\frac{1}{2}U^2 = \frac{c}{R}$$

$$c = \frac{1}{2}RU^2$$

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Review Exercise 1
Exercise A, Question 42

Question:

A projectile P is fired vertically upwards from a point on the Earth's surface. When P is at a distance x from the centre of the Earth its speed is v. Its acceleration is directed towards the centre of the Earth and has magnitude $\frac{k}{x^2}$, where k is a constant. The Earth is assumed to be a sphere of radius R.

a Show that the motion of P may be modelled by the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{gR^2}{x^2}$$

The initial speed of P is U, where $U^2 < 2gR$. The greatest distance of P from the centre of the Earth is X.

b Find X in terms of U, R and g. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$-\frac{k}{x^2} = ma \quad ①$

$a = -\frac{k}{mx^2}$

At $x = R, a = -g$

$-g = -\frac{k}{mR^2}$

$k = mgR^2$

The force is negative in equation ① as the force on P due to gravity is directed towards the centre of the Earth and that is the direction of x decreasing.

You know that the acceleration due to gravity at the surface of the Earth is g and that the direction of the acceleration is towards the centre of the Earth. Substituting $a = -g$ into ① gives you k in terms of m, g and R.

Substituting $k = mgR^2$ into ①

$-\frac{mgR^2}{x^2} = ma$

$a = v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{gR^2}{x^2}$, as required.

$a = v\frac{\mathrm{d}v}{\mathrm{d}x}$ is one of the alternative forms of the acceleration.
$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = v\frac{\mathrm{d}v}{\mathrm{d}x}$ and you must pick out the form of a which you need in any particular question.

b Separating the variables in the printed answer to part **a** and integrating

$\int v\,\mathrm{d}v = -\int \frac{gR^2}{x^2}\,\mathrm{d}x = -\int gR^2x^{-2}\,\mathrm{d}x$

$\frac{1}{2}v^2 = \frac{-gR^2x^{-1}}{-1} + A$

$v^2 = \frac{2gR^2}{x} + B$, where $B = 2A$

At $x = R, v = U$

$U^2 = \frac{2gR^2}{R} + B \Rightarrow B = U^2 - 2gR$

The projectile is fired from a point on the Earth's surface with speed U. This gives you that at $x = R, v = U$.

Hence $v^2 = \frac{2gR^2}{x} + U^2 - 2gR$

When $v = 0, x = X$

$0 = \frac{2gR^2}{X} + U^2 - 2gR$

$0 = 2gR^2 + U^2X - 2gRX$

$X(2gR - U^2) = 2gR^2$

$X = \frac{2gR^2}{2gR - U^2}$

Multiply this equation throughout by X and then make X the subject of the formula.

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Review Exercise 1
Exercise A, Question 43

Question:

A car of mass 800 kg moves along a horizontal straight road. At time t seconds, the resultant force on the car has magnitude $\frac{48\,000}{(t+2)^2}$ N, acting in the direction of motion of the car. When $t = 0$, the car is at rest.

a Show that the speed of the car approaches a limiting value as t increases and find this value.

b Find the distance moved by the car in the first 6s of its motion. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$$\frac{48\,000}{(t+2)^2} = 600a$$

$$a = \frac{dv}{dt} = \frac{60}{(t+2)^2} = 60(t+2)^{-2}$$

When the acceleration is a function of t, the velocity can be found by writing $a = \frac{dv}{dt}$ and integrating with respect to t.

$$v = \int 60(t+2)^{-2}\,dt = \frac{60(t+2)^{-1}}{-1} + A$$

$$= A - \frac{60}{t+2}$$

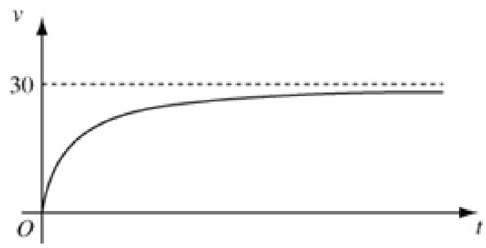
When $t = 0, v = 0$

$$0 = A - \frac{60}{2} \Rightarrow A = 30$$

Hence $v = 30 - \frac{60}{t+2}$

As $t \to \infty, \frac{60}{t+2} \to 0$ and $v \to 30$

As t increases, the car approaches a limiting speed of 30 m s^{-1}.



As the value of t increases, the value of $\frac{60}{t+2}$ decreases and so $30 - \frac{60}{t+2}$ gets closer to 30. The graph of v against t approaches $v = 30$ as an asymptote.

b The distance moved in the first 6 s is given by

$$x = \int_0^6 \left(30 - \frac{60}{t+2}\right)dt$$

$$= \left[30t - 60\ln(t+2)\right]_0^6$$

$$= (180 - 60\ln 8) - (0 - 60\ln 2)$$

$$= 180 - 60\ln 2^3 + 60\ln 2$$

$$= 180 - 180\ln 2 + 60\ln 2$$

$$= 180 - 120\ln 2$$

The car is always travelling in the same direction. It does not turn round and so the distance moved in the interval $0 \le t \le 6$ can be found by evaluating the definite integral $\int_0^6 v\,dt$.

The distance moved by the car in the first 6 s of its motion is $(180 - 120\ln 2)$ m.

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Review Exercise 1
Exercise A, Question 44

Question:

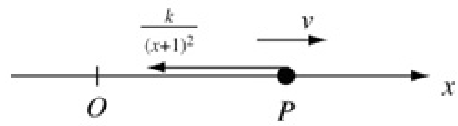
A particle P of mass $\frac{1}{3}$ kg moves along the positive x-axis under the action of a single force. The force is directed towards the origin O and has magnitude $\frac{k}{(x+1)^2}$ N, where $OP = x$ metres and k is a constant. Initially P is moving away from O.
At $x = 1$ the speed of P is 4 m s^{-1}, and at $x = 8$ the speed of P is $\sqrt{2} \text{ m s}^{-1}$.

a Find the value of k.
b Find the distance of P from O when P first comes to instantaneous rest. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$$-\frac{k}{(x+1)^2} f = \frac{1}{3}a$$



The particle is moving along the positive x-axis and the force is directed toward O. So the force is in the direction of x decreasing and force has a negative sign in this equation.

$$a = -\frac{3k}{(x+1)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -3k(x+1)^{-2}$$

$$\frac{1}{2}v^2 = \frac{-3k(x+1)^{-1}}{-1} + A = \frac{3k}{x+1} + A$$

$$v^2 = \frac{6k}{x+1} + B, \text{ where } B = 2A$$

At $x = 1, v = 4$

$$16 = \frac{6k}{2} + B \Rightarrow 3k + B = 16 \quad ①$$

At $x = 8, v = \sqrt{2}$

$$2 = \frac{6k}{9} + B \Rightarrow \frac{2}{3}k + B = 2 \quad ②$$

The information in the question gives you the values of v at two values of x and you use the information to obtain two simultaneous equations, which you solve.

① − ②

$$3k - \frac{2}{3}k = \frac{7}{3}k = 14$$

$$k = 14 \times \frac{3}{7} = 6$$

b Substituting $k = 6$ into ①

$$18 + B = 16 \Rightarrow B = -2$$

Hence $v^2 = \dfrac{36}{x+1} - 2$

To find the value of x for which P comes to rest, substitute $v = 0$ into this equation and solve for x.

When $v = 0$

$$0 = \frac{36}{x+1} - 2 \Rightarrow 2(x+1) = 36$$

$$2x + 2 = 36 \Rightarrow x = \frac{36-2}{2} = 17$$

The distance of P from O when P first comes to instantaneous rest is 17 m.

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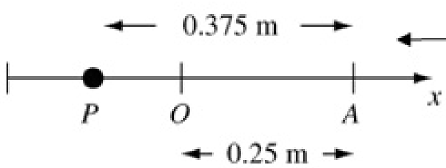
Review Exercise 1
Exercise A, Question 45

Question:

A particle P moves in a straight line with simple harmonic motion about a fixed centre O with period 2 s. At time t seconds the speed of P is v m s^{-1}. When $t = 0$, $v = 0$ and P is at a point A where $OA = 0.25$ m. Find the smallest positive value of t for which $AP = 0.375$ m. **[E]**

Solution:



The motion is simple harmonic with amplitude, a m, given by $a = 0.25$.

At P, $x = 0.25 - 0.375 = -0.125$

When $AP = 0.375$ m, P is 0.125 m from the centre of oscillation O. It is the other side of O from A and, if OA is taken as the direction of x increasing, the displacement of P is -0.125 m.

The period is 2 s

Hence $T = \frac{2\pi}{\omega} = 2 \Rightarrow \omega = \pi$

$x = a\cos\omega t$

$-0.125 = 0.25\cos\pi t$

$\cos\pi t = -\frac{1}{2}$

If the time, t seconds, is measured from the time when the velocity is zero, that is when the distance of P from O is the amplitude, then $x = a\cos\omega t$ is the appropriate formula connecting the displacement with the time.

The smallest positive value of t is given by

$\pi t = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$t = \frac{2}{3}$

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Review Exercise 1
Exercise A, Question 46

Question:

A particle P of mass 0.2 kg oscillates with simple harmonic motion between the points A and B, coming to rest at both points. The distance AB is 0.2 m, and P completes 5 oscillations every second.

a Find, to 3 significant figures, the maximum resultant force exerted on P.

When the particle is at A, it is struck a blow in the direction BA. The particle now oscillates with simple harmonic motion with the same frequency as previously but twice the amplitude.

b Find, to 3 significant figures, the speed of the particle immediately after it has been struck. **[E]**

Solution:

a If P completes 5 oscillations in one second,

then P takes $\frac{1}{5}$ s to complete one oscillation.

$T = \frac{2\pi}{\omega} = \frac{1}{5} \Rightarrow \omega = 10\pi$

The acceleration $\ddot{x}$ m s^{-2} is given by

$\ddot{x} = -\omega^2 x$

In many other topics in Mechanics, it is usual to use the symbol a for the acceleration. With simple harmonic motion, a is often used for the amplitude and it is sensible to use another symbol. Here the calculus symbol $\ddot{x}$, for the acceleration is used.

The amplitude of the oscillation is

$\frac{0.2}{2}$ m $= 0.1$ m

The amplitude is half of the distance between the extreme points of the oscillation.

The greatest magnitude of the acceleration is given by

$|\ddot{x}| = \omega^2 a = (10\pi)^2 \times 0.1 = 10\pi^2$

The maximum magnitude of the force is given by

$|\mathbf{F}| = |m\mathbf{a}| = m\omega^2 a$

$= 0.2 \times 10\pi^2 = 2\pi^2 = 19.739\ldots$

The greatest magnitude of the acceleration, and hence the force of greatest magnitude, occurs when the displacement is the amplitude.

The magnitude of the greatest force exerted on P is 19.7 N (3 s.f.)

b $\omega = 10\pi, a = 2 \times 0.1 = 0.2$

$v^2 = \omega^2(a^2 - x^2)$

$= 100\pi^2(0.2^2 - 0.1^2) = 3\pi^2$

$v = \sqrt{3}\pi = 5.441\ldots$

The blow is struck when P is 0.1 m from the centre of oscillation. So $x = 0.1$.

The speed of P immediately after it has been struck is 5.44 m s^{-1} (3 s.f.)

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Review Exercise 1
Exercise A, Question 47

Question:

A piston P in a machine moves in a straight line with simple harmonic motion about a fixed centre O. The period of the oscillations is π s. When P is 0.5 m from O, its speed is 2.4 m s^{-1}. Find

a the amplitude of the motion,
b the maximum speed of P during its motion,
c the maximum magnitude of the acceleration of P during the motion,
d the total time, in seconds to 2 decimal places, in each complete oscillation for which the speed of P is greater than 2.4 m s^{-1}. **[E]**

Solution:

a $T = \frac{2\pi}{\omega} = \pi \Rightarrow \omega = 2$

When $x = 0.5, v = 2.4$

$$v^2 = \omega^2(a^2 - x^2)$$
$$2.4^2 = 2^2(a^2 - 0.5^2)$$
$$a^2 - 0.25 = \frac{2.4^2}{2^2} = 1.44$$
$$a^2 = 1.69 \Rightarrow a = 1.3$$

The amplitude of the motion is 1.3 m.

b The maximum speed is given by
$v = \omega a = 2 \times 1.3 = 2.6$
The maximum speed of P during its motion is 2.6 m s^{-1}.

As $v^2 = \omega^2(a^2 - x^2)$ and x^2 is positive for all x, the greatest value of v^2 is at $x = 0$. So the greatest value of v^2 is $\omega^2 a^2$ and the greatest value of the speed is ωa.

c The maximum magnitude of the acceleration is given by
$|\ddot{x}| = |\omega^2 a| = 4 \times 1.3 = 5.2$

The maximum magnitude of the acceleration is 5.2 m s^{-2}.

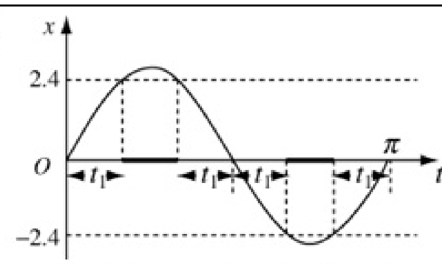
The acceleration is given by $\ddot{x} = -\omega^2 x$ and this has the greatest size when x is the amplitude.

d $x = a\cos\omega t$
Differentiating with respect to t

$$\dot{x} = -a\omega\sin\omega t$$
$$|\dot{x}| = |a\omega\sin\omega t|$$
$$2.4 = 1.3 \times 2\sin 2t_1 = 2.6\sin 2t_1$$
$$\sin 2t_1 = \frac{12}{13}$$
$$2t_1 = \arcsin\left(\frac{12}{13}\right) = 1.176\ldots$$
$$t_1 = 0.588\ldots$$

The required time is given by
$T' = \pi - 4t_1 = \pi - 4 \times 0.588\ldots$
$= 0.789\ldots$

The time for which the speed is greater than 2.4 m is 0.79 s (2 d.p.).



This diagram illustrates that if t_1 is the smallest positive solution of $2.6\sin 2t_1 = 2.4$, the time for which the speed is greater than 2.4 is $\pi - 4t_1$.

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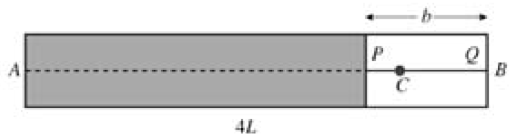
Review Exercise 1
Exercise A, Question 48

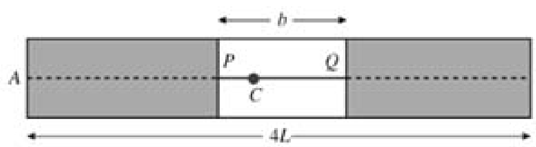
Question:



In a game at a fair, a small target C moves horizontally with simple harmonic motion between the points A and B, where $AB = 4L$. The target moves inside a box and takes 3 s to travel from A to B. A player has to shoot at C, but C is only visible to the player when it passes a window PQ where $PQ = b$. The window is initially placed with Q at the point shown in the figure above. The target takes 0.75 s to travel from Q to P.

a Show that $b = (2 - \sqrt{2})L$.

b Find the speed of C as it passes P.



For advanced players, the window PQ is moved to the centre of AB so that $AP = QB$, as shown in the second figure above.

c Find the time, in seconds to 2 decimal places, taken for C to pass from Q to P in this new position. **[E]**

Solution:

a The period of motion is 6 s.

$T = \frac{2\pi}{\omega}$

$6 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{3}$

> The time taken to move from A to B is half of a complete oscillation. So the period is $2 \times 3\text{ s} = 6\text{ s}$.

Measuring the time, t seconds, from an instant when C is at Q and the displacement from the centre of the oscillation, O say

$x = a\cos\omega t$

> The formula for the displacement in terms of time when time is measured from the instant when a particle is at the amplitude is $x = a\cos\omega t$. If time is measured from the instant when a particle is at the centre of oscillation, then the formula is $x = a\sin\omega t$. In part **a**, it is more efficient to use the formula with the cosine but in part **c**, the formula with the sine gives you a quicker solution.

The amplitude is $2L$ and $\omega = \frac{\pi}{3}$

After 0.75 s, C is at P.

$x = 2L\cos\left(\frac{\pi}{3} \times 0.75\right) = 2L\cos\frac{\pi}{4}$

$= 2L \times \frac{1}{\sqrt{2}} = \sqrt{2}L$

$b = a - x = 2L - \sqrt{2}L$

$= (2 - \sqrt{2})L$, as required

> From part **a**, when C is at P, its displacement from the centre of oscillation is $\sqrt{2}L$ so $x = \sqrt{2}L$.

b The speed of C at P is given by

$v^2 = \omega^2(a^2 - x^2)$

$= \left(\frac{\pi}{3}\right)^2((2L)^2 - (\sqrt{2}L)^2)$

$= \left(\frac{\pi}{3}\right)^2(4L^2 - 2L^2) = 2\left(\frac{\pi}{3}\right)^2 L^2$

$v = \frac{\sqrt{2}\pi L}{3}$

c If the window is centred the displacement of Q from the centre of oscillation is given by

$x = \frac{b}{2} = \frac{2 - \sqrt{2}}{2}L$

Measuring time, t seconds, from the centre of oscillation, at Q

$x = a\sin\omega t$

$\frac{2 - \sqrt{2}}{2}L = 2L\sin\left(\frac{\pi}{3}t\right)$

$\sin\left(\frac{\pi}{3}t\right) = \frac{2 - \sqrt{2}}{4} = 0.146\,446\ldots$

$\frac{\pi}{3}t = 0.146\,975\ldots \Rightarrow t = 0.140\,35\ldots$

> In all SHM questions, it is assumed that angles are measured in radians. It is important that you make sure your calculator is in the correct mode.

The time from P to Q is given by

$T' = 2t = 0.2807\ldots$

The time taken for C to pass from P to Q is 0.28 (2 d.p.).

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Review Exercise 1
Exercise A, Question 49

Question:

The points O, A, B and C lie in a straight line, in that order, with $OA = 0.6\text{ m}$, $OB = 0.8\text{ m}$ and $OC = 1.2\text{ m}$. A particle P, moving in a straight line, has speed $\left(\frac{3}{10}\sqrt{3}\right)\text{m s}^{-1}$ at A, $\left(\frac{1}{5}\sqrt{5}\right)\text{m s}^{-1}$ at B and is instantaneously at rest at C.

a Show that this information is consistent with P performing simple harmonic motion with centre O.

Given that P is performing simple harmonic motion with centre O,

b show that the speed of P at O is 0.6 m s^{-1},

c find the magnitude of the acceleration of P as it passes A,

d find, to 3 significant figures, the time taken for P to move directly from A to B. **[E]**

Solution:

a At C $v^2 = \omega^2(a^2 - x^2)$

$0^2 = \omega^2(a^2 - 1.2^2) \Rightarrow a = 1.2$

At A $v^2 = \omega^2(a^2 - x^2)$

$$\left(\frac{3}{10}\sqrt{3}\right)^2 = \omega^2(1.2^2 - 0.6^2)$$

$$\frac{27}{100} = \omega^2 \times 1.08$$

$$\omega^2 = \frac{27}{108} = \frac{1}{4} \Rightarrow \omega = \frac{1}{2}$$

You show that the information is the question is consistent with SHM by taking the information you have been given about two of the points and using it to find the values of a and ω. You then confirm these values are correct using the information about the third point. As the information about C gives you a directly, it is sensible to start with that point.

Checking $a = 1.2$ and $\omega = \frac{1}{2}$ at B

$$v^2 = \omega^2(a^2 - x^2)$$

$$= \frac{1}{4}(1.2^2 - 0.8^2) = 0.2 = \frac{1}{5}$$

$$v = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{1}{5}\sqrt{5}$$

Using $a = 1.2$ and $\omega = \frac{1}{2}$, you find the speed of P at B. This calculation confirms the speed of P given in the question and you deduce that the information is consistent with P performing simple harmonic motion.

This is consistent with the information in the question. The information is consistent with P performing SHM with centre O.

b At O, $x = 0$. Using $v^2 = \omega^2(a^2 - x^2)$

$$= \frac{1}{4}(1.2^2 - 0^2) = 0.36$$

$$v = \sqrt{0.36} = 0.6$$

The speed of P at O is $0.6\ \text{m s}^{-1}$, as required.

c At A $\ddot{x} = -\omega^2 x = -\frac{1}{4} \times 0.6 = -0.15$

The magnitude of the acceleration at A is $0.15\ \text{m s}^{-2}$.

d At A $x = a\sin\omega t$

$$0.6 = 1.2\sin\frac{1}{2}t_1 \Rightarrow \sin\frac{1}{2}t_1 = \frac{1}{2}$$

$$\frac{1}{2}t_1 = \frac{\pi}{6} \Rightarrow t_1 = \frac{\pi}{3}$$

At B $x = a\sin\omega t$

$$0.8 = 1.2\sin\frac{1}{2}t_2 \Rightarrow \sin\frac{1}{2}t_2 = \frac{2}{3}$$

$$\frac{1}{2}t_2 = 0.729\,727\ldots \Rightarrow t_2 = 1.459\,455\ldots$$

$$t_2 - t_1 = 1.459\,455\ldots - \frac{\pi}{3} = 0.412\,257\ldots$$

In this question, as you need to find the difference between the times at which P is at A and B, it does not matter which of the formulae $x = a\sin\omega t$ or $x = a\cos\omega t$ you use. If you use the formula with cosine, you obtain $\frac{2\pi}{3}$ s and $1.6821\ldots$ s as the times. The difference between these times is again 0.412 (3 s.f.).

The time taken to move directly from A to B is 0.412 s (3 s.f.).

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Review Exercise 1
Exercise A, Question 50

Question:

The rise and fall of the water level in a harbour is modelled as simple harmonic motion. On a particular day the maximum and minimum depths of the water in the harbour are 10 m and 4 m and these occur at 1100 hours and 1700 hours respectively.

a Find the speed, in m h^{-1}, at which the water level in the harbour is falling at 1600 hours on this particular day.

b Find the total time, between 1100 hours and 2300 hours on this particular day, for which the depth in the harbour is less than 5.5 m. **[E]**

Solution:

a $a = \frac{10-4}{2} = 3$

The period of motion is 12 hours

The 6 hours, from 1100 to 1700, are half of a complete oscillation.

$T = \frac{2\pi}{\omega} = 12 \Rightarrow \omega = \frac{\pi}{6}$

At 1600 hours, $t = 5$

$x = a\cos\omega t$

$= 3\cos\frac{5\pi}{6} = 3\times\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$

$v^2 = \omega^2(a^2 - x^2)$

$= \left(\frac{\pi}{6}\right)^2\left(3^2 - \left(-\frac{3\sqrt{3}}{2}\right)^2\right) = \left(\frac{\pi}{6}\right)^2\left(9 - \frac{27}{4}\right)$

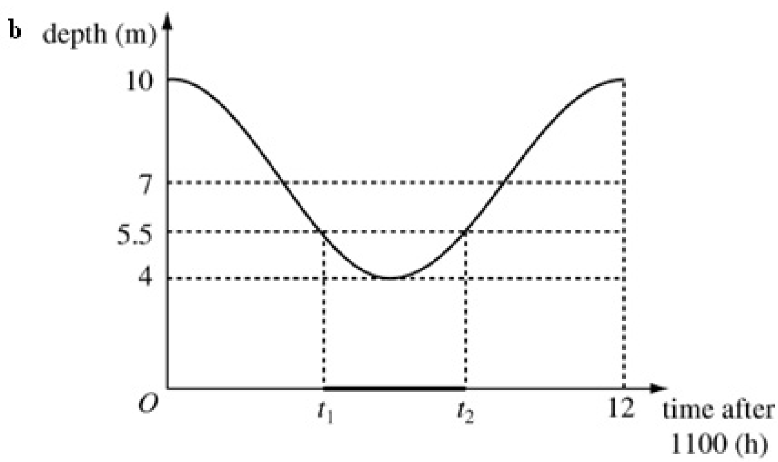
$= \left(\frac{\pi}{6}\right)^2 \times \frac{9}{4}$

$v = (-)\frac{\pi}{6}\times\frac{3}{2} = \frac{\pi}{4}$

The formulae for simple harmonic motion can be used with any consistent set of units. Here metres and hours are used.

At 1600, the water level is falling at a rate of $\frac{\pi}{4}$ m h^{-1}.

b



You find the times, here labelled t_1 and t_2, where the water is at a depth of 5.5 m. The diagram shows that the total time for which the depth of the water is less that 5.5 m is the difference between these times.

5.5 m is 1.5 m below the centre of oscillation

The centre of the oscillation is at a depth of $\frac{10-4}{2}$ m $= 7$ m .

$x = a\cos\omega t$

$-1.5 = 3\cos\left(\frac{\pi}{6}t\right) \Rightarrow \cos\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$

$\frac{\pi}{6}t = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8$

The time for which the depth of water in the harbour is less than 5.5 m is $(8-4)$hours $= 4$ hours .

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 51

Question:

A piston in a machine is modelled as a particle of mass 0.2 kg attached to one end A of a light elastic spring, of natural length 0.6 m and modulus of elasticity 48 N. The other end B of the spring is fixed and the piston is free to move in a horizontal tube which is assumed to be smooth. The piston is released from rest when $AB = 0.9$ m.

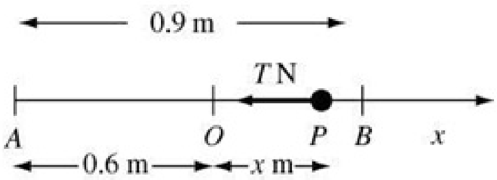
a Prove that the motion of the piston is simple harmonic with period $\frac{\pi}{10}$ s.

b Find the maximum speed of the piston.

c Find, in terms of π, the length of time during each oscillation for which the length of the spring is less than 0.75 m. **[E]**

Solution:

a



Let the piston be modelled by the particle P.
Let O be the point where $AO = 0.6$ m
When P is at a general point in its motion,

let $OP = x$ metres and the force of the spring on P be T newtons.

Displacements in simple harmonic questions are usually measured from the centre of the motion. At the centre, the acceleration of the particle is zero and the forces on the particle are in equilibrium. In this question, the point of equilibrium, O, is where the spring is at its natural length. No horizontal forces will then be acting on the particle.

Hooke's Law

$$T = \frac{\lambda x}{l} = \frac{48x}{0.6} = 80x$$

R$(\rightarrow)$ $\mathbf{F} = ma$

$$-T = 0.2\ddot{x}$$

$$-80x = 0.2\ddot{x}$$

$$\ddot{x} = -400x = -20^2 x$$

When x is positive, the tension in the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

To show that P is moving with simple harmonic motion, you have to show that, at a general point of its motion, the equation of motion of P has the form $\ddot{x} = -kx$, where k is a positive constant. In this case $k = \omega^2 = 100$.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 20$. The period, T seconds, is given by

$$T = \frac{2\pi}{20} = \frac{\pi}{10}, \text{ as required.}$$

b $a = 0.3, \omega = 20$
The maximum speed is given by
$v = a\omega = 0.3 \times 20 = 6$
The maximum speed is 6 m s^{-1}.

c When the length of the spring is 0.75 m

$$x = 0.75 - 0.6 = 0.15$$

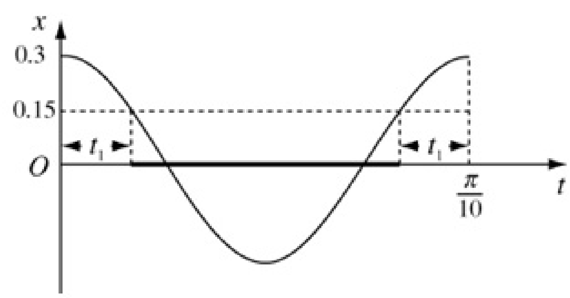
$$x = a\cos\omega t$$

$$0.15 = 0.3\cos 20t_1 \Rightarrow \cos 20t_1 = \frac{1}{2}$$

$$20t_1 = \frac{\pi}{3} \Rightarrow t_1 = \frac{\pi}{60}$$

The total time for which the length of the spring is less that 0.75 m is given by

$$T' = T - 2t_1 = \frac{\pi}{10} - 2 \times \frac{\pi}{60} = \frac{\pi}{15}$$



When the length of the spring is less than 0.75 m, the extension of the spring, x m, is less than 0.15 m. This sketch shows you that if the first time where the extension is 0.15 m is t_1 s, the length of time for which the extension is less than 0.15 m is $\left(\frac{\pi}{10} - 2t_1\right)$s.

The length of time for which the length of the spring is less that 0.75 m is $\frac{\pi}{15}$ s.

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Review Exercise 1
Exercise A, Question 52

Question:

A particle P of mass 0.8 kg is attached to one end A of a light elastic spring OA, of natural length 60 cm and modulus of elasticity 12 N. The spring is placed on a smooth table and the end O is fixed. The particle is pulled away from O to a point B, where $OB = 85\text{ cm}$, and is released from rest.

a Prove that the motion of P is simple harmonic motion with period $\frac{2\pi}{5}\text{s}$.

b Find the greatest magnitude of the acceleration of P during the motion.

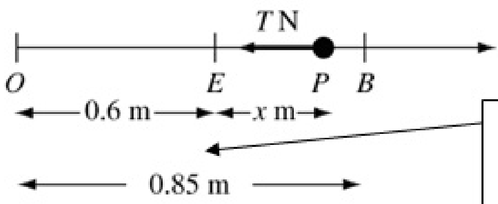
Two seconds after being released from rest, P passes through the point C.

c Find, to 2 significant figures, the speed of P as it passes through C.

d State the direction in which P is moving 2 s after being released. **[E]**

Solution:

a



As you will use Newton's second law in this question, it is safer to use base SI units. So convert the distances in cm to m.

Let E be the point where $OE = 0.6\text{ m}$.

When P is at a general point in its motion, let $EP = x$ metres and the force of the spring on P be T newtons.

Hooke's Law

$$T = \frac{\lambda x}{l} = \frac{12x}{0.6} = 20x$$

$$\text{R}(\rightarrow) \quad \mathbf{F} = ma$$

$$-T = 0.8\ddot{x}$$

$$-20x = 0.8\ddot{x}$$

$$\ddot{x} = -25x = -5^2 x$$

When x is positive, the tension is the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 5$. The period, T seconds, is given by

$T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5}$, as required.

b The amplitude of the motion is 0.25 m.

The maximum magnitude of the acceleration is given by

$$|\ddot{x}| = |\omega^2 a| = 25 \times 0.25 = 6.25$$

The maximum magnitude of the acceleration is 6.25 m s^{-2}.

The acceleration is given by $\ddot{x} = -\omega^2 x$ and this has the greatest size when x is the amplitude.

c $x = a\cos\omega t$

$x = -a\omega \sin \omega t$

At $t = 2$

$$\dot{x} = -0.25 \times 5\sin(5 \times 2) = -1.25\sin 10$$

$$= +0.680\,026\ldots$$

The speed of P as it passes through C is 0.68 m s^{-1} (2 s.f.).

You can derive an equation connecting velocity with time by differentiating $x = a\cos\omega t$ with respect to t. You obtain $v = \dot{x} = \dfrac{\mathrm{d}x}{\mathrm{d}t} = -a\omega\sin\omega t$. This equation is particularly useful when you are asked about the direction of motion of a particle. As the v is squared in $v^2 = \omega^2(a^2 - x^2)$, values of v found using this formula have an ambiguous $\pm$ sign.

d As the sign of $\dot{x}$ in part c is positive, P is travelling in the direction of x increasing as it passes through C.

As it passes through C, P is moving away from O towards B.

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Review Exercise 1
Exercise A, Question 53

Question:

A light elastic string of natural length l has one end attached to a fixed point A. A particle P of mass m is attached to the other end of the string and hangs in equilibrium at the point O, where $AO = \frac{5}{4}l$.

a Find the modulus of elasticity of the string.

The particle P is then pulled down and released from rest. At time t the length of the string is $\frac{5l}{4} + x$.

b Prove that, while the string is taut, $\frac{d^2x}{dt^2} = -\frac{4gx}{l}$

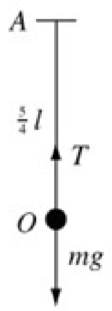
When P is released, $AP = \frac{7}{4}l$. The point B is a distance l vertically below A.

c Find the speed of P at B.

d Describe briefly the motion of P after it has passed through B for the first time until it next passes through O. **[E]**

Solution:

a



At the equilibrium level

R(↑) $T = mg$ ①

Hooke's law

$$T = \frac{\lambda e}{l} = \frac{\lambda \times \frac{1}{4}l}{l} = \frac{\lambda}{4} \quad ②$$

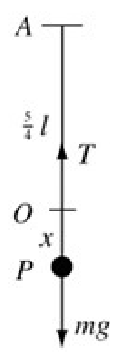
The information you have been given at the equilibrium level enables you to obtain two equations for the tension T, one using Hooke's law and a second by resolving vertically. Eliminating T between the two equations gives you an equation for the modulus of elasticity λ.

Combining ① and ②

$$\frac{\lambda}{4} = mg \Rightarrow \lambda = 4mg$$

The coefficient of elasticity is $4mg$.

b



Hooke's law

$$T = \frac{\lambda e}{l} = \frac{4mg\left(\frac{1}{4}l + x\right)}{l}$$

$$= \frac{mgl + 4mgx}{l} = mg + \frac{4mgx}{l} \quad ③$$

When $AP = \frac{5}{4}l + x$, the extension is $\left(\frac{5}{4}l + x\right) - l = \frac{1}{4}l + x$.

Newton's second law

R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m\frac{d^2x}{dt^2} \quad ④$$

You take the forces in the direction of x increasing. The weight tends to increase the value of x, so mg is positive. The tension tends to decrease the value of x so, in this equation, T has a negative sign.

Substituting ③ into ④

$$mg - \left(mg + \frac{4mgx}{l}\right) = m\frac{d^2x}{dt^2}$$

$$-\frac{4mgx}{l} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{4gx}{l}, \text{ as required.}$$

c Comparing the result of part **b** with the standard formula $\ddot{x} = -\omega^2 x$, while the string is taut, P moves with SHM about O, with $\omega^2 = \frac{4g}{l}$.

When P is released from the point where $AP = \frac{7}{4}l$, the amplitude, a, is given by

$$a = AP - AO = \frac{7}{4}l - \frac{5}{4}l = \frac{1}{2}l$$

At B, $x = -\frac{1}{4}l$

$$v^2 = \omega^2(a^2 - x^2)$$

$$= \frac{4g}{l}\left(\left(\frac{1}{2}l\right)^2 - \left(-\frac{1}{4}l\right)^2\right)$$

$$= \frac{4g}{l}\left(\frac{1}{4}l^2 - \frac{1}{16}l^2\right) = \frac{4g}{l} \times \frac{3l^2}{16} = \frac{3gl}{4}$$

$$v = \frac{1}{2}\sqrt{(3gl)}$$

The speed of P at B is $\frac{1}{2}\sqrt{(3gl)}$.

Alternatively, you can use conservation of energy between the point of release and B to find the velocity at B.
Loss in elastic energy = gain in kinetic energy + gain in potential energy

$$\frac{4mg(\frac{3}{4}l)^2}{2l} = \frac{1}{2}mv^2 + mg \times \frac{3}{4}l.$$

This, of course, leads to the same answer.

d First P moves freely under gravity until it returns to B.
Then it moves with simple harmonic motion about O.

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Review Exercise 1
Exercise A, Question 54

Question:

A light elastic string, of natural length $4a$ and modulus of elasticity $8mg$, has one end attached to a fixed point A. A particle P of mass m is attached to the other end of the string and hangs in equilibrium at the point O.

a Find the distance AO.

The particle is now pulled down to a point C vertically below O, where $OC = d$. It is released from rest. In the subsequent motion the string does not become slack.

b Show that P moves with simple harmonic motion of period $\pi\sqrt{\frac{2a}{g}}$. The greatest speed of P during this motion is $\frac{1}{2}\sqrt{(ga)}$.

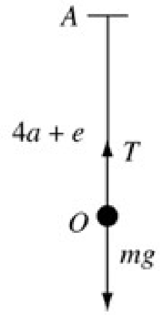
c Find d in terms of a.

Instead of being pulled down a distance d, the particle is pulled down a distance a. Without further calculation,

d Describe briefly the subsequent motion of P. **[E]**

Solution:

a



A particle attached to one end of an elastic string will oscillate about the equilibrium position. When solving problems about vertical oscillations, you often have to begin by finding the point of equilibrium. In this case, the oscillations later in the question have centre O.

At the equilibrium level, let $AO = 4a + e$, where e is the extension of the string.
Hooke's law

$$T = \frac{\lambda e}{l} = \frac{8mge}{4a} = \frac{2mge}{a}$$

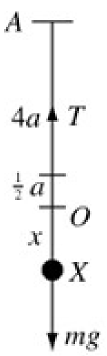
R($\uparrow$) $T = mg$

Hence

$$mg = \frac{2mge}{a} \Rightarrow e = \frac{a}{2}$$

$$AO = 4a + e = 4a + \frac{a}{2} = \frac{9a}{2}$$

b



When P is at a general point, X say, of its motion, let $OX = x$. At this point, the extension of the string is $\frac{1}{2}a + x$

To show that P is moving with simple harmonic motion, you have to show that, at a general point in its motion, the equation of motion of P has the form $\ddot{x} = -\omega^2 x$, where ω is a positive constant.

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{8mg\left(\frac{1}{2}a + x\right)}{4a}$$

$$= \frac{4mga + 8mgx}{4a} = mg + \frac{2mgx}{a} \quad ①$$

Newton's second law

$\text{R}(\downarrow) \quad \mathbf{F} = m\mathbf{a}$

$$mg - T = m\ddot{x} \quad ②$$

Hooke's law and Newton's second law give you two equations from which you eliminate the tension, T.

Substituting ① into ②

$$mg - \left(mg + \frac{2mgx}{a}\right) = m\ddot{x}$$

$$-\frac{2mgx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{2g}{a}x$$

Comparing this equation with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P moves with simple harmonic motion about O and

$$\omega = \sqrt{\left(\frac{2g}{a}\right)}.$$

The period of motion T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{a}{2g}\right)} = \pi\sqrt{\left(\frac{2a}{g}\right)}, \text{ as required.}$$

c The maximum speed is given by $v = a\omega$

$$\frac{1}{2}\sqrt{(ga)} = d\sqrt{\left(\frac{2a}{a}\right)}$$

As the particle is pulled down a distance d from the equilibrium position and released from rest, d is the amplitude of the motion.

$$\frac{1}{4}ga = d^2 \times \frac{2g}{a}$$

Squaring both sides of the previous line.

$$d^2 = \frac{1}{8}a^2$$

$$d = \frac{1}{2\sqrt{2}}a$$

d As $a > \frac{1}{2}a$, the string will become slack during its motion. The subsequent motion of P will be partly under gravity, partly simple harmonic motion.

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Review Exercise 1
Exercise A, Question 55

Question:

A particle P of mass 0.3 kg is attached to one end of a light elastic spring. The other end of the spring is attached to a fixed point O on a smooth horizontal table. The spring has natural length 2 m and modulus of elasticity 21.6 N. The particle P is placed on the table at a point A, where $OA = 2$ m. The particle P is now pulled away from O to the point B, where OAB is a straight line with $OB = 3.5$ m. It is then released from rest.

a Prove that P moves with simple harmonic motion of period $\frac{\pi}{3}$ s.

b Find the speed of P when it reaches A.

The point C is the mid-point of AB.

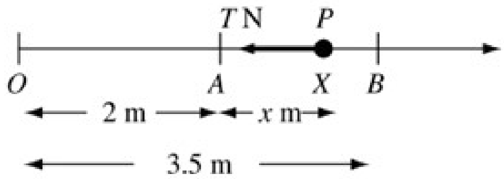
c Find, in terms of π, the time taken for P to reach C for the first time.

Later in the motion, P collides with a particle Q of mass 0.2 kg which is at rest at A. After impact, P and Q coalesce to form a single particle R.

d Show that R also moves with simple harmonic motion and find the amplitude of this motion. **[E]**

Solution:

a



When P is at the point X, where $AX = x$ m, let the tension in the spring be T N.
Hooke's law

$$T = \frac{\lambda x}{l} = \frac{21.6 \times x}{2} = 10.8x$$

R$(\rightarrow)$ $\mathbf{F} = m\mathbf{a}$

$$-T = 0.3\ddot{x}$$

$$-10.8x = 0.3\ddot{x}$$

$$\ddot{x} = -36x = -6^2 x$$

When x is positive, the tension in the spring is acting in the direction of x decreasing, so T has a negative sign in the equation of motion.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P is performing simple harmonic motion about A with $\omega = 6$.

The period of motion Ts is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{6} = \frac{\pi}{3}, \text{ as required.}$$

b At A, $x = 0$

$$v^2 = \omega^2(a^2 - x^2) = 36(1.5^2 - 0^2) = 81$$

$$v = \sqrt{81} = 9$$

The speed of P at A is 9 m s^{-1}.

c At C, $x = \frac{1.5}{2} = 0.75$

$$x = a\cos\omega t$$

$$0.75 = 1.5\cos 6t$$

$$\cos 6t = \frac{1}{2} \Rightarrow 6t = \frac{\pi}{3} \Rightarrow t = \frac{\pi}{18}$$

P reaches C for the first time after $\frac{\pi}{18}$ s.

The time when P first reaches C is the smallest positive value of t for which this equation is true. In principle, in simple harmonic motion, P will reach this point infinitely many times.

d Before impact, the linear momentum of P is
$m_1 u = 0.3 \times 9$ N s $= 2.7$ N s
Let the velocity of the combined particle R immediately after impact be U m s^{-1}.

After impact, the linear momentum of R is
$m_2 U = 0.5U$ N s

Conservation of linear momentum
$0.5U = 2.7 \Rightarrow U = 5.4$

Conservation of linear momentum is an M1 topic and is assumed, and can be tested, in any of the subsequent Mechanics modules.

For R
$$\text{R}(\rightarrow) \quad -T = 0.5\ddot{x}$$
$$-10.8x = 0.5\ddot{x}$$
$$\ddot{x} = -21.6x$$

When R is at X, Hooke's law gives $T = 10.8x$, exactly as in part **a**. There is no need to repeat the working in part **d**.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, after the impact R is performing simple harmonic motion about A with $\omega^2 = 21.6$.
$$v = U = a\omega$$
$$5.4 = a\sqrt{21.6}$$
$$a = \frac{5.4}{\sqrt{21.6}} = 1.161\ldots$$
The amplitude of the motion is 1.16 m (3 s.f.)

As R is performing simple harmonic motion about A, the speed of R immediately after the impact is the maximum speed of R during its motion. The maximum speed during simple harmonic motion is given by $v = a\omega$.

No accuracy is specified in the question and the accurate answer, $\frac{3\sqrt{15}}{10}$ m, or any reasonable approximation would be accepted.

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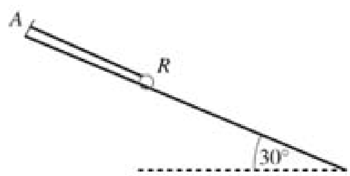
Review Exercise 1
Exercise A, Question 56

Question:



A small ring R of mass m is free to slide on a smooth straight wire which is fixed at an angle of $30°$ to the horizontal. The ring is attached to one end of a light elastic string of natural length a and modulus of elasticity λ. The other end is attached to a fixed point A on the wire, as shown in the figure. The ring rests in equilibrium at the point B, where $AB = \frac{9}{8}a$.

a Show that $\lambda = 4mg$.

The ring is pulled down to a point C, where $BC = \frac{1}{4}a$ and released from rest. At time t after R is released the extension in the string is $\left(\frac{1}{8}a + x\right)$.

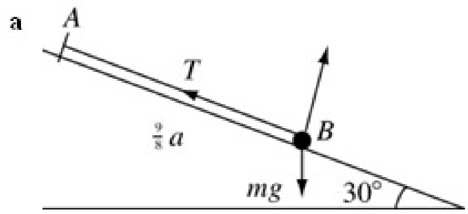
b Obtain a differential equation for the motion of R while the string remains taut, and show that it represents simple harmonic motion with period $\pi\sqrt{\left(\frac{a}{g}\right)}$.

c Find, in terms of g, the greatest magnitude of the acceleration of R while the string remains taut.

d Find, in terms of a and g, the time taken for R to move from the point at which it first reaches a maximum speed to the point where the string becomes slack for the first time. **[E]**

Solution:

a



When R is at B, the extension in the string is

$$\frac{9}{8}a - a = \frac{1}{8}a$$

The extension in the string is AB – the natural length.

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{\lambda \times \frac{1}{8}a}{a} = \frac{\lambda}{8}$$

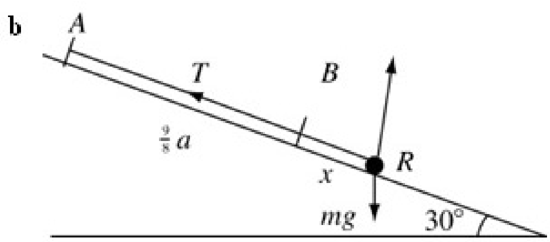
R(↖) $T = mg\sin 30° = \frac{1}{2}mg$

Using $\sin 30° = \frac{1}{2}$.

Hence $\frac{\lambda}{8} = \frac{1}{2}mg \Rightarrow \lambda = 4mg$, as required.

You equate the two expressions for T.

b



When the extension of the string is $\left(\frac{1}{8}a + x\right)$

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{4mg \times \left(\frac{1}{8}a + x\right)}{a}$$

$$= \frac{\frac{1}{2}mga + 4mgx}{a} = \frac{mg}{2} + \frac{4mgx}{a} \quad ①$$

R(↘) $\mathbf{F} = m\mathbf{a}$

$$mg\sin 30° - T = m\ddot{x} \quad ②$$

$mg\sin 30°$ is the component of the weight parallel to the string.

Substituting ② into ①

$$\frac{mg}{2} - \left(\frac{mg}{2} + \frac{4mgx}{a}\right) = m\ddot{x}$$

$$-\frac{4mgx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{4g}{a}x$$

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$,

R is performing simple harmonic motion about B with $\omega = 2\sqrt{\left(\frac{g}{a}\right)}$.

The period of the motion is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\sqrt{\left(\frac{g}{a}\right)}} = \pi\sqrt{\left(\frac{a}{g}\right)}$$, as required.

c The maximum magnitude of the acceleration is given by

$|\ddot{x}| = \omega^2 \times \text{amplitude} = \frac{4g}{a} \times \frac{1}{4}a = g$

The maximum magnitude of the acceleration is g.

As $\ddot{x} = -\omega^2 x$, the magnitude of the acceleration is $\omega^2 |x|$. So the maximum magnitude is where x is a large as possible. That is at the amplitude.

d The maximum speed is at B and the string becomes slack after moving a further distance $\frac{1}{8}a$.

$$x = a\sin\omega t$$

$$\frac{1}{8}a = \frac{1}{4}a\sin\omega t$$

$$\sin\omega t = \frac{1}{2} \Rightarrow \omega t = \frac{\pi}{6}$$

$$t = \frac{\pi}{6\omega} = \frac{\pi}{6 \times 2\sqrt{\left(\frac{g}{a}\right)}} = \frac{\pi}{12}\sqrt{\left(\frac{a}{g}\right)}$$

You find the shortest time for R to move a distance of $\frac{1}{8}a$ from the centre of the simple harmonic motion.

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Review Exercise 2
Exercise A, Question 1

Question:

A circular flywheel of diameter 7 cm is rotating about the axis through its centre and perpendicular to its plane with constant angular speed 1000 revolutions per minute. Find, in m s^{-1} to 3 significant figures, the speed of a point on the rim of the flywheel.

[E]

Solution:

diameter $= 7$ cm

$\therefore$ radius $= 3.5$ cm

i.e. $r = 0.035$ m ← Find the radius in m.

angular speed $= 1000$ revs/min

i.e. $\omega = \dfrac{1000 \times 2\pi}{60}$ radians/second ← Change the units of angular speed to radians per second.

Using $v = r\omega$

$\therefore v = 0.035 \times \dfrac{1000 \times 2\pi}{60}$ m s^{-1} ← Use the formula connecting linear speed and angular speed.

$= 3.67$ m s^{-1} (3 s.f.)

So the speed is 3.67 ms^{-1} (3 s.f.) ← Quote answer to required accuracy.

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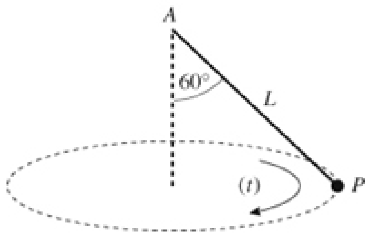
Review Exercise 2
Exercise A, Question 2

Question:

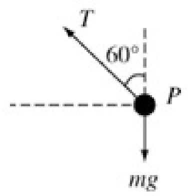


A particle P of mass m is attached to one end of a light string. The other end of the string is attached to a fixed point A. The particle moves in a horizontal circle with constant angular speed ω and with the string inclined at an angle of $60°$ to the vertical, as shown in the diagram above. The length of the string is L.

a Show that the tension in the string is $2mg$.

b Find ω in terms of g and L. **[E]**

Solution:



Draw a diagram showing the forces acting an P.

a $R(\uparrow)$

Resolve vertically

$T\cos 60° - mg = 0$

$\therefore T = \dfrac{mg}{\cos 60°}$

i.e. $T = 2mg$

Make T the subject of the formula.

b Resolve $R(\leftarrow)$

Using the equation of motion:

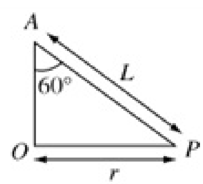
$T\sin 60 = ma$

$= mr\omega^2$

Use Newton's 2nd Law $F = ma$ with $a = r\omega^2$, where r is the radius of the circular motion.

But $r = L\sin 60°$

$\therefore T\sin 60° = mL\sin 60°\,\omega^2$



From the triangle AOP

$r = L\sin 60°$.

$\therefore \omega^2 = \dfrac{T}{mL}$

$= \dfrac{2mg}{mL}$

$= \dfrac{2g}{L}$

$\therefore \omega = \sqrt{\dfrac{2g}{L}}$

Substitute the value for T from part **a**.

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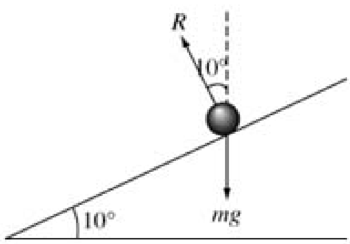
Review Exercise 2
Exercise A, Question 3

Question:

A car moves round a bend which is banked at a constant angle of 10° to the horizontal. When the car is travelling at a constant speed of 18 m s^{-1}, there is no sideways frictional force on the car. The car is modelled as a particle moving in a horizontal circle of radius r metres. Calculate the value of r. **[E]**

Solution:



Draw a diagram showing the forces acting on the car – its weight mg and the normal reaction R.

R(↑)

$R\cos 10° - mg = 0$

$\therefore R = \dfrac{mg}{\cos 10°}$ ①

Using $F = ma$ horizontally gives

Use Newton's Second Law.

$R\sin 10° = \dfrac{mv^2}{r} = \dfrac{m \times 18^2}{r}$

Substituting for R

Use the result in equation ①.

$\dfrac{mg}{\cos 10°} \times \sin 10° = \dfrac{m \times 18^2}{r}$

$\therefore r = \dfrac{18^2}{g \times \tan 10°}$

$= 187$ (3 s.f.) or 190 (2 s.f.)

$\dfrac{\sin 10°}{\cos 10°} = \tan 10°$

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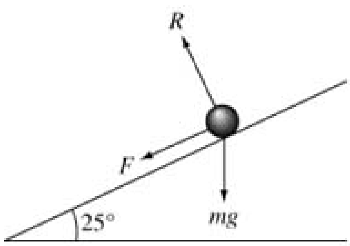
Review Exercise 2
Exercise A, Question 4

Question:

A cyclist is travelling around a circular track which is banked at $25°$ to the horizontal. The coefficient of friction between the cycle's tyres and the track is 0.6. The cyclist moves with constant speed in a horizontal circle of radius 40 m, without the tyres slipping.
Find the maximum speed of the cyclist. **[E]**

Solution:



Draw a diagram showing the weight mg, the normal reaction R and the friction acting down the slope for maximum speed.

R(↑)

$$R\cos 25° - F\sin 25° - mg = 0 \quad ①$$

R(←)

$$R\sin 25° + F\cos 25° = \frac{mv^2}{40} \quad ②$$

Use $F = ma$ with $a = \frac{v^2}{r}$ in a horizontal direction.

Also $F = 0.6R$

The maximum friction is μR for no slipping.

$$\therefore R(\cos 25° - 0.6\sin 25°) = mg$$

i.e. $R = \dfrac{mg}{\cos 25° - 0.6\sin 25°}$

Make R the subject of the formula in equation ①.

Also

$$R(\sin 25° + 0.6\cos 25°) = \frac{mv^2}{40}$$

$$\therefore \frac{mg(\sin 25° + 0.6\cos 25°)}{\cos 25° - 0.6\sin 25°} = \frac{mv^2}{40}$$

Substitute into equation ②.

$$\therefore v^2 = 40g\frac{(\sin 25° + 0.6\cos 25°)}{(\cos 25° - 0.6\sin 25°)}$$

$$= 580$$

$$\therefore v = 24.1 \text{ or } 24$$

So maximum speed is 24 m s^{-1} (2 s.f.)

Make v^2 the subject of the formula and obtain v.

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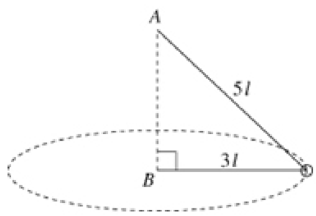
Review Exercise 2
Exercise A, Question 5

Question:



A light inextensible string of length $8l$ has its ends fixed to two points A and B, where A is vertically above B. A small smooth ring of mass m is threaded on the string. The ring is moving with constant speed in a horizontal circle with centre B and radius $3l$, as shown in the diagram. Find

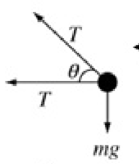
a the tension in the string,

b the speed of the ring.

c State briefly in what way your solutions might no longer be valid if the ring were firmly attached to the string. **[E]**

Solution:

a

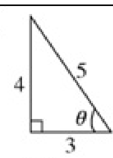


Draw a diagram showing the weight of the ring and the tensions of equal magnitude T.

R(↑)

$T\sin\theta - mg = 0$

$$\therefore T = \frac{mg}{\sin\theta} = \frac{5mg}{4}$$



θ is an angle in a 3-4-5 triangle.

b R(←)

$$T + T\cos\theta = \frac{mv^2}{3l}$$

Use Newton's 2nd Law with $a = \frac{v^2}{3l}$. v is the speed of the ring.

$$\therefore T\left(1+\frac{3}{5}\right) = \frac{mv^2}{3l}$$

$\cos\theta = \frac{3}{5}$ from 3-4-5 triangle.

$$\text{i.e. } \frac{5mg}{4}\times\frac{8}{5} = \frac{mv^2}{3l}$$

Substitute value for T from part **a**.

$$\therefore v^2 = 6gl$$

$$v = \sqrt{6gl}$$

Make v the subject of the formula.

c The tensions could not be assumed to have the same magnitude.

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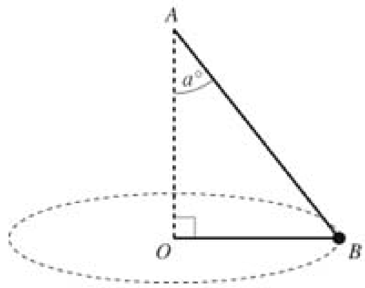
Review Exercise 2
Exercise A, Question 6

Question:



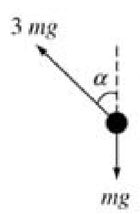
A metal ball B of mass m is attached to one end of a light inextensible string. The other end of the string is attached to a fixed point A. The ball B moves in a horizontal circle with centre O vertically below A, as shown in the diagram. The string makes a constant angle $\alpha°$ with the downward vertical and B moves with constant angular speed $\sqrt{(2gk)}$, where k is a constant. The tension in the string is $3mg$. By modelling B as a particle, find

a the value of α,

b the length of the string. **[E]**

Solution:

a



Draw a diagram showing the forces acting on the metal ball.

R(↑)

$3mg\cos\alpha - mg = 0$

$\therefore \cos\alpha = \frac{1}{3}$

$\therefore \alpha = 70.5°$

Resolve vertically and make α the subject of the equation.

b R(←)

$3mg\sin\alpha = mr\omega^2$

$= mr \times 2gk$

But $r = l\sin\alpha$

$\therefore 3mg = ml \times 2gk$

$\therefore l = \frac{3}{2k}$

Use Newton's 2nd Law $F = ma$ with $a = r\omega^2$.

Use ΔAOB to express the radius r in terms of the length of the string l and the angle α.

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Review Exercise 2
Exercise A, Question 7

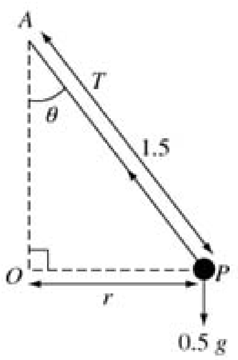
Question:

A particle P of mass 0.5 kg is attached to one end of a light inextensible string of length 1.5 m. The other end of the string is attached to a fixed point A. The particle is moving, with the string taut, in a horizontal circle with centre O vertically below A. The particle is moving with constant angular speed 2.7 rad s^{-1}. Find

a the tension in the string,

b the angle, to the nearest degree, that AP makes with the downward vertical. **[E]**

Solution:



Draw a diagram. Let the tension in the string be T and let the string make an angle θ with the vertical.

a $r = 1.5\sin\theta$

Let $OP = r$.

R(←)

$T\sin\theta = mr\omega^2$

$= 0.5 \times 1.5\sin\theta \times 2.7^2$

Use Newton's 2nd Law $F = ma$ with $a = r\omega^2$.

$\therefore T = 5.4675 \text{ N}$

$= 5.5 \text{ N (2 s.f.)}$

b R(↑)

Resolve vertically.

$T\cos\theta - 0.5g = 0$

$\therefore T\cos\theta = 0.5g$

Substitute the value for T found in part **a**.

$\therefore \cos\theta = \dfrac{0.5g}{5.4675}$

$= 0.896$

$\therefore \theta = 26^\circ$ (nearest degree)

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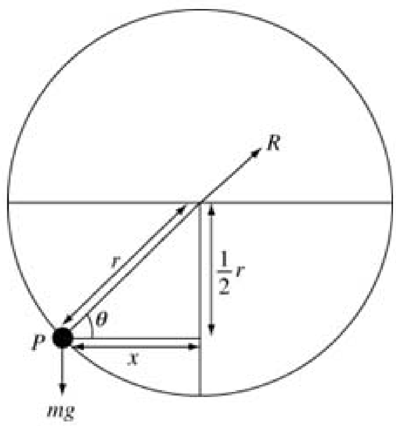
Review Exercise 2
Exercise A, Question 8

Question:

A particle P of mass m moves on the smooth inner surface of a spherical bowl of internal radius r. The particle moves with constant angular speed in a horizontal circle, which is at a depth $\frac{1}{2}r$ below the centre of the bowl.

a Find the normal reaction of the bowl on P.
b Find the time for P to complete one revolution of its circular path. **[E]**

Solution:



Draw a diagram showing the forces acting an the particle P. θ is the angle between the normal reaction and the horizontal.

a $\sin\theta = \dfrac{\frac{1}{2}r}{r} = \dfrac{1}{2}$

$\therefore \theta = 30°$

R(↑)

Then $R\sin\theta - mg = 0$

$\therefore R = \dfrac{mg}{\sin\theta} = 2mg$

Find θ and use this to find R.

b R(→)

$R\cos\theta = mx\omega^2$

$= m(r\cos\theta)\omega^2$

As $R = 2mg$

$2mg\cos\theta = mr\cos\theta\omega^2$

$\therefore 2g = r\omega^2$

$\omega = \sqrt{\dfrac{2g}{r}}$

Use Newton's 2nd Law '$F = ma$' with $a = x\omega^2$ $x = \dfrac{\sqrt{3}}{2}r$ but you do not need to find this.

Time to complete one revolution $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{r}{2g}}$

You should learn this formula.

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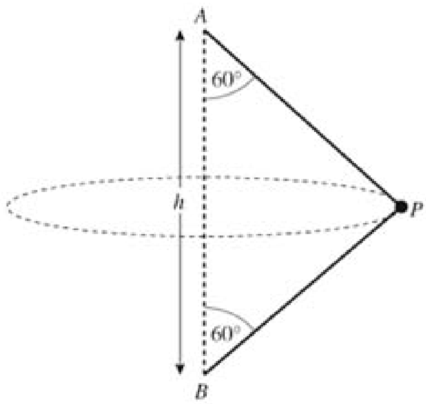
Review Exercise 2
Exercise A, Question 9

Question:



A particle P of mass m is attached to two light inextensible strings. The other ends of the string are attached to fixed points A and B. The point A is a distance h vertically above B. The system rotates about the line AB with constant angular speed ω. Both strings are taut and inclined at $60°$ to AB, as shown in the diagram. The particle moves in a circle of radius r.

a Show that $r = \frac{\sqrt{3}}{2}h$.

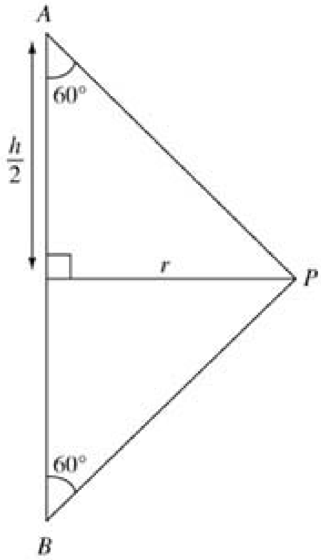
b Find, in terms of m, g, h and ω, the tension in AP and the tension in BP.

The time taken for P to complete one circle is T.

c Show that $T < \pi\sqrt{\left(\frac{2h}{g}\right)}$. **[E]**

Solution:

a



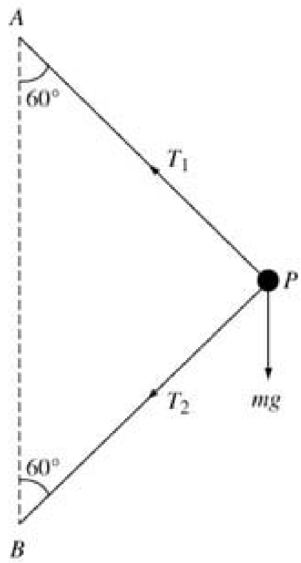
Divide the equilateral triangle into two right angled triangles. Use trigonometry to express r in terms of h.

$$\tan 60^\circ = \frac{r}{\frac{h}{2}}$$

$$\therefore r = \frac{h}{2} \times \tan 60^\circ$$

$$\text{i.e. } r = \frac{\sqrt{3}h}{2}$$

b



Draw another diagram showing the forces acting on P. Let T_1 be the tension in AP and T_2 be the tension in BP.

R(↑)

$T_1 \cos 60° - T_2 \cos 60° - mg = 0$

$\therefore \frac{1}{2}T_1 - \frac{1}{2}T_2 = mg$ ①

R(←)

$T_1 \sin 60° + T_2 \sin 60° = mr\omega^2$

$\therefore \frac{\sqrt{3}}{2}T_1 + \frac{\sqrt{3}}{2}T_2 = m\frac{\sqrt{3}}{2}h\omega^2$

i.e. $\frac{1}{2}T_1 + \frac{1}{2}T_2 = \frac{1}{2}mh\omega^2$ ②

Use equation of motion $F = ma$ with $a = r\omega^2$.

Adding ① and ②

$T_1 = mg + \frac{1}{2}mh\omega^2$

Subtracting ② − ①

$T_2 = \frac{1}{2}mh\omega^2 - mg$

Solve the simultaneous equations to find the two tensions.

c $T_2 > 0 \Rightarrow \omega > \sqrt{\frac{2g}{h}}$

$T_1 > 0$ for all values of ω so consider the condition $T_2 > 0$.

As $T = \frac{2\pi}{\omega}, T < 2\pi\sqrt{\frac{h}{2g}}$

i.e. $T < \pi\sqrt{\frac{2h}{g}}$

Use $T = \frac{2\pi}{\omega}$ to find the time to complete one circle.

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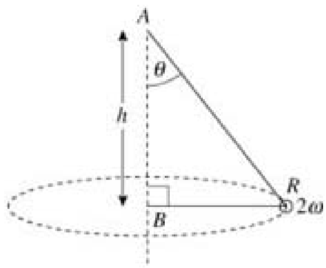
Review Exercise 2
Exercise A, Question 10

Question:



One end of a light inextensible string is attached to a fixed point A. The other end of the string is attached to a fixed point B, vertically below A, where $AB = h$. A small smooth ring R of mass m is threaded on the string. The ring R moves in a horizontal circle with centre B, as shown in the diagram. The upper section of the string makes a constant angle θ with the downward vertical and R moves with constant angular speed ω. The ring is modelled as a particle.

a Show that $\omega^2 = \frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right)$.

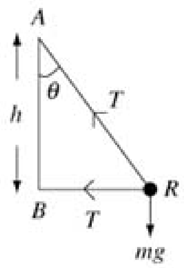
b Deduce that $\omega > \sqrt{\frac{2g}{h}}$.

Given that $\omega = \sqrt{\frac{3g}{h}}$,

c find, in terms of m and g, the tension in the string.

Solution:

a



Draw a diagram showing the forces acting on the ring.

R(↑)

$T\cos\theta - mg = 0$

$\therefore\ T\cos\theta = mg$ ①

Resolve vertically.

R(←)

$T + T\sin\theta = mr\omega^2$

But $r = h\tan\theta$

Resolve horizontally using $F = ma$ with $a = r\omega^2$, where $r = h\tan\theta$.

$\therefore \frac{mg}{\cos\theta}(1+\sin\theta) = mh\frac{\sin\theta}{\cos\theta}\omega^2$

$\therefore \omega^2 = \frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right)$

Show each line of working as there is a printed answer.

b As $\omega^2 = \frac{g}{h}\left(\frac{1}{\sin\theta}+1\right)$ and $\sin\theta < 1$ so that $\frac{1}{\sin\theta} > 1$

$\therefore \omega^2 > \frac{g}{h}\times 2$

So $\omega > \sqrt{\frac{2g}{h}}$

Express ω^2 in this way, as it is clear that when $\sin\theta$ is a maximum, ω^2 is a minimum.

c Given $\omega = \sqrt{\frac{3g}{h}}$

Then $\frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right) = \frac{3g}{h}$

Use the expression obtained in part **a** to find the angle θ.

$\therefore 1+\sin\theta = 3\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$

$\therefore \cos\theta = \frac{\sqrt{3}}{2}$

and $T = \frac{2\sqrt{3}}{3}mg$ or $1.15mg$.

From equation ① $T = \frac{mg}{\cos\theta}$.

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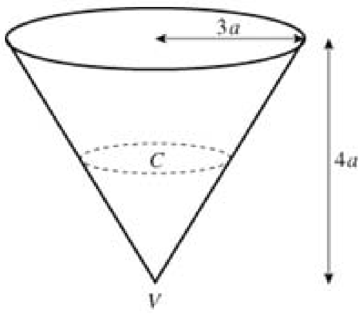
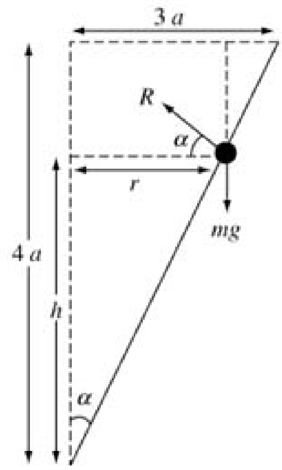
Review Exercise 2
Exercise A, Question 11

Question:



A hollow cone, of base radius $3a$ and height $4a$, is fixed with its axis vertical and vertex V downwards, as shown in the diagram. A particle moves in a horizontal circle with centre C, on the smooth inner surface of the cone with constant angular speed $\sqrt{\frac{8g}{9a}}$. Find the height of C above V. **[E]**

Solution:



$\tan \alpha = \frac{3a}{4a} = \frac{3}{4}$

$\tan \alpha = \frac{r}{h}$

Draw a diagram showing angle α between the slant side and the axis of the cone.

Also show the radius r and the height h.

R(↑)

$R \sin \alpha - mg = 0$

$\therefore R \sin \alpha = mg$ ①

R(←)

$R \cos \alpha = mr\omega^2$

$R \cos \alpha = mr \times \frac{8g}{9a}$ ②

Use the equation of motion $F = ma$ with $a = r\omega^2$.

Divide equation ① by equation ②

Then $\tan \alpha = mg \div \frac{8mrg}{9a}$

$= \frac{9a}{8r}$

$\therefore \frac{9a}{8r} = \frac{3}{4} \Rightarrow r = \frac{3a}{2}$

$\therefore h = \frac{r}{\tan \alpha} = \frac{3a}{2} \times \frac{4}{3}$

$= 2a$

The height of C above V is $2a$.

Use the result that $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$.

Use $\tan \alpha = \frac{3}{4}$ to obtain r.

Use $\tan \alpha = \frac{r}{h}$ to obtain h.

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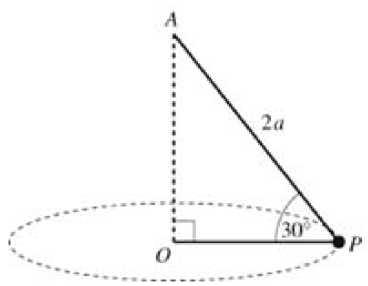
Review Exercise 2
Exercise A, Question 12

Question:



A particle P of mass m is attached to one end of a light inextensible string of length $2a$. The other end of the string is fixed to a point A which is vertically above the point O on a smooth horizontal table. The particle P remains in contact with the surface of the table and moves in a circle with centre O and with angular speed $\sqrt{\frac{kg}{3a}}$, where k is a constant. Throughout the motion the string remains taut and $\angle APO = 30°$, as shown in the diagram.

a Show that the tension in the string is $\frac{2kmg}{3}$.

b Find, in terms of m, g and k, the normal reaction between P and the table.

c Deduce the range of possible values of k.

The angular speed of P is changed to $\sqrt{\left(\frac{2g}{a}\right)}$. The particle P now moves in a horizontal circle above the table. The centre of this circle is X.

d Show that X is the mid-point of OA. **[E]**

Solution:

a R(←)

Use equation of motion

$T\cos 30^\circ = m(2a\cos 30^\circ)\left(\frac{kg}{3a}\right)$

$\therefore T = m \times 2a \times \frac{kg}{3a}$

$T = \frac{2kmg}{3}$

Let the tension in the string be T
Use $F = ma$ with $a = r\omega^2$,
noting that $r = 2a\cos 30^\circ$.

b R(↑)

$R + T\sin 30^\circ - mg = 0$

$\therefore R = mg - \frac{2kmg}{3} \times \frac{1}{2}$

$= mg\left(1 - \frac{k}{3}\right)$

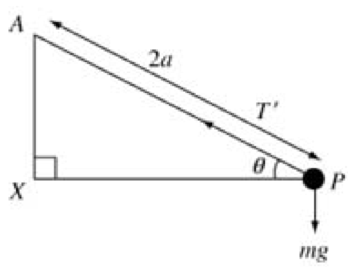
Let the normal reaction be R and resolve vertically.

c As $R > 0$, $1 - \frac{k}{3} > 0$

$\therefore k < 3$

Use the condition $R > 0$ to find k.

d



Let the new tension be T' and let AP make an angle θ with the horizontal.

$PX = 2a\cos\theta$

R(←)

$T'\cos\theta = m \times 2a\cos\theta \times \left(\frac{2g}{a}\right)$

$\therefore T' = 4mg$

Use $F = ma$ for the new horizontal circular motion.

R(↑)

$T'\sin\theta - mg = 0$

$\therefore \sin\theta = \frac{mg}{T'} = \frac{mg}{4mg} = \frac{1}{4}$

Find the expression for θ.

As $AX = 2a\sin\theta$

$AX = 2a \times \frac{1}{4} = \frac{1}{2}a$

But $AO = 2a\sin 30^\circ = a$

$\therefore AX = \frac{1}{2}AO$ as required.

Find the lengths AX and AO to show the result which is asked.

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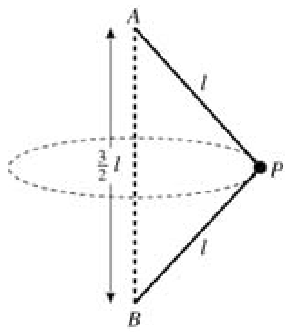
Review Exercise 2
Exercise A, Question 13

Question:



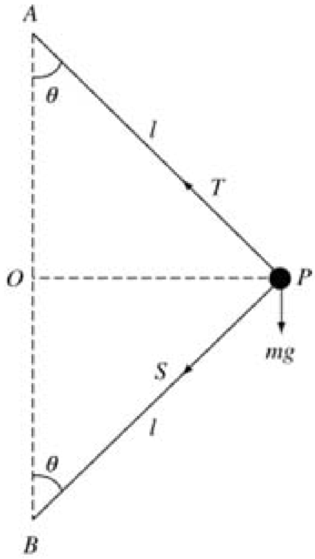
A particle P of mass m is attached to the ends of two light inextensible strings AP and BP each of length l. The ends A and B are attached to fixed points, with A vertically above B and $AB = \frac{3}{2}l$, as shown in the diagram above. The particle P moves in a horizontal circle with constant angular speed ω. The centre of the circle is the mid-point of AB and both strings remain taut.

a Show that the tension in AP is $\frac{1}{6}m(3l\omega^2 + 4g)$.

b Find, in terms of m, l, ω and g, an expression for the tension in BP.

c Deduce that $\omega^2 \geq \frac{4g}{3l}$. **[E]**

Solution:

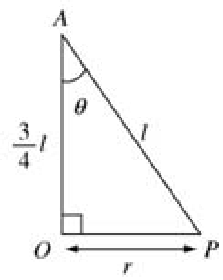


Let the tension in AP be T and in BP be S.
Draw a diagram showing the forces acting.

a R(↑)

$T\cos\theta - S\cos\theta - mg = 0$

$\therefore T - S = \dfrac{mg}{\cos\theta} = \dfrac{4mg}{3}$ ①



From triangle APO $\cos\theta = \dfrac{3}{4}$

R(←)

$T\sin\theta + S\sin\theta = mr\omega^2$

$= ml\sin\theta\omega^2$

$\therefore T + S = ml\omega^2$ ②

Also from triangle APO $r = l\sin\theta$.

Adding equations ① and ② gives

$2T = \dfrac{4}{3}mg + ml\omega^2$

$\therefore T = \dfrac{1}{6}m(3l\omega^2 + 4g)$

Solve the simultaneous equation to obtain T first.

b Subtracting ② − ① gives

$2S = ml\omega^2 - \dfrac{4mg}{3}$

$\therefore S = \dfrac{1}{6}m(3l\omega^2 - 4g)$

Also use these equations to find S.

c As $S \geq 0$, $\omega^2 \geq \dfrac{4g}{3l}$ as required.

As string BP is taut it is under tension and $S \geq 0$

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Review Exercise 2
Exercise A, Question 14

Question:

A rough disc rotates in a horizontal plane with constant angular velocity ω about a fixed vertical axis. A particle P of mass m lies on the disc at a distance $\frac{4}{3}a$ from the axis. The coefficient of friction between P and the disc is $\frac{3}{5}$. Given that P remains at rest relative to the disc,

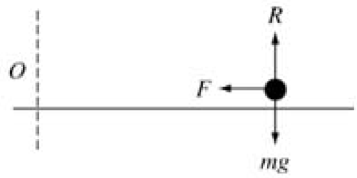
a prove that $\omega^2 \leq \frac{9g}{20a}$.

The particle is now connected to the axis by a horizontal light elastic string of natural length a and modulus of elasticity $2mg$. The disc again rotates with constant angular velocity ω about the axis and P remains at rest relative to the disc at a distance $\frac{4}{3}a$ from the axis.

b Find the greatest and least possible values of ω^2. **[E]**

Solution:

a



R(↑)

$R - mg = 0 \quad \therefore R = mg$

R(←)

$F = mr\omega^2$

$= m\left(\frac{4}{3}a\right)\omega^2$

As P remains at rest $F \le \mu R$

$\therefore m\left(\frac{4}{3}a\right)\omega^2 \le \frac{3}{5}mg$

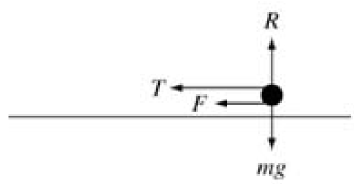
$\therefore \omega^2 \le \frac{9g}{20a}$

> Use $F = ma$ with $a = r\omega^2$ for circular motion.

> The radius of the circular motion is $\frac{4a}{3}$.

> Substitute $R = mg$ and $F = m\left(\frac{4}{3}a\right)\omega^2$ into $F \le \mu R$.

b



> Draw a diagram showing friction acting towards the centre of the motion.
> Case **i** maximum value for ω:

Case **i** minimum value for ω:

$T = \frac{2mg}{a} \times \frac{a}{3}$

> Use Hooke's Law $T = \frac{\lambda e}{l}$ with $\lambda = 2mg$, $l = a$ and $e = \frac{4a}{3} - a = \frac{a}{3}$.

R(←)

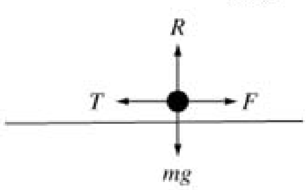
$T + F = mr\omega_{max}^2$

$\therefore \frac{2mg}{3} + \frac{3}{5}mg = m \times \frac{4a}{3}\omega_{max}^2$

$\therefore \omega_{max}^2 = \frac{19g}{20a}$

> $F = \mu R = \frac{3}{5}mg$, as in **a**.

> This is the greatest possible value of ω^2.



> Draw a diagram showing friction acting away from the centre of the circular motion.

Case **ii** minimum value for ω:

R(←)

$T - F = mr\omega_{min}^2$

$\therefore \frac{2mg}{3} - \frac{3mg}{5} = m \times \frac{4a}{3}\omega_{min}^2$

$\therefore \omega_{min}^2 = \frac{g}{20a}$

> This is the least possible value of ω^2.

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Review Exercise 2
Exercise A, Question 15

Question:

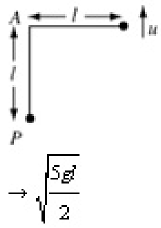
One end of a light inextensible string of length l is attached to a particle P of mass m. The other end is attached to a fixed point A. The particle is hanging freely at rest with the string vertical when it is projected horizontally with speed $\sqrt{\frac{5gl}{2}}$.

a Find the speed of P when the string is horizontal.
When the string is horizontal it comes into contact with a small smooth fixed peg which is at the point B, where AB is horizontal, and $AB < l$. Given that the particle then describes a complete semi-circle with centre B,

b find the least possible value of the length AB. **[E]**

Solution:

a



Let u be the speed of P when the string is horizontal.

Conservation of energy:

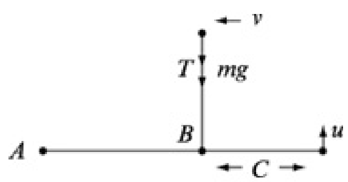
$$\frac{1}{2}m\left[\frac{5gl}{2}-u^2\right]=mgl$$

Using loss of kinetic energy = gain in potential energy.

$$\therefore u^2=\frac{5gl}{2}-2gl=\frac{gl}{2}$$

$$\therefore u=\sqrt{\frac{gl}{2}}$$

b



Let the particle move in a semi-circle of radius r.

Conservation of energy

$$\frac{1}{2}m(u^2-v^2)=mgr$$

$$\therefore v^2=u^2-2gr \quad ①$$

Write down an equation of motion $F=ma$ when the particle is at the highest point.

$$R(\downarrow): T+mg=\frac{mv^2}{r}$$

$$\therefore T=m\frac{(u^2-2gr)}{r}-mg$$

Substitute from equation ①.

$$=\frac{mu^2}{r}-3mg$$

Use the value of u from part **a**.

$$=\frac{mgl}{2r}-3mg$$

As $T\geq 0 \Rightarrow \frac{mgl}{2r}\geq 3mg$

$$\therefore \frac{l}{6}\geq r$$

As the string does not go slack $T\geq 0$. Use this condition to find the least possible value of the length AB.

$\therefore$ least value of AB is $l-\frac{l}{6}=\frac{5l}{6}$

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Review Exercise 2
Exercise A, Question 16

Question:

One end of a light inextensible string of length l is attached to a fixed point A. The other end is attached to a particle P of mass m which is hanging freely at rest at point B. The particle P is projected horizontally from B with speed $\sqrt{3gl}$. When AP makes an angle θ with the downward vertical and the string remains taut, the tension in the string is T.

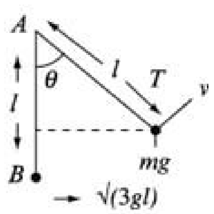
a Show that $T = mg(1+3\cos\theta)$.

b Find the speed of P at the instant when the string becomes slack.

c Find the maximum height above the level of B reached by P. **[E]**

Solution:

a



Use conservation of energy:

$$\frac{1}{2}m(u^2 - v^2) = mgl(1-\cos\theta)$$

$$\therefore v^2 = u^2 - 2gl(1-\cos\theta)$$

$$= 3gl - 2gl + 2gl\cos\theta$$

$$\text{i.e. } v^2 = gl + 2gl\cos\theta \quad ①$$

Resolve along the string

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$= \frac{mgl + 2mgl\cos\theta}{l}$$

$$\therefore T = mg(1+3\cos\theta)$$

Use loss in kinetic energy = gain in potential energy.

Make v^2 the subject of the formula.

Use Newton's 2nd Law $F = ma$ with $a = \frac{v^2}{r}$ and $r = l$.

Substitute v^2 from equation ①.

b Put $T = 0$

Then $1+3\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{3}$

$$\therefore v^2 = gl + 2gl\left(-\frac{1}{3}\right)$$

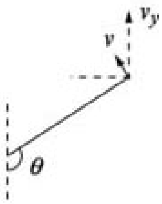
$$= \frac{gl}{3}$$

$$\therefore v = \sqrt{\frac{gl}{3}}$$

When the string becomes slack, $T = 0$.

Solve to find $\cos\theta$ and substitute into equation ① to give v^2.

c



The particle now moves as a projectile, under gravity. Maximum height is achieved when the vertical component of the velocity is zero.

$v_y = v\sin\theta = \sqrt{\frac{gl}{3}} \times \frac{2\sqrt{2}}{3}$

Consider vertical motion

$u = v_y, v = 0, s = h, a = -g$

use $v^2 = u^2 - 2gh$

$\therefore h = \frac{v_y^2}{2g} = \frac{gl}{3} \times \frac{8}{9} \times \frac{1}{2g}$

This is the height above the point at which the string becomes slack.

i.e. $h = \frac{4l}{27}$

$\therefore H = l(1-\cos\theta) + \frac{4l}{27} = \frac{4l}{3} + \frac{4l}{27}$

i.e. maximum height above B is $\frac{40l}{27}$.

You need to add $l(1-\cos\theta)$ to obtain the height above B.

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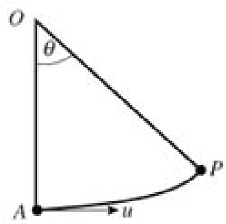
Review Exercise 2
Exercise A, Question 17

Question:



A particle of mass m is attached to one end of a light inextensible string of length l. The other end of the string is attached to a fixed point O. The particle is hanging at the point A, which is vertically below O. It is projected horizontally with speed u. When the particle is at the point P, $\angle AOP = \theta$, as shown in the diagram. The string oscillates through an angle α on either side of OA where $\cos\alpha = \frac{2}{3}$.

a Find u in terms of g and l.

When $\angle AOP = \theta$, the tension in the string is T.

b Show that $T = \frac{mg}{3}(9\cos\theta - 4)$.

c Find the range of values of T. **[E]**

Solution:

a Using conservation of energy

$$\frac{1}{2}mu^2 = mgl(1-\cos\alpha)$$

$$= \frac{mgl}{3}$$

$$\therefore u^2 = \frac{2}{3}gl \text{ and } u = \sqrt{\frac{2gl}{3}}$$

Use loss of kinetic energy = gain in potential energy and substitute $\cos\alpha = \frac{7}{3}$.

b Resolve along the string

Use Newton's 2nd Law $F = ma$ with $a = \frac{v^2}{r}$ and $r = l$.

$$T - mg\cos\theta = \frac{mv^2}{l} \quad ①$$

Conservation of energy:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgl(1-\cos\theta) \quad ②$$

Use loss of K.E. = gain in P.E.

$$\therefore T = mg\cos\theta + \frac{mu^2}{l} - 2mg(1-\cos\theta)$$

$$= 3mg\cos\theta + \frac{2mg}{3} - 2mg$$

$$= \frac{mg}{3}(9\cos\theta - 4)$$

Eliminate v^2 from equations ① and ② and substitute the value for u^2 from part **a**.

c Maximum value of T is when $\theta = 0$

$$T_{\max} = \frac{5mg}{3}$$

When $\theta = 0$, $\cos\theta = 1$ which is the maximum value that $\cos\theta$ can take.

Minimum value of T is when

$$\cos\theta = \frac{2}{3}$$

$$T_{\min} = \frac{2mg}{3}$$

$\cos\theta = \frac{2}{3}$ is the minimum value that $\cos\theta$ can take.

$$\therefore \frac{2mg}{3} \le T \le \frac{5mg}{3}$$

State the range of values of T, as requested.

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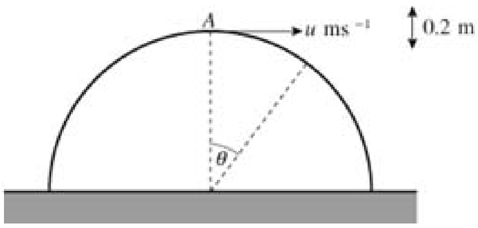
Review Exercise 2
Exercise A, Question 18

Question:



A smooth solid hemisphere, of radius 0.8 m and centre O, is fixed with its plane face on a horizontal table. A particle of mass 0.5 kg is projected horizontally with speed u m s^{-1} from the highest point A of the hemisphere. The particle leaves the hemisphere at the point B, which is a vertical distance of 0.2 m below the level of A. The speed of the particle at B is v m s^{-1} and the angle between OA and OB is θ, as shown in the diagram.

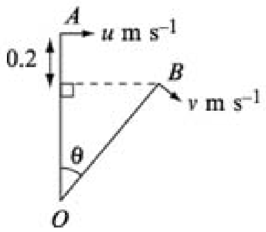
a Find the value of $\cos\theta$.

b Show that $v^2 = 5.88$.

c Find the value of u. **[E]**

Solution:

a



Draw a diagram indicating the lengths given in the question.

As AO is a radius, $AO = 0.8\,\text{m}$

$AN = 0.2\,\text{m}$

But $\therefore\ ON = 0.8 - 0.2$

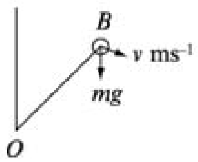
$= 0.6\,\text{m}$

OB is a radius, $OB = 0.8\,\text{m}$

$\therefore\ \cos\theta = \dfrac{0.6}{0.8} = \dfrac{3}{4}$ or 0.75

Identify the lengths of sides of ΔOBN where BN is perpendicular to OA.

b



As the particle leaves the hemisphere at B the normal reaction $R = 0$.

Resolve along the radius OB:

$mg\cos\theta = \dfrac{mv^2}{l} = \dfrac{mv^2}{0.8}$

$\therefore\ v^2 = 0.8g\cos\theta = 0.6g$

$= 5.88\ (3\text{ s.f.})$

Use Newton's 2nd Law $F = ma$ with $a = \dfrac{v^2}{l}$.

This is the answer printed on the question so give 3 s.f.

c $\dfrac{1}{2}m \times 5.88 - \dfrac{1}{2}mu^2 = mg \times 0.2$

$\therefore\ u^2 = 5.88 - 0.4g$

$\therefore\ u^2 = 0.2g$

$u = 1.4$

Use gain in kinetic energy = loss in potential energy.

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Review Exercise 2
Exercise A, Question 19

Question:

A smooth solid sphere, with centre O and radius a, is fixed to the upper surface of a horizontal table. A particle P is placed on the surface of the sphere at a point A, where OA makes an angle α with the upward vertical, and $0 < \alpha < \frac{\pi}{2}$. The particle is released from rest. When OP makes an angle θ with the upward vertical, and P is still on the surface of the sphere, the speed of P is v.

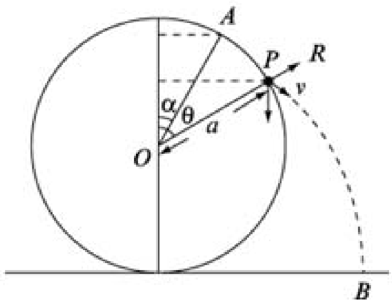
a Show that $v^2 = 2ga(\cos\alpha - \cos\theta)$.

Given that $\cos\alpha = \frac{3}{4}$, find

b the value of θ when P loses contact with the sphere.

c the speed of P as it hits the table. **[E]**

Solution:



Draw a diagram.

a $\frac{1}{2}mv^2 = mg(a\cos\alpha - a\cos\theta)$

$\therefore v^2 = 2ga(\cos\alpha - \cos\theta)$

Use gain in kinetic energy = loss in potential energy.

b Resolve along the radius

$mg\cos\theta - R = \frac{mv^2}{a}$ and $R = 0$

When P losses contact with the sphere the normal reaction $R = 0$.

$\therefore g\cos\theta = 2g(\cos\alpha - \cos\theta)$

$= 2g\left(\frac{3}{4} - \cos\theta\right)$

Use Newton's 2nd Law.

$\therefore 3g\cos\theta = \frac{3g}{2} \Rightarrow \cos\theta = \frac{1}{2}$

i.e. $\theta = 60°$

Substitute $\cos\alpha = \frac{3}{4}$ and find the value of $\cos\theta$.

c From A to B

$\frac{1}{2}m\omega^2 = mg(a + a\cos\alpha)$

You may use gain in kinetic energy = loss in potential energy.

$\therefore \omega^2 = 2ga\left(1 + \frac{3}{4}\right)$

$\therefore \omega = \left(\frac{7ga}{2}\right)^{\frac{1}{2}}$

There are alternative methods involving projectiles, but this is the shortest method.

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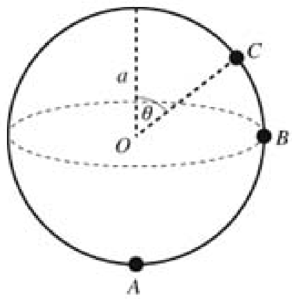
Review Exercise 2
Exercise A, Question 20

Question:



The diagram shows a fixed hollow sphere of internal radius a and centre O. A particle P of mass m is projected horizontally from the lowest point A of a sphere with speed $\sqrt{\left(\frac{7}{2}ag\right)}$. It moves in a vertical circle, centre O, on the smooth inner surface of the sphere. The particle passes through the point B, which is in the same horizontal plane as O. It leaves the surface of the sphere at the point C, where OC makes an angle θ with the upward vertical.

a Find, in terms of m and g, the normal reaction between P and the surface of the sphere at B.

b Show that $\theta = 60°$.

After leaving the surface of the sphere, P meets it again at the point A.

c Find, in terms of a and g, the time P takes to travel from C to A. **[E]**

Solution:

a Conservation of energy:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga$$

Use loss in K.E. = gain in P.E.

i.e. $\frac{1}{2}m \times \frac{7ag}{2} - mga = \frac{1}{2}mv^2$ or $v^2 = \frac{3}{2}ga$

Resolve along radius ←

$$R = \frac{mv^2}{a}$$

When the particle is at B, the radius is horizontal use $F = ma$.

i.e. $R = \frac{3}{2}mg$

b $\frac{1}{2}m \times \frac{7ga}{2} - \frac{1}{2}mV^2 = mga(1+\cos\theta)$ ①

Loss in K.E. = gain in P.E.

Resolving along radius

$mg\cos\theta = \frac{mV^2}{a}$ ②

The normal reaction is zero at this point C, as the particle leaves the sphere.

Eliminate V^2 from equations ① and ②

$$ag\cos\theta = \frac{7ga}{2} - 2ga(1+\cos\theta)$$

Solve to find $\cos\theta$.

$$\therefore 3ga\cos\theta = \frac{3ga}{2} \Rightarrow \cos\theta = \frac{1}{2}$$

$$\therefore \theta = 60°$$

c Consider motion in horizontal direction.

Distance $= a\sin 60°$

$V_x = V\cos 60°$

The particle moves as a projectile.

$$\therefore \text{Time } t = \frac{a\sin 60°}{V\cos 60°}$$

But $V^2 = ag\cos 60° = \frac{ag}{2}$

$$\therefore t = a\tan 60° \div \sqrt{\frac{ag}{2}}$$

This comes from equation ② in part **b**.

$$= \sqrt{\frac{6a}{g}}$$

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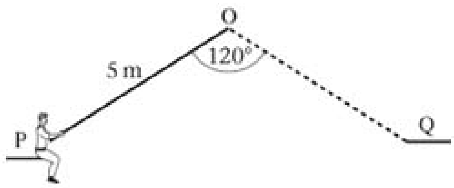
Review Exercise 2
Exercise A, Question 21

Question:



A trapeze artiste of mass 60 kg is attached to the end A of a light inextensible rope OA of length 5 m. The artiste must swing in an arc of a vertical circle, centre O, from a platform P to another platform Q, where PQ is horizontal. The other end of the rope is attached to the fixed point O which lies in the vertical plane containing PQ, with $\angle POQ = 120°$ and $OP = OQ = 5$ m, as shown in the diagram.

As part of her act, the artiste projects herself from P with speed $\sqrt{15}$ m s^{-1} in a direction perpendicular to the rope OA and in the plane POQ. She moves in a circular arc towards Q. At the lowest point of her path she catches a ball of mass m kg which is travelling towards her with speed 3 m s^{-1} and parallel to QP. After catching the ball, she comes to rest at the point Q.

By modelling the artiste and the ball as particles and ignoring her air resistance, find

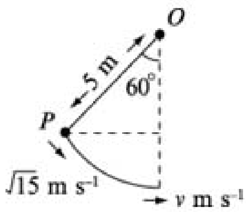
a the speed of the artiste immediately before she catches the ball,

b the value of m,

c the tension in the rope immediately after she catches the ball. **[E]**

Solution:

a



$$\frac{1}{2}mv^2 - \frac{1}{2}m(\sqrt{15})^2 = mg \times 5(1 - \cos 60°)$$

Gain in kinetic energy = loss in potential energy.

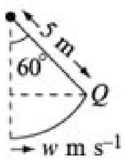
$$\therefore v^2 = 15 + 10g \times \frac{1}{2}$$

i.e. $v = 8$

Speed is 8 m s^{-1}.

b Let her velocity after catching the ball be ω m s^{-1}.

Consider motion after catching the ball.



$$\therefore \frac{1}{2}M\omega^2 - 0 = Mg5(1 - \cos 60°)$$

$$\therefore \omega^2 = 49$$

i.e. $\omega = 7$

Loss in K.E. = Gain in P.E.

Then use C.L.M. when she catches the ball:

	(60)	(m)
Before impact	$\rightarrow 8 \text{ m s}^{-1}$	$\leftarrow 3 \text{ m s}^{-1}$
After impact	(60 + m) $\rightarrow 7 \text{ m s}^{-1}$	

Use conservation of linear momentum at the instant when she catches the ball.

$$\therefore 60 \times 8 - 3 \times m = (60 + m) \times 7$$

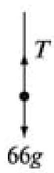
i.e. $480 - 3m = 420 + 7m$

i.e. $60 = 10m$

i.e. $m = 6$

The 3 m s^{-1} is in the opposite direction from the 8 m s^{-1}, so use a minus sign.

c



Use Newton's 2nd Law $F = ma$ where $a = \frac{v^2}{r}$.

R(↑)

$$T - 66g = 66 \times \frac{7^2}{5}$$

i.e. $T = 132g$ or 1300 N (2 s.f.)

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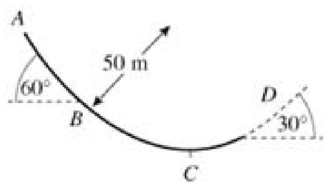
Review Exercise 2
Exercise A, Question 22

Question:



The diagram represents the path of a skier of mass 70 kg moving on a ski-slope *ABCD*. The path lies in a vertical plane. From *A* to *B*, the path is modelled as a straight line inclined at $60°$ to the horizontal. From *B* to *D*, the path is modelled as an arc of a vertical circle of radius 50 m. The lowest point of the arc *BD* is *C*.
At *B*, the skier is moving downwards with speed 20 m s^{-1}. At *D*, the path is inclined at $30°$ to the horizontal and the skier is moving upwards. By modelling the slope as smooth and the skier as a particle, find

a the speed of the skier at *C*,
b the normal reaction of the slope on the skier at *C*,
c the speed of the skier at *D*,
d the change in the normal reaction of the slope on the skier as she passes *B*.

The model is refined to allow for the influence of friction on the motion of the skier.
e State briefly, with a reason, how the answer to part **b** would be affected by using such a model. (No further calculations are expected.) **[E]**

Solution:

a Let the speed at C be v m s^{-1}

$$\frac{1}{2}mv^2 - \frac{1}{2}m \times 20^2 = mg \times 50(1-\cos 60^\circ)$$

Gain in K.E. = loss of P.E.

$v^2 = 20^2 + 50g \Rightarrow v^2 = 890$

$\therefore v = 30$ (2 s.f.) $\therefore$ Speed is 30 m s^{-1}.

b $\uparrow$ at C: $R - mg = \frac{m \times 890}{50}$

$\therefore R = 1900$ (2 s.f.)

Normal reaction is 1900 Newtons.

Use $F = ma$ where $a = \frac{v^2}{r}$.

c Consider motion C to D. Let speed of skier at D be ω m s^{-1}.

Then $\frac{1}{2}m \times 890 - \frac{1}{2}m\omega^2 = mg \times 50(1-\cos 30^\circ)$

Use loss of K.E. = gain in P.E.

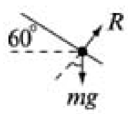
$\therefore \omega^2 = 890 - 100g(1-\cos 30^\circ)$

$= 759$

$\therefore \omega = 28$ (to 2 s.f.)

$\therefore$ Speed of D is 28 m s^{-1}.

d Before:



$R = mg\cos 60^\circ$

After:

$R - mg\cos 60^\circ = \frac{m \times 20^2}{50}$

i.e.: $R = mg\cos 60^\circ + \frac{m \times 20^2}{50}$

Resolve perpendicular to the slope at B just before the circular motion and just as circular motion begins.

$\therefore$ Change in R is $\frac{70 \times 20^2}{50} = 560$ N

So change in normal reaction is 560 N.

e Lower speed at $C \Rightarrow$ the normal reaction is reduced.

Starting 'lower speed' gives the reason for the reduction in normal reaction.

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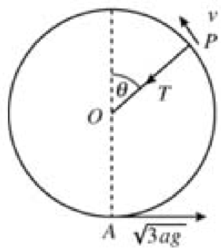
Review Exercise 2
Exercise A, Question 23

Question:



A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a point O. The point A is vertically below O, and $OA = a$. The particle is projected horizontally from A with speed $\sqrt{(3ag)}$. When OP makes an angle θ with the upward vertical through O and the string is still taut, the tension in the string is T and the speed of P is v, as shown in the diagram.

a Find, in terms of a, g and θ, an expression for v^2.

b Show that $T = (1 - 3\cos\theta)mg$.

The string becomes slack when P is at the point B.

c Find, in terms of a, the vertical height of B above A.

After the string becomes slack, the highest point reached by P is C.

d Find, in terms of a, the vertical height of C above B. **[E]**

Solution:

a Conservation of energy:

$$\frac{1}{2}m3ag - \frac{1}{2}mv^2 = mga(1+\cos\theta)$$

$$\therefore v^2 = ag(1-2\cos\theta)$$

Use loss of K.E. = gain in P.E.

b Resolve ↙ along radius:

$$T + mg\cos\theta = \frac{mv^2}{a}$$

$$\therefore T = (1-3\cos\theta)mg$$

Use Newton's 2nd Law $F = ma$ with $a = \frac{v^2}{r}$.

c Use $T = 0$, then $\cos\theta = \frac{1}{3}$

String becomes slack when $T = 0$.

$$\therefore \text{height above } A = a + \frac{1}{3}a = \frac{4}{3}a$$

Substitute into $h = a(1+\cos\theta)$.

d At point B, $v^2 = \frac{1}{3}ag$

Consider vertical motion

$$(v\sin\theta)^2 = 2gh$$

As $\cos\theta = \frac{1}{3}, \cos^2\theta = \frac{1}{9}$ and $\sin^2\theta = \frac{8}{9}$

$$\therefore \frac{1}{3}ag \times \frac{8}{9} = 2gh$$

$$\therefore h = \frac{4}{27}a \text{ or } 0.148a$$

This method considers motion under gravity but the solution could be found using energy.

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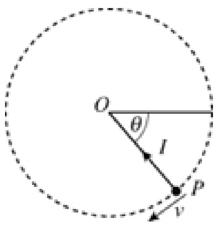
Review Exercise 2
Exercise A, Question 24

Question:



A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is fixed at a point O. The particle is held with the string taut and OP horizontal. It is then projected vertically downwards with speed u, where $u^2 = \frac{3}{2}ga$. When OP has turned through an angle θ and the string is still taut, the speed of P is v and the tension in the string is T, as shown in the diagram above.

a Find an expression for v^2 in terms of a, g and θ.

b Find an expression for T in terms of m, g and θ.

c Prove that the string becomes slack when $\theta = 210°$.

d State, with a reason, whether P would complete a vertical circle if the string were replaced by a light rod.

After the string becomes slack, P moves freely under gravity and is at the same level as O when it is at the point A.

e Explain briefly why the speed of P at A is $\sqrt{\left(\frac{3}{2}ga\right)}$.

The direction of motion of P at A makes an angle ϕ with the horizontal.

f Find ϕ. **[E]**

Solution:

a $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga\sin\theta$

$\therefore\ v^2 = u^2 + 2ga\sin\theta$

$= \frac{3}{2}ga + 2ga\sin\theta$ ①

Gain in K.E. = loss in P.E. from conservation of energy.

b Radial equation:

$T - mg\sin\theta = \frac{mv^2}{a}$

$T = mg\sin\theta + \frac{3}{2}mg + 2mg\sin\theta$

$= \frac{3mg}{2}(1 + 2\sin\theta)$

Use Newton's 2nd Law in the direction of the radius.

c Put $T = 0$, then $\sin\theta = -\frac{1}{2}$ so $\theta = 210^\circ$

When the string is *slack* $T = 0$.

d Set $v = 0$ in ①

Then $\sin\theta = -\frac{3}{4}$, so not a complete circle.

i.e. P would not complete vertical circle.

To complete the circle $v \neq 0$ before reaching the top point.

e Consider motion at start and at A:

no change in potential energy $\Rightarrow$

no change in kinetic energy

so $v = u = \sqrt{\frac{3}{2}ga}$.

The particle began its motion at the same level as O and thus at the same level as A.

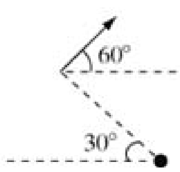
f When the string becomes slack

$v^2 = \frac{3}{2}ga + 2ga\left(-\frac{1}{2}\right) = \frac{1}{2}ga$

Its horizontal component of velocity

is $\sqrt{\frac{1}{2}ga}\cos 60^\circ$

Substitute $\sin\theta = -\frac{1}{2}$ from **c** into the equation obtained in part **a**.



When P reaches point A, horizontal component of velocity is $\sqrt{\frac{3}{2}ga}\cos\phi$

$\therefore \sqrt{\frac{3ga}{2}}\cos\phi = \sqrt{\frac{1}{2}ga}\cos 60^\circ$

i.e. $\cos\phi = \frac{\cos 60^\circ}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$

i.e. $\phi = 73.2^\circ$ (3 s.f.)

There are a number of possible methods but conservation of horizontal component of velocity is direct and short.

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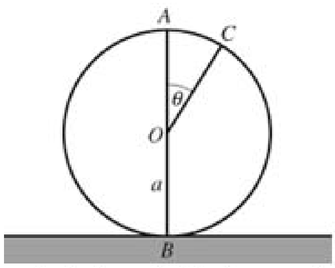
Review Exercise 2
Exercise A, Question 25

Question:



A particle is at the highest point A on the outer surface of a fixed smooth sphere of radius a and centre O. The lowest point B of the sphere is fixed to a horizontal plane. The particle is projected horizontally from A with speed u, where $u < \sqrt{(ag)}$. The particle leaves the sphere at the point C, where OC makes an angle θ with the upward vertical, as shown in the diagram above.

a Find an expression for $\cos\theta$ in terms of u, g and a.

The particle strikes the plane with speed $\sqrt{\left(\frac{9ag}{2}\right)}$.

b Find, to the nearest degree, the value of θ. **[E]**

Solution:

a $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga(1-\cos\theta)$ ← Gain in K.E. = Loss in P.E.

Resolve in radical direction:

$-R + mg\cos\theta = \frac{mv^2}{a}$ ① ← Use equation on motion $F = ma$.

When particle leaves the sphere $R = 0$

$\therefore mg\cos\theta = \frac{mu^2}{a} + 2mg(1-\cos\theta)$ ← Put $R = 0$ and eliminate v^2.

$\therefore 3\cos\theta = \frac{u^2}{ag} + 2$ ← Make $\cos\theta$ the subject of the formula.

i.e. $\cos\theta = \frac{2}{3} + \frac{u^2}{3ag}$

b Conservation of energy between C and the ground

$\frac{1}{2}m\left(\frac{9ag}{2}\right) - \frac{1}{2}mv^2 = mga(1+\cos\theta)$ ← Energy is the most direct method, though you could use projectiles.

Putting $R = 0$ into equation ① gives

$v^2 = ag\cos\theta$ ← Find an expression for the velocity when the particle leaves the sphere.

$\therefore \frac{9}{4}ag - \frac{1}{2}ag\cos\theta = ag + ag\cos\theta$

$\therefore \frac{5}{4}ag = \frac{3}{2}ag\cos\theta$ ← Substitute v^2 into the energy equation to find $\cos\theta$.

$\therefore \cos\theta = \frac{5}{6}$

So $\theta = 34^\circ$ (nearest degree) ← Give θ to the nearest degree.

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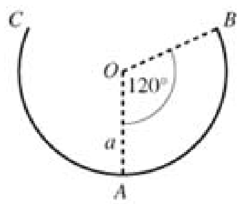
Review Exercise 2
Exercise A, Question 26

Question:



Part of a hollow spherical shell, centre O and radius a, is removed to form a bowl with a plane circular rim. The bowl is fixed with the circular rim uppermost and horizontal. The point A is the lowest point of the bowl. The point B is on the rim of the bowl and $\angle AOB = 120°$, as shown in the diagram above. A smooth small marble of mass m is placed inside the bowl at A and given an initial horizontal speed u. The direction of motion of the marble lies in the vertical plane AOB. The marble stays in contact with the bowl until it reaches B. When the marble reaches B, its speed is v.

a Find an expression for v^2.

b For the case when $u^2 = 6ga$, find the normal reaction of the bowl on the marble as the marble reaches B.

c Find the least possible value of u for the marble to reach B.

The point C is the other point on the rim of the bowl lying in the vertical plane OAB.

d Find the value of u which will enable the marble to leave the bowl at B and meet it again at the point C. **[E]**

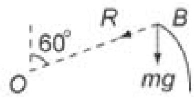
Solution:

a $\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga(1+\cos 60°)$

$\therefore v^2 = u^2 - 3ga$ ①

Loss in K.E. = Gain in P.E.

b Resolve in a radial direction:



Use equation of motion: $F = ma$ with $a = \frac{v^2}{r}$.

$R + mg\cos 60° = \frac{mv^2}{a}$ ②

When $u^2 = 6ga, v^2 = 3ga$

Use equation ①.

$\therefore R = 3mg - \frac{mg}{2}$

i.e. $R = \frac{5mg}{2}$

Substitute into equation ②.

The least value for u will enable particle to teach B, but to leave the surface at B, so $R = 0$.

c $R = 0$ at point B $\Rightarrow \frac{mg}{2} = \frac{mv^2}{a}$

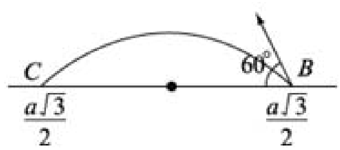
$\Rightarrow v^2 = \frac{ag}{2}$

Use equation ①.

$\therefore \frac{ag}{2} = u^2 - 3ga$

i.e. $u^2 = \frac{7ga}{2} \Rightarrow u = \sqrt{\frac{7ga}{2}}$

d



Consider motion of a projectile under gravity.

$(\rightarrow) B$ to $C: v\cos 60° \times t = a\sqrt{3}$

$\therefore t = \frac{2a\sqrt{3}}{v}$

First consider horizontal component of motion.

$(\uparrow) B$ to $C: 0 = v\sin 60° t - \frac{1}{2}gt^2$

$\Rightarrow t = \frac{2v\sin 60°}{g} = \frac{v\sqrt{3}}{g}$

Then consider the vertical component.

$\therefore \frac{2a\sqrt{3}}{v} = \frac{v\sqrt{3}}{g} \Rightarrow v^2 = 2ag$

Equate the two expressions for t and find v^2.

$\therefore$ From ①, $u^2 = 5ag$, i.e. $u = \sqrt{5ag}$

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Review Exercise 2
Exercise A, Question 27

Question:

A uniform solid is formed by rotating the region enclosed between the curve with equation $y = \sqrt{x}$, the x-axis and the line $x = 4$, through one complete revolution about the x-axis. Find the distance of the centre of mass of the solid from the origin O. **[E]**

Solution:

$$\bar{x} = \frac{\pi \int y^2 x \, dx}{\pi \int y^2 \, dx}$$

You should learn this formula.

Substitute $y = \sqrt{x}$ into the formula.

$$= \frac{\pi \int_0^4 x^2 \, dx}{\pi \int_0^4 x \, dx}$$

Integrate the numerator and the denominator.

$$= \frac{\left[\frac{1}{3}\pi x^3\right]_0^4}{\left[\frac{1}{2}\pi x^2\right]_0^4}$$

Evaluate and simplify.

$$= \frac{64}{3} \div 8$$

$$= \frac{8}{3}$$

$\therefore$ The centre of mass of the solid is at a distance $\frac{8}{3}$ from O.

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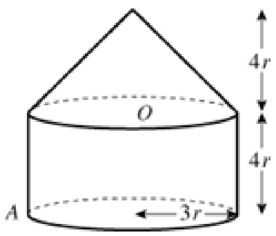
Review Exercise 2
Exercise A, Question 28

Question:



A toy is formed by joining a uniform solid right circular cone, of base radius $3r$ and height $4r$, to a uniform solid cylinder, also of radius $3r$ and height $4r$. The cone and the cylinder are made from the same material, and the plane face of the cone coincides with a plane face of the cylinder, as shown in the diagram. The centre of this plane face is O.

a Find the distance of the centre of mass of the toy from O.

The point A lies on the edge of the plane face of the cylinder which forms the base of the toy. The toy is suspended from A and hangs in equilibrium.

b Find, in degrees to one decimal place, the angle between the axis of symmetry of the toy and the vertical.

The toy is placed with the curved surface of the cone on horizontal ground.

c Determine whether the toy will topple. **[E]**

Solution:

a

Shape	Mass	Mass ratio	Distance of centre of mass from O
Cylinder	$36\pi\rho r^3$	3	$2r$
Cone	$12\pi\rho r^3$	1	$-r$
Toy	$48\pi\rho r^3$	4	$\bar{x}$

Let ρ be the mass per unit volume.

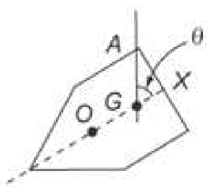
Set out table – downward from O is positive here.

$\circlearrowleft M(O): 3\times 2r + 1\times(-r) = 4\bar{x}$

$x = \frac{5r}{4}$

Centre of mass is $\frac{5r}{4}$ below point O on the figure.

b



In ΔAGX, AX is the radius $3r$ of the cylinder and X is the centre of the base of the cylinder.

AG is vertical.
In ΔAGX,

$GX = OX - OG$.

$$\tan\theta = \frac{AX}{GX}$$

$$\therefore \tan\theta = \frac{3r}{4r - \bar{x}}$$

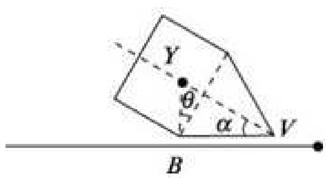
$$= \frac{3r}{4r - \frac{5r}{4}}$$

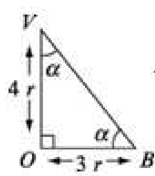
$$= \frac{12}{11}$$

$\therefore \theta = 47.5^\circ$ (1 d.p.)

The question asked for the angle to 1 decimal place.

c





The toy will not topple if G (the centre of mass) is vertically above a point between B and V.

$\tan\alpha = \frac{3r}{4r}$ from ΔVOB

$\tan\alpha = \frac{OY}{OB} = \frac{OY}{3r}$ from ΔBOY

Let Y be the point directly above B on the axis of symmetry of the toy.

$\therefore \frac{OY}{3r} = \frac{3}{4} \Rightarrow OY = \frac{9r}{4}$

But $OG = \frac{5r}{4}$ so $OG < OY$

This means that the toy will not topple.

Then if $VG < VY$ toy will not topple.
i.e. $OG < OY$ – no toppling.

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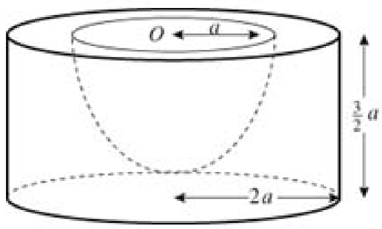
Review Exercise 2
Exercise A, Question 29

Question:



A uniform solid cylinder has radius $2a$ and height $\frac{3}{2}a$. A hemisphere of radius a is removed from the cylinder. The plane face of the hemisphere coincides with the upper plane face of the cylinder, and the centre O of the hemisphere is also the center of this plane face, as shown in the diagram above. The remaining solid is S.

a Find the distance of the centre of mass of S from O.

The lower plane face of S rests in equilibrium on a desk lid which is inclined at an angle θ to the horizontal. Assuming that the lid is sufficiently rough to prevent S from slipping, and that S is on the point of toppling when $\theta = \alpha$,

b find the value of α.

Given instead that the coefficient of friction between S and the lid is 0.8, and that S is on the point of sliding down the slide when $\theta = \beta$,

c find the value of β. **[E]**

Solution:

a

Shape	Mass	Mass ratios	Distance of centre of mass from O
Cylinder	$\pi\rho(2a)^2\left(\frac{3}{2}a\right)$	6	$\frac{3}{4}a$
Hemi-sphere	$\frac{2}{3}\pi\rho a^3$	$\frac{2}{3}$	$\frac{3}{8}a$
Remainder	$\pi\rho\left[6a^3-\frac{2}{3}a^3\right]$	$\frac{16}{3}$	$\bar{x}$

Draw a table giving mass ratios and distances.

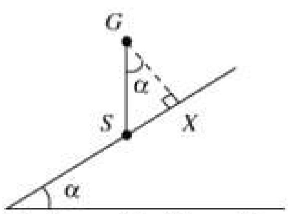
$$\circlearrowleft M(O): 6\times\frac{3}{4}a-\frac{2}{3}\times\frac{3}{8}a = \frac{16}{3}\bar{x}$$

$$\therefore \frac{9}{2}a-\frac{1}{4}a=\frac{16}{3}\bar{x}$$

$$\therefore \bar{x}=\frac{51a}{64} \text{ or } 0.797a\ (3\text{ s.f.})$$

Take moments and find $\bar{x}$, the distance of the centre of mass from O.

b



Draw a diagram showing G, the position of the centre of mass, above S.

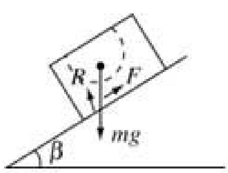
On the point of toppling: G is above S – the lowest point on the bottom circular face.

$$\tan\alpha = \frac{SX}{XG}=\frac{2a}{\frac{3}{2}a-\bar{x}}=\frac{2a}{\frac{45a}{64}}$$

Let X be the centre of the base of the cylinder.

$$\therefore \tan\alpha=\frac{128}{45}\Rightarrow \alpha=70.6^\circ$$

c



Draw a diagram showing the forces acting on the solid.

$R(\nwarrow): R-mg\cos\beta=0$

$\therefore R=mg\cos\beta$

$R(\nearrow): F-mg\sin\beta=0$

$\therefore F=mg\sin\beta$

Resolve perpendicular to and parallel to the plane

Using $F=\mu R$: $mg\sin\beta=0.8mg\cos\beta$

$\therefore \tan\beta=0.8$

So $\beta=38.7^\circ$

Use the condition for sliding that $F=\mu R$.

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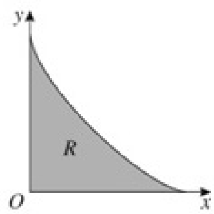
Review Exercise 2
Exercise A, Question 30

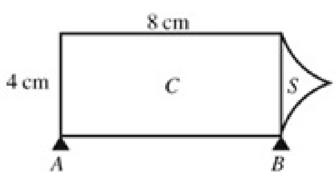
Question:



The shaded region R is bounded by part of the curve with equation $y=\frac{1}{2}(x-2)^2$, the x-axis and the y-axis, as shown above. The unit of length on both axis is 1 cm. A uniform solid S is made by rotating R through $360°$ about the x-axis. Using integration,

a calculate the volume of the solid S, leaving your answer in terms of π,

b show that the centre of mass of S is $\frac{1}{3}$ cm from its plane face.



A tool is modelled as having two components, a solid uniform cylinder C and the solid S. The diameter of C is 4 cm and the length of C is 8 cm. One end of C coincides with the plane face of S. The components are made of different materials. The weight of C is $10W$ newtons and the weight of S is $2W$ newtons. The tool lies in equilibrium with its axis of symmetry horizontal on two smooth supports A and B, which are at the ends of the cylinder, as shown above.

c Find the magnitude of the force of the support A on the tool. **[E]**

Solution:

a $V = \pi \int y^2 \, dx$

$= \pi \int_0^2 \frac{1}{4}(x-2)^4 \, dx$ — Substitute $y = \frac{1}{2}(x-2)^2$ into equation.

$= \frac{\pi}{4}\left[\frac{1}{5}(x-2)^5\right]_0^2$

$= \frac{\pi}{4}\left[0 - \left(\frac{-32}{5}\right)\right]$ — Use limits 0 and 2 as $y = 0 \Rightarrow x = 2$.

$= \frac{32\pi}{20} = \frac{8\pi}{5}$

b $\bar{x} = \dfrac{\pi \int xy^2 \, dx}{\pi \int y^2 \, dx}$ — Use the standard formula.

$= \dfrac{\pi \int_0^2 \frac{1}{4}x(x-2)^4 \, dx}{\frac{8\pi}{5}}$ — Use a substitution $u = x-2, \frac{du}{dx} = 1$ and $x = (u+2)$

$= \frac{5}{8\pi} \times \frac{\pi}{4} \int_{-2}^0 (u+2)u^4 \, du$ — Change the limits of integration.

$= \frac{5}{32} \int_{-2}^0 u^5 + 2u^4 \, du$

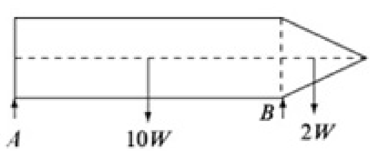
$= \frac{5}{32}\left[\frac{1}{6}u^6 + \frac{2}{5}u^5\right]_{-2}^0$

$= \frac{5}{32}\left[0 - \left(\frac{64}{6} - \frac{64}{5}\right)\right]$ — The centre of mass lies on the axis of symmetry at a distance $\frac{1}{3}$ cm from the plane base.

$= \frac{5}{32} \times \frac{64}{30}$

$\therefore \bar{x} = \frac{1}{3}$

c



Let the reaction at A be A, and at B be B.

$\circlearrowleft M(B): A \times 8 + 2W \times \frac{1}{3} = 10W \times 4$

$\therefore A = \dfrac{\left(40 - \frac{2}{3}\right)W}{8} = \frac{118W}{24}$ — Take moments about B to eliminate the force B.

$= \frac{59W}{12}$ — This is the force of the support A on the tool.

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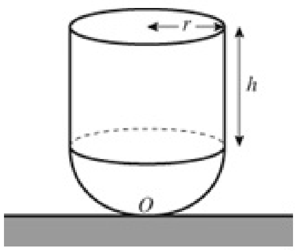
Review Exercise 2
Exercise A, Question 31

Question:



A child's toy consists of a uniform solid hemisphere attached to a uniform solid cylinder. The plane face of the hemisphere coincides with the plane face of the cylinder, as shown in the diagram above. The cylinder and the hemisphere each have radius r and the height of the cylinder is h. The material of the hemisphere is six times as dense as the material of the cylinder. The toy rests in equilibrium on a horizontal plane with the cylinder above the hemisphere and the axis of the cylinder vertical.

a Show that the distance d of the centre of mass of the toy from its lowest point O is given by $d = \frac{h^2 + 2hr + 5r^2}{2(h + 4r)}$

When the toy is placed with any point of the curved surface of the hemisphere resting on the plane it will remain in equilibrium.

b Find h in terms of r. **[E]**

Solution:

a

Shape	Mass	Mass ratio	Distance of centre of mass from O
Hemisphere	$\frac{2}{3}\pi r^3 6\rho$	$4r$	$\frac{5r}{8}$
Cylinder	$\pi r^2 h\rho$	h	$\frac{h}{2}+r$
Toy	$\pi r^2\rho(4r+h)$	$4r+h$	$\bar{x}$

Draw a table showing masses and position of centre of mass.

$$\circlearrowleft M(O)\quad 4r\times\frac{5r}{8}+h\left(\frac{h}{2}+r\right)=(4r+h)\bar{x}$$

$$\therefore \bar{x}=\frac{\frac{5}{2}r^2+\frac{h^2}{2}+rh}{4r+h}$$

$$\text{So } d=\frac{h^2+2hr+5r^2}{2(h+4r)}$$

Use moments equation and make $\bar{x}$ the subject of the formula.

b As $\bar{x}=r$

$$\therefore h^2+2hr+5r^2=2r(h+4r)$$

$$\text{i.e. } h^2+2hr+5r^2=2rh+8r^2$$

$$\therefore h^2=3r^2$$

$$\text{i.e. } h=\sqrt{3}r$$

The centre of mass must be at the centre of the hemisphere, so $\bar{x}=r$.

Solve the resulting equation to give h in terms of r.

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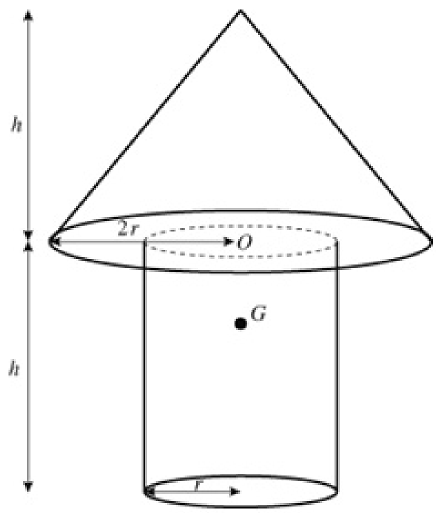
Review Exercise 2
Exercise A, Question 32

Question:



A model tree is made by joining a uniform solid cylinder to a uniform solid cone made of the same material. The centre O of the base of the cone is also the centre of one end of the cylinder, as shown in the diagram. The radius of the cylinder is r and the radius of the base of the cone is $2r$. The height of the cone and the height of the cylinder are each h. The centre of mass of the model is at the point G.

a Show that $OG = \frac{1}{14}h$.

The model stands on a desk top with its plane face in contact with the desk top. The desk top is tilted until it makes an angle α with the horizontal, where $\tan\alpha = \frac{7}{26}$. The desk top is rough enough to prevent slipping and the model is about to topple.

b Find r in terms of h. **[E]**

Solution:

a

Shape	Mass	Mass ratio	Position – distance of centre of mass from O
Cylinder	$\rho\pi r^2 h$	3	$+\frac{h}{2}$
Cone	$\frac{1}{3}\rho\pi(2r)^2 h$	4	$-\frac{h}{4}$
Tree	$\rho\pi r^2 h\left(1+\frac{4}{3}\right)$	7	$\bar{x}$

The centre of mass lies on the axis of symmetry.

Draw a table showing masses and positions of centre of mass.

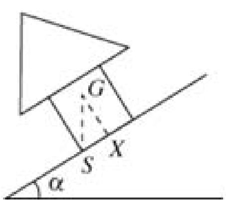
$$\circlearrowleft M(0): 3\times\frac{h}{2} - 4\times\frac{h}{4} = 7\bar{x}$$

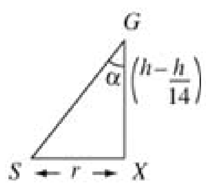
$$\therefore \frac{3h}{2} - h = 7\bar{x}$$

$$\therefore \bar{x} = \frac{h}{14}$$

Take moments and let $OG = \bar{x}$.

b





Draw a diagram showing G directly above S, the lowest point of the base of the cylinder.

Enlarge ΔGSX, where X is the centre of the circular base.

$$\tan\alpha = \frac{r}{h-\frac{h}{14}} = \frac{7}{26}$$

$$\therefore r = \frac{7}{26}\left(\frac{13h}{14}\right)$$

$$r = \frac{1}{4}h$$

Use $\tan\alpha$ to find r in terms of h.

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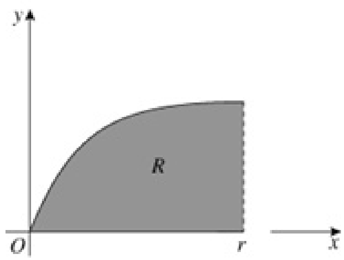
Review Exercise 2
Exercise A, Question 33

Question:



The diagram shows the region R bounded by the curve with equation $y^2 = rx$, where r is a positive constant, the x-axis and the line $x = r$. A uniform solid of revolution S is formed by rotating R through one complete revolution about the x-axis.

a Show that the distance of the centre of mass of S from O is $\frac{2}{3}r$.

The solid is placed with its plane face on a plane which is inclined at an angle α to the horizontal. The plane is sufficiently rough to prevent S from sliding. Given that S does not topple,

b find, to the nearest degree, the maximum value of α. **[E]**

Solution:

a The centre of mass lies on the axis of symmetry OX. Let the distance of the centre of mass of S from O be $\bar{x}$ and let it be at point G.

Use $\bar{x} = \dfrac{\pi \int y^2 x \, dx}{\pi \int y^2 \, dx}$ — Use the standard formula.

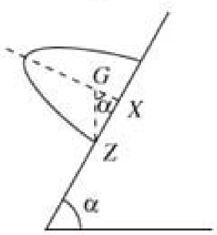
$= \dfrac{\int_0^r r x^2 \, dx}{\int_0^r r x \, dx}$ — Substitute $y^2 = rx$ into the formula.

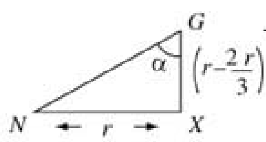
$= \dfrac{\left[\frac{1}{3} r x^3\right]_0^r}{\left[\frac{1}{2} r x^2\right]_0^r}$ — Integrate numerator and denominator.

$= \frac{1}{3} r^4 \div \frac{1}{2} r^3$ — Substitute the limits ϕ and r.

$\therefore \bar{x} = \frac{2}{3} r$

b





Draw a sketch showing G vertically above the lowest point of the plane face.

Show an enlargement of ΔGXN.
$GX = OX - OG$
$= r - \frac{2}{3} r$

$\tan \alpha = \dfrac{r}{r - \frac{2}{3} r}$

$= r \div \frac{1}{3} r$

$= 3$

$\therefore \alpha = 72^\circ$ (nearest degree)

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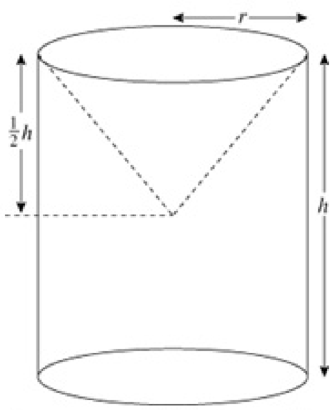
Review Exercise 2
Exercise A, Question 34

Question:



An ornament S is formed by removing a solid right circular cone, of radius r and height $\frac{1}{2}h$, from a solid uniform cylinder, of radius r and height h, as shown in the diagram.

a Show that the distance of the centre of mass S from its plane face is $\frac{17}{40}h$.

The ornament is suspended from a point on the circular rim of its open end. It hangs in equilibrium with its axis of symmetry inclined at an angle α to the horizontal. Given that $h = 4r$,

b find, in degrees to one decimal place, the value of α. **[E]**

Solution:

a

Shape	Mass	Mass ratios	Distance of centre of mass from base
Cylinder	$\pi\rho r^2 h$	1	$\frac{h}{2}$
Cone	$\frac{1}{3}\pi\rho r^2\left(\frac{h}{2}\right)$	$\frac{1}{6}$	$h-\frac{1}{4}\left(\frac{h}{2}\right)$
Ornament	$\frac{5}{6}\pi\rho r^2 h$	$\frac{5}{6}$	$\bar{x}$

Draw a table showing masses and position of centre of mass on axis of symmetry.

Take moments about O, centre of plane base:

$$1\times\frac{h}{2}-\frac{1}{6}\times\frac{7h}{8}=\frac{5}{6}\bar{x}$$

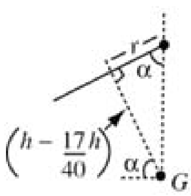
$$\therefore \frac{5}{6}\bar{x}=\frac{h}{2}-\frac{7h}{48}$$

$$=\frac{17h}{48}$$

$$\therefore \bar{x}=\frac{17h}{48}\div\frac{5}{6}=\frac{17h}{48}\times\frac{6}{5}=\frac{17h}{40}$$

Take moments and solve to find $\bar{x}$.

b



$$\tan\alpha=\frac{h-\frac{17h}{40}}{r}$$

Also $h=4r$

$$\therefore \tan\alpha=\frac{\frac{23}{40}\times 4r}{r}$$

$$=2.3$$

$$\therefore \alpha=66.5^\circ \text{ (1 d.p.)}$$

Draw a diagram showing the centre of mass G below the point of suspension.

Use $\tan\alpha$ and inverse tan to find the angle α.

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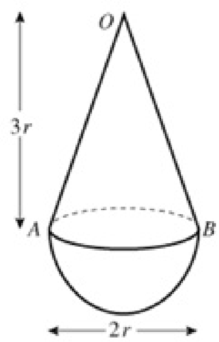
Review Exercise 2
Exercise A, Question 35

Question:



A child's toy consists of a uniform solid hemisphere, of mass M and base radius r, joined to a uniform solid right circular cone of mass m, where $2m < M$. The cone has vertex O, base radius r and height $3r$. Its plane face, with diameter AB, coincides with the plane face of the hemisphere, as shown in the diagram above.

a Show that the distance of the centre of mass of the toy from AB is $\frac{3(M-2m)}{8(M+m)}r$.

The toy is placed with OA on a horizontal surface. The toy is released from rest and does not remain in equilibrium.

b Show that $M > 26m$. **[E]**

Solution:

a

Shape	Mass	Distance of centre of mass from AB
Hemisphere	M	$+\frac{3}{8}r$
Cone	m	$-\frac{1}{4}\times 3r$
Toy	$m+M$	$\bar{x}$

Draw a table showing mass and distance of centre of mass from AB.

$\circlearrowleft M(AB)$

$$(m+M)\bar{x} = +\frac{3}{8}Mr - \frac{3}{4}mr$$

Take moments about AB.

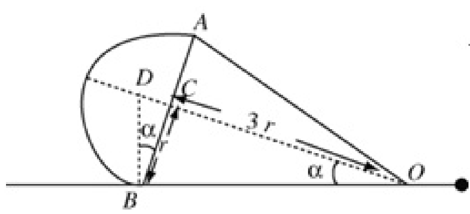
$$= \frac{3r}{8} - (2m+M)$$

$$\therefore \bar{x} = \frac{3(M-2m)}{8(M+m)}r$$

Make $\bar{x}$ the subject of the formula.

where the centre of mass is on the axis of symmetry at a distance $\bar{x}$ from AB in the direction away from O.

b



Draw a diagram with point D on the axis of symmetry above point B.

No equilibrium $\Rightarrow \bar{x} > CD$

$$\tan\alpha = \frac{r}{3r} = \frac{CD}{r}$$

This is the condition for toppling.

$$\therefore CD = \frac{1}{3}r$$

Express CD in terms of r.

So $\bar{x} > CD \Rightarrow \frac{3(M-2m)}{8(M+m)}r > \frac{1}{3}r$

i.e. $9(M-2m) > 8(M+m)$

$\therefore M > 26$ m

Substitute and express M in terms of m.

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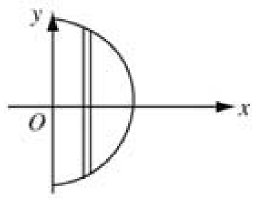
Review Exercise 2
Exercise A, Question 36

Question:

Use integration to show that the centre of mass of a uniform semi-circular lamina, of radius a, is a distance $\frac{4a}{3\pi}$ from the mid-point of its straight edge, O. A semi-circular lamina, of radius b with O as the mid-point of its straight edge, is removed from the first lamina. Show that the centre of mass of the resulting lamina is at a distance $\bar{x}$ from O, where $\bar{x} = \frac{4}{3\pi}\frac{(a^2+ab+b^2)}{(a+b)}$

Hence find the position of the centre of mass of a uniform semi-circular arc of radius a. **[E]**

Solution:



The centre of mass lies an the x-axis from symmetry.
An elemental strip of area is $2y\delta x$

$$\therefore\ \bar{x} = \frac{\rho\int 2xy\delta x}{\rho\int 2y\delta x}$$

Substitute $y = (a^2 - x^2)^{\frac{1}{2}}$ into the formula for $\bar{x}$.

The boundary of the semi-circle has equation $x^2 + y^2 = a^2\ 0 \le x \le a$

$$\therefore\ \bar{x} = \frac{\rho\int_0^a 2x(a^2 - x^2)^{\frac{1}{2}}\,dx}{\rho \times \frac{\pi a^2}{2}}$$

The denominator is the area of semi-circle times ρ. i.e. $\rho \times \frac{\pi a^2}{2}$.

$$= \frac{2}{\pi a^2}\left[-\frac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a$$

Integrate using the reverse of the chain rule.

$$= \frac{4}{3\pi a^2}\left[a^3\right] = \frac{4a}{3\pi}$$, as required.

Shape	Mass	Ratio of mass	Distance of centre of mass from O
Semi-circle radius a	$\frac{1}{2}\pi\rho a^2$	a^2	$\frac{4a}{3\pi}$
Semi-circle radius b	$\frac{1}{2}\pi\rho b^2$	b^2	$\frac{4b}{3\pi}$
Remainder	$\frac{1}{2}\pi\rho(a^2 - b^2)$	$a^2 - b^2$	$\bar{x}$

Draw a table and complete with mass ratios and distances of centres of mass from O.

$$\circlearrowleft M(O): a^2 \times \frac{4a}{3\pi} - b^2 \times \frac{4b}{3\pi} = (a^2 - b^2)\bar{x}$$

Take moments to find $\bar{x}$.

$$\therefore\ \bar{x} = \frac{4}{3\pi}\frac{\left[a^3 - b^3\right]}{(a^2 - b^2)} = \frac{4}{3\pi}\frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)}$$

$$= \frac{4}{3\pi}\frac{(a^2 + ab + b^2)}{(a+b)}$$

Factorise numerator and denominator to simplify the fraction.

As $b \to a$, the area becomes a circular arc and

$$\bar{x} \to \frac{4}{3\pi} \times \frac{3a^2}{2a} = \frac{2a}{\pi}$$

Let $b = a$ in the formula and obtain the limiting value.

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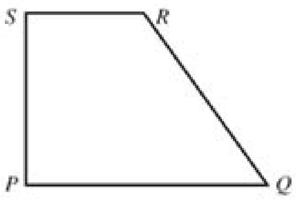
Review Exercise 2
Exercise A, Question 37

Question:

A uniform triangular lamina ABC has $\angle ABC = 90°$ and $AB = c$. Using integration show that the centre of mass of the lamina is at a distance $\frac{1}{3}c$ from BC.

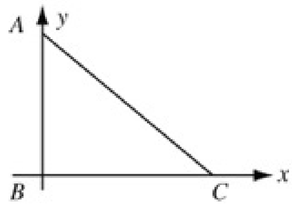


The diagram shows a uniform lamina in which $PQ = PS = 2a, SR = a$.
The centre of mass of the lamina is G.

a Show that the distance of G from PS is $\frac{7}{9}a$.

b Find the distance of G from PQ. **[E]**

Solution:



Choose B as the origin. The direction BA is the y-axis and the direction BC is the x-axis.

Let the equation of the line AC be $y = c - mx$.

Then $\bar{x} = \dfrac{\frac{1}{2}\int y^2 \,\mathrm{d}x}{\int y \,\mathrm{d}x}$

i.e. $\bar{y} = \dfrac{\frac{1}{2}\int_0^{\frac{c}{m}} (c - mx)^2 \,\mathrm{d}x}{\int_0^{\frac{c}{m}} c - mx \,\mathrm{d}x}$

$y = c - mx$ so when $y = 0$ $x = \dfrac{c}{m}$, which gives the upper limit for the integral.

$= \dfrac{\frac{1}{2}\left[\frac{-1}{3m}(c - mx)^3\right]_0^{\frac{c}{m}}}{\left[\frac{-1}{2m}(c - mx)^2\right]_0^{\frac{c}{m}}}$

Integrate numerator and denominator.

$= \dfrac{1}{6}\dfrac{c^3}{m} \div \dfrac{1}{2}\dfrac{c^2}{m}$

Substitute limits.

$= \dfrac{1}{3}c$ as required.

Shape	Mass	Position of centre of mass
Rectangle	$2a^2\rho$	$\left(\frac{a}{2}, a\right)$
Triangle	$a^2\rho$	$\left(\frac{4a}{3}, \frac{2a}{3}\right)$
Lamina	$3a^2\rho$	$(\bar{x}, \bar{y})$

Complete a table with mass and positions of centres of mass.

$$↻ 2a^2\rho\begin{pmatrix}\frac{a}{2}\\ a\end{pmatrix} + a^2\rho\begin{pmatrix}\frac{4a}{3}\\ \frac{2a}{3}\end{pmatrix} = 3a^2\rho\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\therefore \begin{pmatrix}a + \frac{4}{3}a\\ 2a + \frac{2}{3}a\end{pmatrix} = \begin{pmatrix}3\bar{x}\\ 3\bar{y}\end{pmatrix}$$

Use a vector equation an as in $M2$.

$$\therefore \bar{x} = \frac{7}{9}a,\ \bar{y} = \frac{8}{9}a$$

a Distance of G from PS is $\dfrac{7}{9}a$

b Distance of G from PQ is $\dfrac{8}{9}a$

Give the answers clearly.

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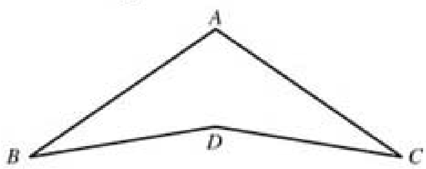
Review Exercise 2
Exercise A, Question 38

Question:

A uniform triangular lamina XYZ has $XY = XZ$ and the perpendicular distance of X from YZ is h. Prove, by integration, that the centre of mass of the lamina is at a distance $\frac{2h}{3}$ from X.



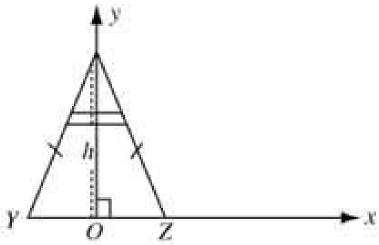
A uniform triangular lamina ABC has $AB = AC = 5a, BC = 8a$ and D is the centre of mass of the lamina. The triangle BCD is removed from the lamina, leaving the plate $ABDC$ shown in the diagram.

a Show that the distance of the centre of mass of the plate from A is $\frac{5a}{3}$.

The plate, which is of mass M, has a particle of mass M attached at B. The loaded plate is suspended from C and hangs in equilibrium.

b Prove that in this position CB makes an angle of $\arctan\frac{1}{9}$ with the vertical. **[E]**

Solution:



Let O be the mid-point of YZ.
Let OZ be the x-axis and OX the y-axis

$$\bar{y} = \frac{\int y2x \, dy}{\int 2x \, dy} \text{ where}$$

The triangle is isosceles and the y-axis is the line of symmetry.

$$y = h - mx \Rightarrow x = \frac{h-y}{m}$$

The equation describes XZ which gives x in terms of y.

$$\therefore \bar{y} = \frac{2\int_0^h y\left(\frac{h-y}{m}\right) dy}{2\int_0^h \frac{h-y}{m} dy}$$

Integrate numerator and denominator.

$$= \frac{\left[\frac{hy^2}{2m} - \frac{y^3}{3m}\right]_0^h}{\left[\frac{hy}{m} - \frac{y^2}{2m}\right]_0^h}$$

Substitute limits.

$$= \frac{h^3}{6m} \div \frac{h^2}{2m}$$

$$= \frac{h}{3}$$

You obtained distance from O.

$\therefore$ The centre of mass of the lamina is at a distance $h - \frac{h}{3} = \frac{2h}{3}$ from X.

Subtract to give distance from X.

a

Shape	Mass	Distance of centre of mass from A
ΔABC	$12\rho a^2$	$2a$
ΔBDC	$4\rho a^2$	$2a+\frac{2a}{3}$
Remainder	$8\rho a^2$	$\bar{x}$

Draw a table completing masses and positions of centres of mass.

$$\circlearrowleft M(A): 12\rho a^2 \times 2a - 4\rho a^2\left(\frac{8a}{3}\right) = 8\rho a^2\bar{x}$$

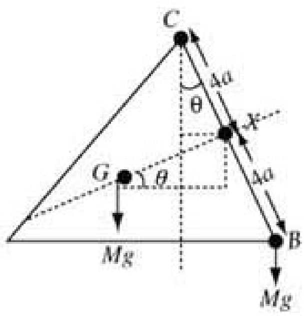
Take moments and make $\bar{x}$ the subject of the formula.

$$\therefore 24\rho a^2 - \frac{32\rho a^3}{3} = 8\rho a^2\ \bar{x}$$

$$\therefore 8\bar{x} = \frac{40a}{3} \Rightarrow \bar{x} = \frac{5a}{3}$$

Draw a diagram showing B, C and G and X the mid-point of BC

b



The distance $GX = 3a - \frac{5a}{3}$

$= \frac{4a}{3}$

$$\circlearrowleft M(C): Mg \times 8a\sin\theta = Mg\left[\frac{4a}{3}\cos\theta - 4a\sin\theta\right]$$

$$\therefore 12Mga\sin\theta = \frac{4}{3}Mga\cos\theta$$

CB makes angle θ with vertical. Also GX makes an angle θ with the horizontal.

$$\therefore \frac{\sin\theta}{\cos\theta} = \frac{4}{3} \div 12$$

$$\text{i.e. } \tan\theta = \frac{1}{9}$$

Divide by $\cos\theta$ as $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

So CB makes an angle $\arctan\left(\frac{1}{9}\right)$ with the vertical.

This is the required answer.

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Review Exercise 2
Exercise A, Question 39

Question:

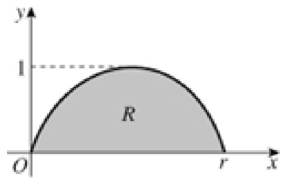


Figure 1

A uniform lamina occupies the region R bounded by the x-axis and the curve $y = \sin x, 0 \leq x \leq \pi$, as shown in Figure 1.

a Show, by integration, that the y-coordinate of the centre of mass of the lamina is $\frac{\pi}{8}$.

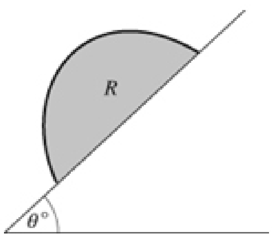


Figure 2

A uniform prism S has cross section R. The prism is placed with its rectangular face on a table which is inclined at an angle θ to the horizontal. The cross section R lies in a vertical plane as shown in Figure 2. The table is sufficiently rough to prevent S sliding. Given that S does not topple,

b find the largest possible value of θ. **[E]**

Solution:

a

$$\bar{y} = \frac{\frac{1}{2}\int y^2\, dx}{\int y\, dx}$$

Substitute $y = \sin x$ into the standard formula

$$= \frac{\frac{1}{2}\int_0^{\pi} \sin^2 x\, dx}{\int_0^{\pi} \sin x\, dx}$$

Use double angle formula to integrate $\sin^2 x$.

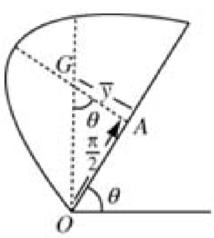
$$= \frac{\frac{1}{4}\int_0^{\pi} 1 - \cos 2x\, dx}{\left[-\cos x\right]_0^{\pi}}$$

Integrate and substitute limits.

$$= \frac{1}{4}\,\frac{\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}}{1+1}$$

$$= \frac{1}{8}[\pi] = \frac{\pi}{8}$$

b



When S is on the point of toppling, G is above O.
Let A be the point mid-way along the base of S.

$$\tan\theta = \frac{\frac{\pi}{2}}{\bar{y}} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}}$$

Use triangle GOA.

$$= 4$$

$\therefore\ \theta = 76.0^\circ$ (3 s.f.) or 76° (nearest degree)

Give your answer to 3 s.f. or to the nearest degree.

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Review Exercise 2
Exercise A, Question 40

Question:

A closed container C consists of a thin uniform hollow hemispherical bowl of radius a, together with a lid. The lid is a thin uniform circular disc, also of radius a. The centre O of the disc coincides with the centre of the hemispherical bowl. The bowl and its lid are made of the same material.

a Show that the centre of mass of C is at a distance $\frac{1}{3}a$ from O.

The container C has mass M. A particle of mass $\frac{1}{2}M$ is attached to the container at a point P on the circumference of the lid. The container is then placed with a point of its curved surface in contact with a horizontal plane. The container rests in equilibrium with P, O and the point of contact in the same vertical plane.

b Find, to the nearest degree, the angle made by the line PO with the horizontal. **[E]**

Solution:

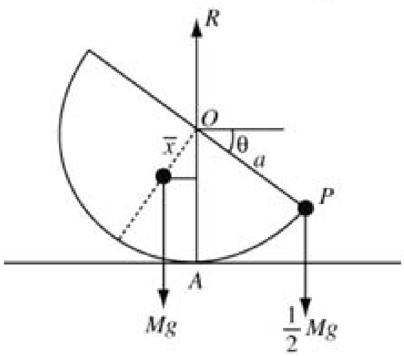
Shape	Mass	Distance of centre of mass from O
Circular disc	$\pi a^2 \rho$	0
Hemispherical bowl	$2\pi a^2 \rho$	$\frac{1}{2}a$
Closed container	$3\pi a^2 \rho$	$\bar{x}$

Draw a table with masses or mass ratios and distances of centres of mass from O.

$$\circlearrowleft M(O): 0 + 2\pi a^2 \rho \times \frac{a}{2} = 3\pi a^2 \rho \bar{x}$$

$$\therefore \bar{x} = \frac{a}{3}$$

Take moments and solve to find $\bar{x}$, the distance of the centre of mass of C from O.



Draw a diagram showing O above the point of contact with the weight of P acting to one side and the weight of C balancing on the other side.

$$\circlearrowleft M(A): Mg \times \frac{a}{3}\sin\theta = \frac{1}{2}Mg \times a\cos\theta$$

$$\therefore \tan\theta = \frac{3}{2}$$

$$\theta \approx 56^\circ$$

Use $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Give your answer to the nearest degree, as requested.

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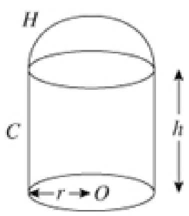
Review Exercise 2
Exercise A, Question 41

Question:



A body consists of a uniform solid circular cylinder C, together with a uniform solid hemisphere H which is attached to C. The plane face of H coincides with the upper plane face of C, as shown in the diagram. The cylinder C has base radius r, height h and mass $3M$. The mass of H is $2M$. The point O is the centre of the base of C.

a Show that the distance of the centre of mass of the body from O is $\frac{14h+3r}{20}$

The body is placed with its plane face on a rough plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{4}{3}$. The plane is sufficiently rough to prevent slipping. Given that the body is on the point of toppling,

b find h in terms of r.

Solution:

a

Shape	Mass	Distance of centre of mass from O
H	$2M$	$h+\frac{3}{8}r$
C	$3M$	$\frac{h}{2}$
Total body	$5M$	$\bar{x}$

Draw a table showing masses and positions of centres of mass.

$$\circlearrowleft M(O):\ 2M\left(h+\frac{3}{8}r\right)+3M\times\frac{h}{2}=5M\bar{x}$$

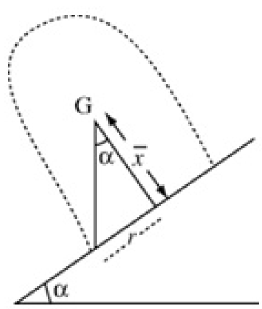
$$\therefore 5\bar{x}=2h+\frac{3}{4}r+\frac{3h}{2}$$

$$=\frac{7h}{2}+\frac{3r}{4}$$

$$\therefore \bar{x}=\frac{14h+3r}{20}$$

Take moments and make $\bar{x}$ the subject of the formula.

b



Draw a diagram showing the centre of mass G above the lowest point of the plane circular base.

$$\tan\alpha=\frac{r}{\bar{x}}=\frac{20r}{14h+3r}$$

$$\text{As }\tan\alpha=\frac{4}{3}$$

$$\therefore \frac{20r}{14h+3r}=\frac{4}{3}$$

$$\therefore 60r=56h+12r$$

$$\therefore 48r=56h$$

$$\therefore h=\frac{48}{56}r=\frac{6}{7}r$$

Use trigonometry on the triangle shown in the figure to find α.

Make h the subject of the formula.

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Review Exercise 2
Exercise A, Question 42

Question:

A bowl consists of a uniform solid metal hemisphere, of radius a and centre O, from which is removed the solid hemisphere of radius $\frac{1}{2}a$ with the same centre O.

a Show that the distance of the centre of mass of the bowl from O is $\frac{45}{112}a$.

The bowl is fixed with its plane face uppermost and horizontal. It is now filled with liquid. The mass of the bowl is M and the mass of the liquid is kM, where k is a constant. Given that the distance of the centre of mass of the bowl and liquid together from O is $\frac{17}{48}a$,

b find the value of k. **[E]**

Solution:

a

Shape	Mass	Mass ratios	Distance of centre of mass from O
Large hemisphere	$\frac{2}{3}\pi\rho a^3$	8	$\frac{3}{8}a$
Small hemisphere	$\frac{2}{3}\pi\rho\left(\frac{a}{2}\right)^3$	1	$\frac{3}{16}a$
Remainder	$\frac{2}{3}\pi\rho\frac{7a^3}{8}$	7	$\bar{x}$

Complete a table showing the mass ratios and positions of the centres of mass.

$$\circlearrowleft M(O): 8\times\frac{3}{8}a - 1\times\frac{3}{16}a = 7\bar{x}$$
$$\therefore \frac{45}{16}a = 7\bar{x}$$
$$\therefore \bar{x} = \frac{45a}{112}$$

Take moments and make $\bar{x}$ the subject of the formula.

b

Shape	Mass ratios	Distance of centre of mass from O
Bowl	M	$\frac{45}{112}a$
Liquid	kM	$\frac{3}{16}a$
Bowl + liquid	$(k+1)M$	$\frac{17a}{48}$

Complete a second table.

$$\circlearrowleft M(O): M\times\frac{45a}{112} + kM\times\frac{3}{16}a = (k+1)M\times\frac{17a}{48}$$
$$\therefore M\left(\frac{45a}{112}-\frac{17a}{48}\right) = kM\left(\frac{17a}{48}-\frac{3a}{16}\right)$$
$$\therefore k = \frac{2}{7}$$

Take moments and make k the subject of the formula.

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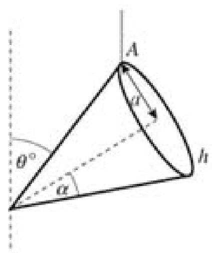
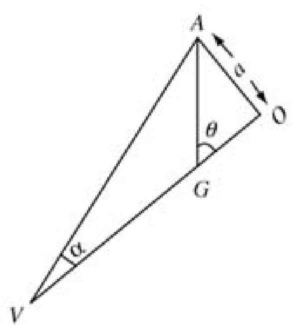
Review Exercise 2
Exercise A, Question 43

Question:



A uniform solid right circular cone has base radius a and semi-vertical angle α, where $\tan\alpha = \frac{1}{3}$. The cone is freely suspended by a string attached at a point A on the rim of its base, and hangs in equilibrium with its axis of symmetry making an angle of $\theta°$ with the upward vertical, as shown in the diagram. Find, to one decimal place, the value of θ. **[E]**

Solution:



Let V be the vertex of the cone and O be the centre of its base. Let G be the position of its centre of mass. Draw a diagram showing G vertically below A.

From ΔVAO, $\tan\alpha = \frac{OA}{OV} = \frac{a}{h}$

where h is the height of the cone.

$\therefore \frac{1}{3} = \frac{a}{h}$

Using $\tan\alpha = \frac{1}{3}$

$\therefore h = 3a$

$\therefore OG = \frac{1}{4} \times 3a$

$= \frac{3a}{4}$

Use the known result for the centre of mass of a solid cone.

Then from ΔGAO

$\tan\theta = \frac{a}{\frac{3a}{4}} = \frac{4}{3}$

$\therefore \theta = 53.1°$ (1 d.p.)

Give your answer to 1 decimal place as requested.

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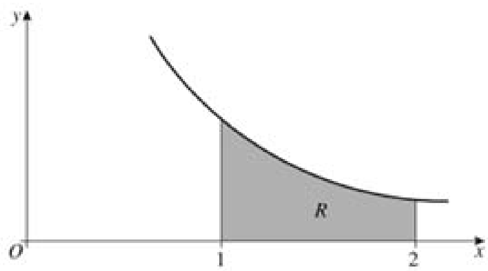
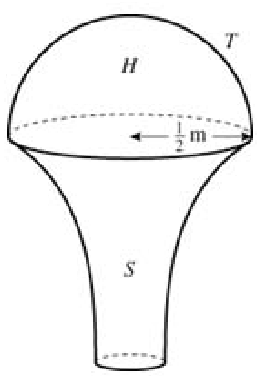
Review Exercise 2
Exercise A, Question 44

Question:



The shaded region R is bounded by the curve with equation $y = \frac{1}{2x^2}$, the x-axis and the lines $x = 1$ and $x = 2$, as shown above. The unit of length on each axis is 1 m. A uniform solid S has the shape made by rotating R through $360°$ about the x-axis.

a Show that the centre of mass of S is $\frac{2}{7}$ m from its larger plane face.



A sporting trophy T is a uniform solid hemisphere H joined to the solid S. The hemisphere has radius $\frac{1}{2}$ m and its plane face coincides with the larger plane face of S, as shown above. Both H and S are made of the same material.

b Find the distance of the centre of mass of T from its plane face. **[E]**

Solution:

a Use the formula

$$\bar{x} = \frac{\int \pi x y^2 \, dx}{\int \pi y^2 \, dx}$$

Substitute $y = \frac{1}{2x^2}$ into the standard formula.

$$= \frac{\pi \int_1^2 x \times \frac{1}{4} x^{-4} \, dx}{\pi \int_1^2 \frac{1}{4} x^{-4} \, dx}$$

Integrate and use the limits $x = 1$ and $x = 2$.

$$= \frac{\frac{1}{4} \int_1^2 x^{-3} \, dx}{\frac{1}{4} \int_1^2 x^{-4} \, dx}$$

$$= \frac{\left[\frac{-1}{2} x^{-2}\right]_1^2}{\left[\frac{-1}{3} x^{-3}\right]_1^2}$$

$$= \frac{\left[\frac{-1}{8} + \frac{1}{2}\right]}{\left[\frac{-1}{24} + \frac{1}{3}\right]}$$

$$= \frac{3}{8} \div \frac{7}{24}$$

$$= \frac{9}{7}$$

The centre of mass lies on the axis of symmetry at a distance $\frac{9}{7}$ from O.

Subtract 1 to give the required distance.

$\therefore$ The centre of mass is $\left(\frac{9}{7} - 1\right) = \frac{2}{7}$ m from the larger plane face.

b

Shape	Mass	Distance of centre of mass from common face
Solid S	$\frac{7}{96}\pi\rho$	$\frac{2}{7}$
Hemisphere H	$\frac{2}{3}\pi\rho\times\left(\frac{1}{2}\right)^3$	$\frac{-3}{8}\times\frac{1}{2}$
Trophy T	$\pi\rho\left(\frac{7}{96}+\frac{1}{12}\right)$	$\bar{x}$

Draw a table of masses and positions of centre of mass.

The mass of solid S, i.e. $\frac{7}{96}\pi\rho$ is obtained from the denominator in part **a**.

$\circlearrowleft M$ (common face):

$$\frac{7}{96}\pi\rho\times\frac{2}{7}-\frac{1}{12}\pi\rho\times\frac{3}{16}=\pi\rho\times\frac{5}{32}\bar{x}$$

$$\text{i.e. } \frac{1}{48}\pi\rho-\frac{1}{64}\pi\rho=\pi\rho\times\frac{5}{32}\bar{x}$$

$$\therefore \frac{1}{192}=\frac{5}{32}\bar{x}$$

$$\therefore \bar{x}=\frac{1}{192}\div\frac{5}{32}=\frac{1}{30}$$

This is the distance of the centre of mass from the common face.

$\therefore$ Distance of centre of mass from the plane face is $1-\frac{1}{30}=\frac{29}{30}$ m or 0.967 m.

As the height of S is 1 m subtract $\bar{x}$ from 1 m to give the required answer.

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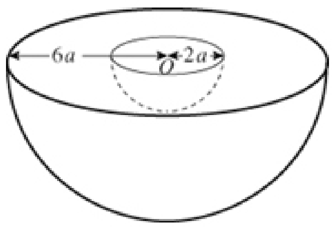
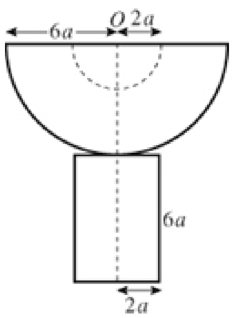
Review Exercise 2
Exercise A, Question 45

Question:



A uniform solid hemisphere, of radius $6a$ and centre O, has a solid hemisphere of radius $2a$, and centre O, removed to form a bowl B as shown.

a Show that the centre of mass of B is $\frac{30}{13}a$ from O.



The bowl B is fixed to a plane face of a uniform solid cylinder made from the same material as B. The cylinder has radius $2a$ and height $6a$ and the combined solid S has an axis of symmetry which passes through O, as shown above.

b Show that the centre of mass of S is $\frac{201}{61}a$ from O.

The plane surface of the cylindrical base of S is placed on a rough plane inclined at $12°$ to the horizontal. The plane is sufficiently rough to prevent slipping.

c Determine whether or not S will topple. **[E]**

Solution:

a

Shape	Mass	Mass ratio	Distance of centre of mass from O
Large hemisphere	$\frac{2}{3}\pi\rho(6a)^3$	27	$\frac{3}{8}\times 6a$
Small hemisphere	$\frac{2}{3}\pi\rho(2a)^3$	1	$\frac{3}{8}\times 2a$
Remainder	$\frac{2}{3}\pi\rho(6^3-2^3)a^3$	26	$\bar{x}$

Complete a table of mass and positions of centres of mass.

$$\circlearrowleft M(O): 26\bar{x} = 27\times\frac{3}{8}\times 6a - 1\times\frac{3}{8}\times 2a$$

$$\text{i.e. } 26\bar{x} = \frac{243a}{4} - \frac{3a}{4}$$

$$= 60a$$

$$\therefore \bar{x} = \frac{30a}{13}$$

Take moments and make $\bar{x}$ the subject of the formula.

b

Shape	Mass	Mass ratio	Distance of centre of mass from O
Bowl B	$\frac{416}{3}\pi\rho a^3$	52	$\frac{30a}{13}$
Cylinder	$24\pi\rho a^3$	9	$6a+3a$
Combined solid	$\frac{488}{3}\pi\rho a^3$	61	$\bar{y}$

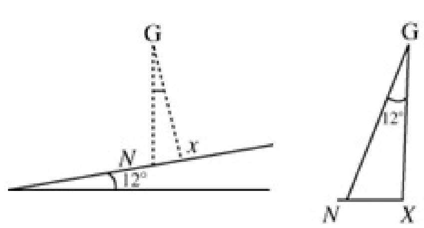
Complete a second table of mass and positions of centres of mass.

$$\circlearrowleft M(O): 52\times\frac{30a}{13} + 9\times 9a = 61\bar{y}$$

$$\therefore 120a + 81a = 61\bar{y}$$

$$\therefore \bar{y} = \frac{201}{61}a$$

c



Draw a clear diagram showing position of centre of mass G.

Let N be the point on the plane vertically below G, and let X be the centre of the cylindrical base of S.

$$GX = 12a - \frac{201}{61}a = \frac{531}{61}a$$

$$\therefore NX = GX\tan 12^{\circ} = 1.85a \text{ (3 s.f.)}$$

As $NX < 2a$, S will not topple.

S will topple when point N is outside the base circle.

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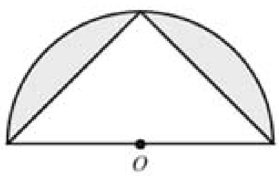
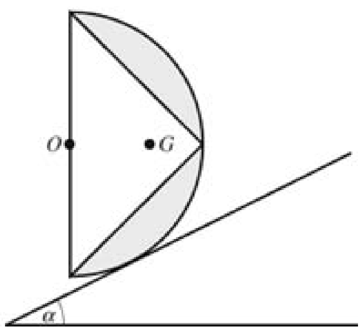
Review Exercise 2
Exercise A, Question 46

Question:



The diagram shows a cross section of a solid formed by the removal of a right circular cone, of base radius a and height a, from a uniform solid hemisphere of base radius a. The plane bases of the cone and the hemisphere are coincident, both having centre O.

Show that G, the centre of mass of the solid, is at a distance $\frac{a}{2}$ from O.



The second diagram shows a cross section of the solid resting in equilibrium with a point of its curved surface in contact with a rough inclined plane of inclination α. Given that O and G are in the same vertical plane through a line of greatest slope of the inclined plane, and that OG is horizontal, show that $\alpha = \frac{\pi}{6}$. Given that $\alpha = \frac{\pi}{6}$, find the smallest possible value of the coefficient of friction between the solid and the plane. **[E]**

Solution:

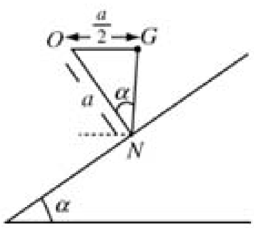
Shape	Mass	Mass ratio	Distance from O of centre of mass
Hemisphere	$\frac{2}{3}\pi\rho a^3$	2	$\frac{3}{8}a$
Cone	$\frac{1}{3}\pi\rho a^3$	1	$\frac{a}{4}$
Remainder	$\frac{1}{3}\pi\rho a^3$	1	$\bar{x}$

Draw a table.

$$\circlearrowleft M(O): 2\times\frac{3}{8}a - 1\times\frac{a}{4} = 1\bar{x}$$

$$\therefore \bar{x} = \frac{a}{2}$$

Take moments.



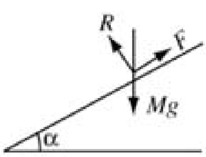
Draw a figure with the centre of mass G above the point of contact N.

From ΔOGN $\sin\alpha = \frac{\frac{a}{2}}{a} = \frac{1}{2}$

$$\therefore \alpha = \frac{\pi}{6}$$

Use trigonometry to find α.

For limiting equilibrium, when the solid is about to slip $F = \mu R$



Draw another figure showing the normal reaction force R and the friction force F.

R(↗) $F = mg\sin\alpha$

R(↖) $R = mg\cos\alpha$

$$F \le \mu R \Rightarrow mg\sin\alpha \le \mu mg\cos\alpha$$

$$\Rightarrow \mu \ge \frac{\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \mu \ge \tan\alpha$$

As $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$\alpha = \frac{\pi}{6} \Rightarrow \mu \ge \frac{1}{\sqrt{3}}$ so $\frac{1}{\sqrt{3}}$ is the smallest value of μ

Use $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

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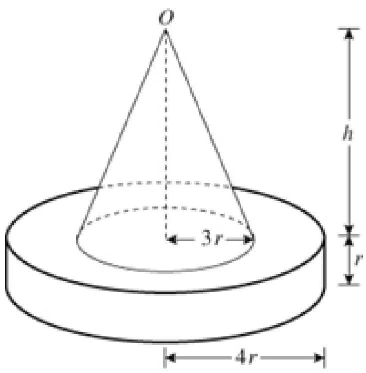
Review Exercise 2
Exercise A, Question 47

Question:



An experimental plastic traffic bollard B is made by joining a uniform solid cylinder to a uniform solid right circular cone of the same density. They are joined to form a symmetrical solid, in such a way that the centre of the plane face of the cone coincides with the centre of one of the plane faces of the cylinder, as shown in the diagram.
The cylinder has radius $4r$ and height r. The cone has vertex O, base radius $3r$ and height h.

a Show that the distance from O of the centre of mass of B is $\dfrac{32r^2+64rh+9h^2}{4(16r+3h)}$

The bollard is placed on a rough plane which is inclined at an angle α to the horizontal. The circular base of B is in contact with the inclined plane. Given that $h=4r$ and that B is on the point of toppling,

b find α, to the nearest degree. **[E]**

Solution:

a

Shape	Mass	Ratio of mass	Distance from O of centre of mass
Cone	$\frac{1}{3}\pi\rho(3r)^2 h$	$3h$	$\frac{3}{4}h$
Cylinder	$\pi\rho(4r)^2 r$	$16r$	$h+\frac{r}{2}$
Bollard	$\pi\rho(16r+3h)r^2$	$16r+3h$	$\bar{x}$

Complete the table showing mass and positions of centre of mass.

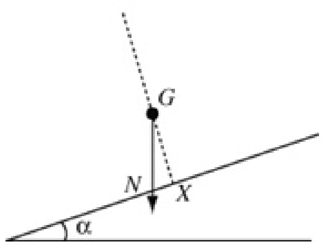
$$\circlearrowleft M(O): 3h\times\frac{3h}{4}+16r\left(h+\frac{r}{2}\right) = (16r+3h)\bar{x}$$

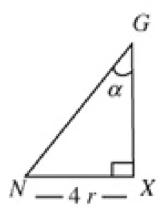
Take moments about O.

$$\therefore \frac{9h^2}{4}+16rh+8r^2 = (16r+3h)\bar{x}$$

$$\therefore \bar{x} = \frac{32r^2+64rh+9h^2}{4(16r+3h)}$$

b





Draw a diagram showing the centre of mass G above N a point on the plane which is the lowest point on the base of the bollard.

Let X be the centre point of the base

$GX = OX - \bar{x}$

Also $h = 4r, \therefore OX = 5r$ and

$$\bar{x} = \frac{32r^2+256r^2+144r^2}{4(28r)}$$

$$= \frac{432r}{112} = \frac{27r}{7}$$

$$\therefore GX = 5r - \frac{27r}{7} = \frac{8r}{7}$$

Substitute $h = 4r$ into the expression for $\bar{x}$ and evaluate $h + r - \bar{x}$.

From ΔGNX: $\tan\alpha = \dfrac{4r}{\frac{8r}{7}} = \dfrac{28}{8}$

$= 3.5$

Use trigonometry to find angle α.

$\therefore \alpha = 74°$ (nearest degree)

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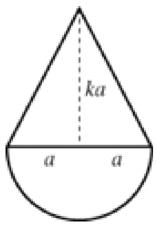
Review Exercise 2
Exercise A, Question 48

Question:

a Show, by integration, that the centre of mass of a uniform solid hemisphere, of radius R, is at a distance $\frac{3}{8}R$ from its plane face.



The diagram shows a uniform solid top made from a right circular cone of base radius a and height ka and a hemisphere of radius a. The circular plane faces of the cone and hemisphere are coincident.

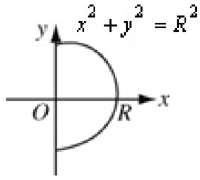
b Show that the distance of the centre of mass of the top from the vertex V of the cone is $\frac{(3k^2+8k+3)a}{4(k+2)}$

The manufacturer requires the top to have its centre of mass situated at the centre of the coincident plane faces.

c Find the value of k for this requirement. **[E]**

Solution:

a



Use $\bar{x} = \dfrac{\rho\int \pi x y^2 \, dx}{\frac{2}{3}\pi R^3 \rho}$

A hemisphere is generated when a semicircle is notated through $180°$ about the x-axis.

i.e. $\bar{x} = \dfrac{\int_0^R \pi x (R^2 - x^2)\, dx}{\frac{2}{3}\pi R^3}$

Substitute $y^2 = R^2 - x^2$ into the standard equation.

$= \left[\pi R^2 \dfrac{x^2}{2} - \dfrac{\pi x^4}{4}\right]_0^R \div \dfrac{2}{3}\pi R^3$

Integrate and substitute the limits.

$= \dfrac{\pi R^4}{4} \times \dfrac{3}{2\pi R^3}$

$= \dfrac{3}{8}R$

b

Shape	Mass	Mass ratio	Position of centre of mass
Cone	$\frac{1}{3}\pi\rho a^2 ka$	k	$\frac{3}{4}ka$
Hemisphere	$\frac{2}{3}\pi\rho a^3$	2	$ka + \frac{3}{8}a$
Top	$\frac{1}{3}\pi\rho a^3(k+2)$	$k+2$	$\bar{x}$

Draw a table with component masses and positions of centres of mass, measured from V.

$\circlearrowleft M(V): k\left(\dfrac{3}{4}ka\right) + 2\left(ka + \dfrac{3a}{8}\right) = (k+2)\bar{x}$

Take moments about V.

$\therefore \dfrac{3}{4}k^2 a + 2ka + \dfrac{3a}{4} = (k+2)\bar{x}$

i.e. $\bar{x} = \dfrac{(3k^2 + 8k + 3)a}{4(k+2)}$

c i.e. $\bar{x} = ka$

$\therefore \dfrac{3k^2 + 8k + 3}{4(k+2)} = k$

i.e. $3k^2 + 8k + 3 = 4k^2 + 8k$

$\therefore k^2 = 3$ so $k = \sqrt{3}$

Put $\bar{x} = ka$ and solve resulting equation to find k.

Solutionbank M3
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Review Exercise 2
Exercise A, Question 49

Question:

a A uniform solid hemisphere H has base radius a and the centre of its plane circular face is C.

The plane face of a second hemisphere K, of radius $\frac{a}{2}$, and made of the same material as H, is stuck to the plane face of H, so that the centres of the two plane faces coincide at C, to form a uniform composite body S. Given that the mass of K is M, show that the mass of S is $9M$, and find, in terms of a, the distance of the centre of mass of the body S from C.

b A particle P, of mass M, is attached to a point on the edge of the circular face of H of the body S. The body S with P attached is placed with a point of the curved surface of the part H in contact with a horizontal plane and rests in equilibrium. Find the tangent of the acute angle made by the line PC with the horizontal. **[E]**

Solution:

a

Shape	Mass	Distance of C of mass from C
H	$8M$	$\frac{3a}{8}$
K	M	$\frac{-3a}{16}$
S	$M+8M$	$\bar{x}$

The hemisphere K has mass M. H is a similar figure with double the radius so it has 8 times the volume and mass $8M$. S has mass $M+8M=9M$

Draw a table listing masses and positions of centres of mass.

$$\circlearrowleft M(C): 8M\times\frac{3a}{8}-M\times\frac{3a}{16}=9M\bar{x}$$

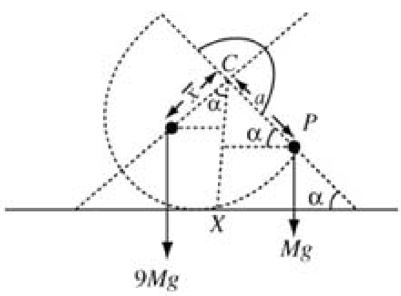
$$\therefore\ 3a-\frac{3}{16}a=9\bar{x}$$

$$\therefore\ \bar{x}=\frac{45a}{16}\div 9$$

$$=\frac{5a}{16}$$

Take moments and make $\bar{x}$ the subject of the formula.

b



Draw a diagram showing the composite body with particle P attached resting in equilibrium with C above the point of contact with the plane, X.

$$\circlearrowleft M(X): Mg\times a\cos\alpha = 9Mg\bar{x}\sin\alpha$$

Take moments about X.

$$\therefore\ \tan\alpha=\frac{a}{9\bar{x}}$$

$$=\frac{16}{45}$$

Use $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$

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Review Exercise 2
Exercise A, Question 50

Question:

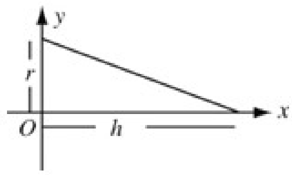
a Prove, by integration, that the position of the centre of mass of a uniform solid right circular cone is one quarter of the way up the axis from the base.

b From a uniform solid right circular cone of height H is removed a cone with the same base and height h, the two axes coinciding. Show that the centre of mass of the remaining solid S is a distance $\frac{1}{4}(3H - h)$ from the vertex of the original cone.

c The solid S is suspended by two vertical strings, one attached to the vertex and the other attached to a point on the bounding circular base. Given that S is in equilibrium, with its axis of symmetry horizontal, find, in terms of H and h, the ratio of the magnitude of the tension in the string attached to the vertex to that in the other string. **[E]**

Solution:

a



Draw a diagram and find the equation of the generator of the cone.

The line shown is rotated around the x-axis.
The triangular region generates a solid cone.

The equation of the line is $y = r - \frac{r}{h}x$

Use $\bar{x} = \frac{\int \pi x y^2 \, dx}{\int \pi y^2 \, dx}$

$$= \frac{\pi \int_0^h x\left(r^2 - \frac{2r^2}{h}x + \frac{r^2x^2}{h^2}\right)dx}{\frac{1}{3}\pi r^2 h}$$

Substitute $y = \left(r - \frac{rx}{h}\right)$ into the standard equation.

$$= \frac{3\pi}{\pi r^2 h}\int_0^h xr^2 - \frac{2r^2}{h}x^2 + \frac{r^2}{h^2}x^3 \, dx$$

Expand.

$$= \frac{3\pi}{\pi r^2 h}\left[\frac{x^2r^2}{2} - \frac{2r^2x^3}{3h} + \frac{r^2x^4}{4h^2}\right]_0^h$$

Integrate

$$= \frac{3\pi}{\pi r^2 h}\left[\frac{h^2r^2}{2} - \frac{2h^2r^2}{3h} + \frac{h^4r^2}{4h^2}\right]$$

Substitute limits.

$$= \frac{3}{r^2 h} \times \frac{h^2 r^2}{12}$$

$$= \frac{h}{4}$$

b

Shape	Mass	Mass ratio	Distance from base of centre of mass
Large cone	$\frac{1}{3}\pi\rho r^2 h$	H	$\frac{H}{4}$
Small cone	$\frac{1}{3}\pi\rho r^2 h$	h	$\frac{h}{4}$
Remainder	$\frac{1}{3}\pi\rho r^2 (H-h)$	$H-h$	$\bar{x}$

Complete the table.

$$H \times \frac{H}{4} - h \times \frac{h}{4} = (H - h)\bar{x}$$

Take moments.

$$\therefore \bar{x} = \frac{1}{4}\frac{(H^2 - h^2)}{(H - h)} = \frac{1}{4}(H + h)$$

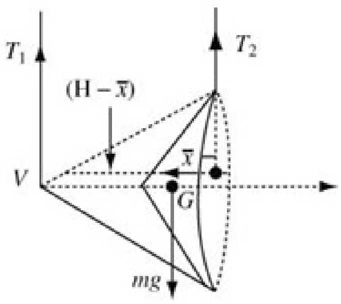
Factorise, using difference of two squares and simplify.

$\therefore$ Distance from the vertex is $H - \frac{1}{4}(H + h)$

Calculate $H - \bar{x}$.

$$= \frac{3}{4}H - \frac{1}{4}h$$

$$= \frac{1}{4}(3H - h)$$



Let the tension in the string attached to the vertex be T_1 and tension in the other string be T_2.

$$\circlearrowleft M(G): T_2\bar{x} = T_1(H - \bar{x})$$

Take moments about G, the position of the centre of mass.

$$\therefore \frac{T_1}{T_2} = \frac{\bar{x}}{H - \bar{x}}$$

$$= \frac{\frac{1}{4}(H + h)}{\frac{1}{4}(3H - h)}$$

This is distance of G from vertex.

This is distance of G from base.

$$= \frac{H + h}{3H - h}$$