Solutionbank D1
Edexcel AS and A Level Modular Mathematics

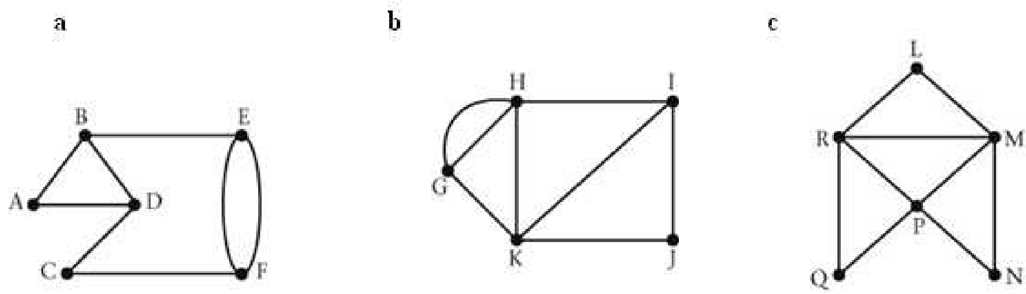
Exercise A, Question 1

Question:

List the valency of each vertex and hence determine if each of the graphs below are

i Eulerian
ii semi-Eulerian
iii neither.

For those that are Eulerian or semi-Eulerian, find a route that traverses each arc just once.



Solution:

a

vertex	A	B	C	D	E	F
valency	2	3	2	3	3	3

There are 4 nodes with odd valency so the graph is *neither* Eulerian nor Semi-Eulerian.

b

vertex	G	H	I	J	K
valency	3	4	3	2	4

There are precisely 2 nodes of odd degree (G and I) so the graph is *semi-Eulerian*.
A possible route starting at G and finishing at I is:
G – H – K – I – J – K – G – H – I.

c

vertex	L	M	N	P	Q	R
valency	2	4	2	4	2	4

All vertices have even valency, so the graph is *Eulerian*.
A possible route starting and finishing at L is:
L – M – N – P – M – R – P – Q – R – L.

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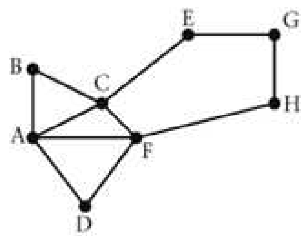
Exercise A, Question 2

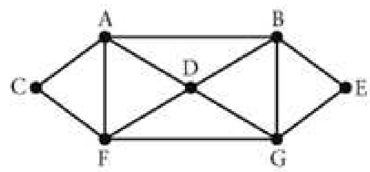
Question:

a Show that each of the graphs below is Eulerian.
b In each case, find a route that starts and finishes at A and traverses each arc just once.

i



ii



Solution:

a **i**

vertex	A	B	C	D	E	F	G	H
valency	4	2	4	2	2	4	2	2

ii

vertex	A	B	C	D	E	F	G
valency	4	4	2	4	2	4	4

b **i**

A possible route is: A – B – C – A – F – C – E – G – H – F – D – A.

ii

A possible route is: A – C – F – A – B – E – G – B – D – G – F – D – A.

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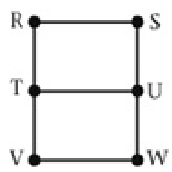
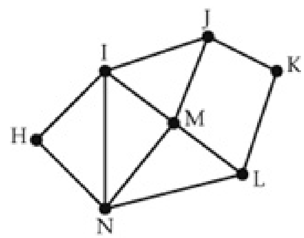
Exercise A, Question 3

Question:

i



ii



a Show that each of these graphs is semi-Eulerian.

b In each case, find a route, starting and finishing at different vertices, that traverses each edge just once.

Solution:

a **i**

vertex	R	S	T	U	V	W
valency	2	2	3	3	2	2

Precisely 2 vertices of odd valency (T and U) so semi-Eulerian.

ii

vertex	H	I	J	K	L	M	N
valency	2	4	3	2	3	4	4

Precisely 2 nodes of odd degree (J and L) so semi-Eulerian.

b **i** A possible route starting of T and finishing at U is:
T – R – S – U – W – V – T – U.

ii A possible route starts at J and finishes at L:
J – K – L – M – J – I – M – N – I – H – N – L.

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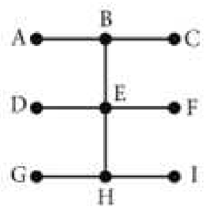
Exercise A, Question 4

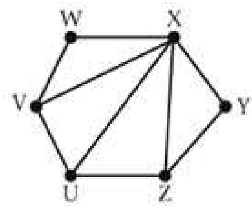
Question:

Explain why each of the graphs below is not traversable.

a



b



Solution:

a

vertex	A	B	C	D	E	F	G	H	I
valency	1	3	1	1	4	1	1	3	1

There are more than 2 vertices of odd degree so the graph is not traversable.

b

vertex	U	V	W	X	Y	Z
valency	3	3	2	5	2	3

There are more than 2 nodes of odd degree so the graph is not traversable.

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Exercise A, Question 5

Question:

Considering the valencies of each vertex in Questions **1** to **4**, verify the hand-shaking lemma.

Solution:

In each case there are either zero, or an even number of, vertices with odd valency.

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Exercise A, Question 6

Question:

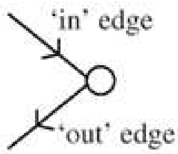
Explain why a traversable graph has either

a all of its vertices with even valency or

b precisely two vertices of odd valency, these being the start and finish points.

Solution:

If a graph is traversable we will approach each vertex on one edge and must leave on a different one.



a This means that the edges must be in pairs at each vertex, an 'out' edge for each 'in' edge, and since the graph is traversable there will be no edges left over. So the valency of each vertex will be even if we return to the start.

b For routes that start and finish at different vertices there will be an unpaired 'out' edge from the start vertex which will be balanced by an unpaired 'in' edge at the finish vertex. So these two vertices will have odd valency, but all others will be even.

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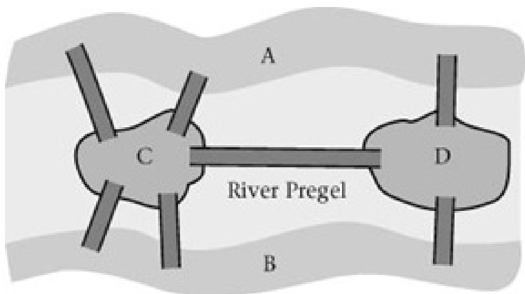
Exercise A, Question 7

Question:

The diagram represents the city of Königsberg (Prussia, now Kaliningrad, Russia). The Pregel river runs through the city and creates two large islands in the centre. The two islands (C and D) were linked to each other and the mainland (A and B) by seven bridges.



The problem for the citizens of Königsberg was to decide whether or not it was possible to walk a route that crossed each bridge just once and returned to its starting point.

This is the famous 'Bridges of Königsberg' problem.

a Using four vertices A, B, C and D to represent the four parts of the city, and seven arcs to represent the bridges, draw a graph to model the problem.

b Show that the graph is not traversable.

There is a continuation of the problem. Johannes works at A, Gregor works at B and Peter works at D. There is a hotel at C.

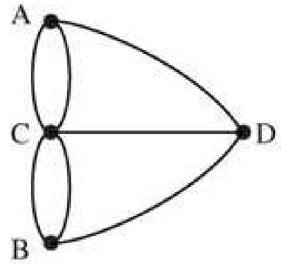
c Johannes builds an eighth bridge so that he can start at A and finish at his home at C, crossing each bridge once. However, he does not want Gregor to be able to find a similar route from B to C. Where should Johannes build his eighth bridge?

d Gregor decides to build a ninth bridge so that he can start at B and finish at his home near C, crossing each bridge once. He does not want Johannes to be able to find a similar route from A to C. Where should Gregor build his ninth bridge?

e Peter decides to build a tenth bridge, so that every person in the city can cross all the bridges in turn and return to their starting point. Where should Peter build the tenth bridge?

Solution:

a

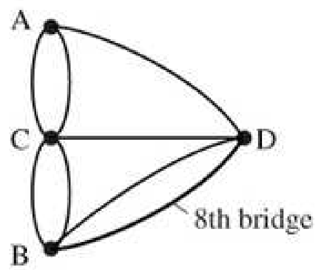


b

vertex	A	B	C	D
valency	3	3	5	3

There are more than two odd nodes, so the graph is *not* traversable.

c

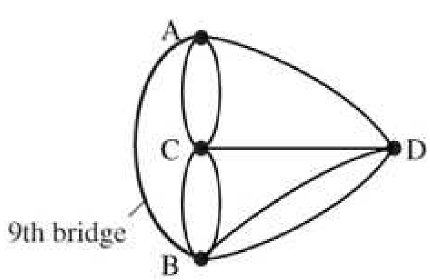


We will start of A and finish at C so these still need to have odd valency. We can only have two odd valencies so B and D must have even valencies (see table).

We need to change the valency of B and of D. So we build a bridge from B to D.

vertex	A	B	C	D
valency with 7 bridges	odd	*odd*	odd	*odd*
valency wanted	odd	*even*	odd	*even*

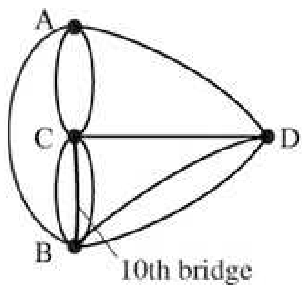
d



We will start at B and finish at C so these vertices need to be the two vertices with odd valency. We need A and D to have even valency (see table).
We need to change the valency of node A and of node B.
So we build a bridge from A to B.

vertex	A	B	C	D
valency with 8 bridges	*odd*	*even*	odd	even
valency wanted	*even*	*odd*	odd	even

e



All vertices now need to have even valency.
This means we need to change the valencies of nodes B and C.
So the 10th bridge needs to be built from B to C.

vertex	A	B	C	D
valency with 9 bridges	even	*odd*	*odd*	even
valency wanted	even	*even*	*even*	even

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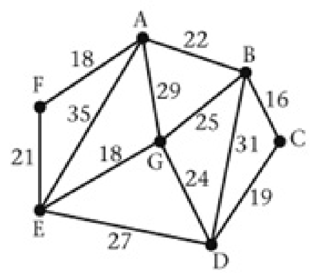
Exercise B, Question 1

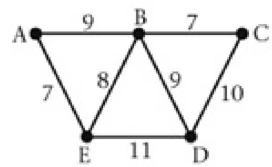
Question:

Solve the route inspection problem for each of the networks below. In each case, state your minimal route and its length.

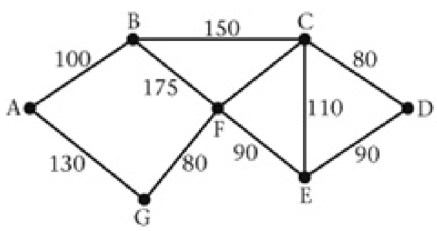
a



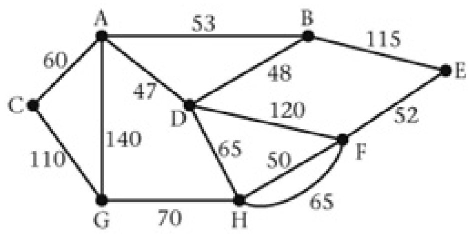
b



c

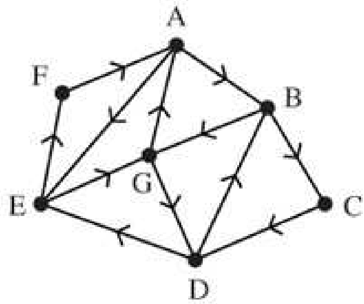


d



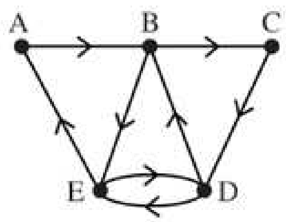
Solution:

a All valencies are even, so the network is traversable and can return to its start.



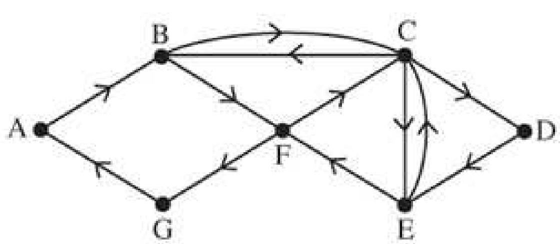
A possible route is:
A – B – C – D – B – G – D – E – G – A – E – F – A.
length of route = weight of network
= 285

b The valencies of D and E are odd, the rest are even.
We must repeat the shortest path between D and E, which is the direct path DE.
We add this extra arc to the diagram.



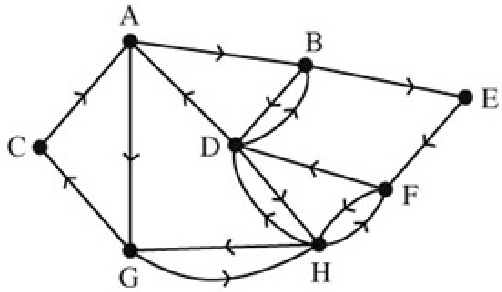
A possible route is:
A – B – C – D – E – D – B – E – A.
length of route = weight of network + arc DE
= 61 + 11
= 72

c The degrees of B and E are odd, the rest are even.
We must repeat the shortest path from B to E.
By inspection this is BCE, length 260.
We add these extra arcs to the diagram.



A possible route is:
A – B – C – D – E – C – B – F – C – E – F – G – A.
length of route = weight of network + BCE
= 1055 + 260
= 1315

d The order of B and G are odd, the rest are even.
We must repeat the shortest path from B to G.
By inspection this is BDHG, length 183.
We add these arcs to the diagram.



A possible route is:
A – B – E – F – D – B – D – A – G – H – F – H – D – H – G – C – A.

length of route = weight of network + BDHG
= 995 + 183
= 1178

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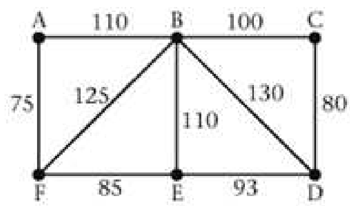
Exercise B, Question 2

Question:

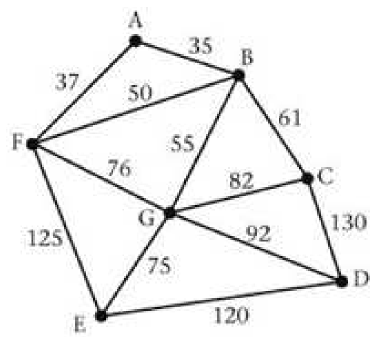
Each of the diagrams below show a network of roads that need to be inspected. In each case, find the length of the shortest route that traverses each arc at least once and returns to the start vertex. State your routes.

a



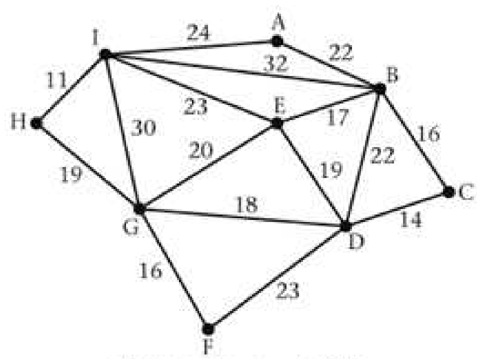
(Weight of network 908)

b



(Weight of network 938)

c



(Weight of network 326)

Solution:

a Odd valencies at B, D, E and F.

Considering all possible pairings and their weights.

$BD + EF = 130 + 85 = 215 \leftarrow$ least weight

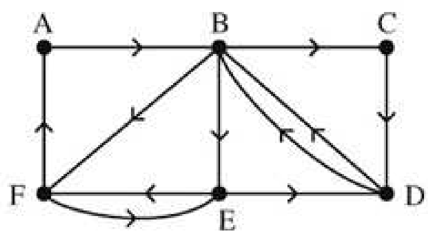
$BE + DF = 110 + 178 = 288$ ← Shortest route D to F, is DEF.

$BF + DE = 125 + 93 = 218$

We need to repeat arcs BD and EF.

$$\text{The length of the shortest route} = \text{weight of network} + 215 = 908 + 215 = 1123$$

Adding BD and EF to the diagram gives.



A possible route is:

A – B – C – D – B – E – D – B – F – E – F – A.

b Odd valencies at C, D, E and G

Considering all possible pairings and their weights

$CD + EG = 130 + 75 = 205$ ← Shortest route from C to E is CGE.

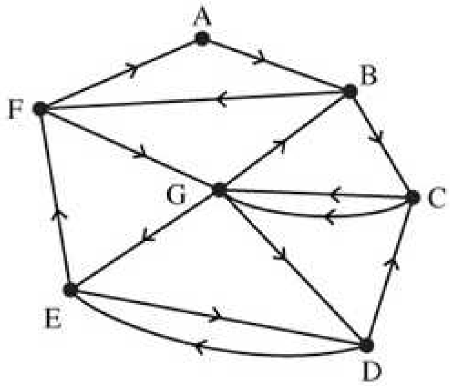
$CE + DG = 157 + 92 = 249$

$CG + DE = 82 + 120 = 202 \leftarrow$ least weight

We need to repeat arcs CG and DE.

$$\text{The length of the shortest route} = \text{weight of network} + 202 = 938 + 202 = 1140$$

Adding CG and DE to the diagram gives.



A possible route is:
A – B – C – G – D – C – G – E – D – E – F – G – B – F – A.

c Odd degrees at B, D, G and I.
Considering all possible pairings and their weight

$BD + GI = 22 + 30 = 52$

$BG + DI = 37 + 42 = 79$

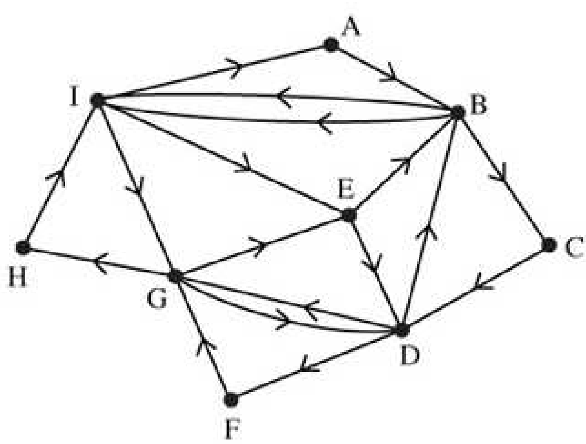
$BI + DG = 32 + 18 = 50 \leftarrow$ least weight

We need to repeat arcs BI and DG.

Shortest routes
BG = BEG
DI = DEI

The length of the shortest route = weight of network + 50
= 326 + 50
= 376

Adding BI and DG to the diagram gives.



A possible route is
A – B – C – D – B – I – E – B – I – G – D – F – G – E – D – G – H – I – A.

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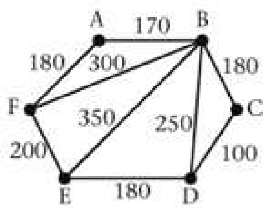
Exercise B, Question 3

Question:

The diagram shows the paths in a park. The number on each arc gives the length, in metres, of that path. The vertices show the park entrances, A, B, C, D, E and F.



A gardener needs to inspect each path for weeds.
She will walk along each path once and wishes to minimise her route.

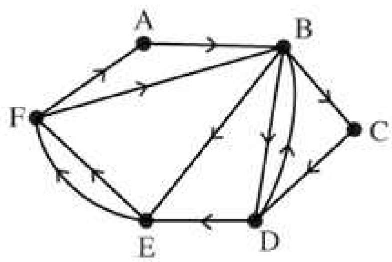
a Use the route inspection algorithm to find a minimum route, starting and finishing at entrance A. State the length of your route.

Given that it is now permitted to start and finish at two different entrances,

b find the start and finish points that would give shortest route, and state the length of the route.

Solution:

a Odd degrees are B, D, E and F.
Considering all possible pairings and their weight
$BD + EF = 250 + 200 = 450 \leftarrow$ least weight
$BE + DF = 350 + 380 = 730$
$BF + DE = 300 + 180 = 480$
We need to repeat arcs BD and EF.
Adding these to the diagram gives



A possible route is:
A – B – C – D – B – D – E – F – B – E – F – A.
length=1910+450=2360.

b We will still have two odd valencies.
We need to select the pair that gives the least path.
From part **a** our six choices are
BD (250), EF (200), BE (350), DF (380), BF (300) and DE (180).
The shortest is DE (180) so we choose to repeat this.
It is the other two vertices (B and F) that will be our start and finish.
For example, start at B, finish at F
length of route $= 1910 + 180 = 2090$

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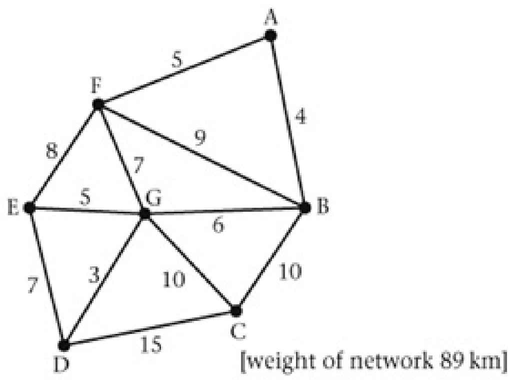
Exercise B, Question 4

Question:

The diagram represents a system of roads.



The number on each arc gives the distance, in kilometres, of that road.
The town council need to renew the road markings.
Cherry will be renewing the kerbside markings and Mac will renew the centre road markings.

Cherry needs to travel along each road twice, once on each side of the road.

a Explain how this differs from the standard route inspection problem and find the length of Cherry's route.

Mac must travel along each road once.

b Use the route inspection algorithm to find a minimal route. You should state the roads he will traverse twice and the length of his route.

Road EG is being resurfaced soon and it is decided not to renew its road markings until after the resurfacing.

Given that EG may be omitted from his route,

c find the length of Mac's minimal route.

Solution:

a Each arc must be traversed twice, whereas in the standard problem each arc need only be visited once.
This has the same effect as doubling up all the edges
The length of the route $= 2\times$ weight of network

$$= 2\times 89 = 178 \text{ km}$$

b Odd nodes C, D, E, G.
Consider all possible pairings.

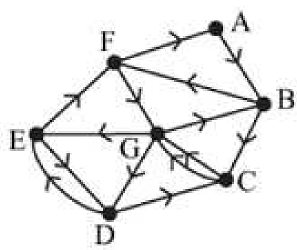
$CD + EG = 13 + 5 = 18$

$CE + DG = 15 + 3 = 18$

$CG + DE = 10 + 7 = 17 \leftarrow$ least weight

Shortest routes
CD is CGD
CE is CGE

We need to repeat arcs CG and DE.
Adding these to the network



A possible route is:
A – B – C – G – D – C – G – E – D – E – F – G – B – F – A.
Length $= 89 + 17 = 106$ km

c If EG is omitted E and G become even and the only odd valencies are at C and D.
We must repeat the shortest path between C and D, CGD.
The new length $= 89 + 13 = 102$ km

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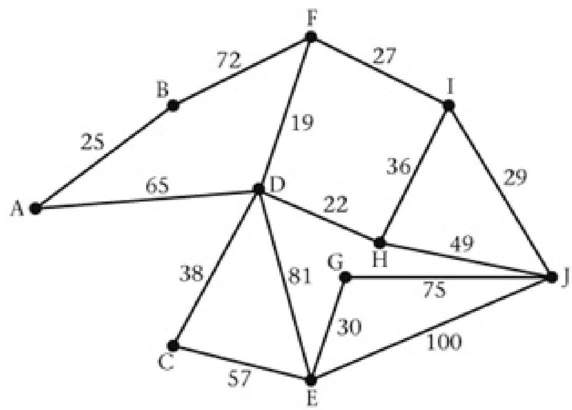
Exercise C, Question 1

Question:

The network of paths in a garden is shown below. The numbers on the paths give their lengths in metres. The gardener wishes to inspect each of the paths to check for broken paving slabs so that they can be repaired before the garden is opened to the public. The gardener has to walk along each of the paths at least once.



a Write down the degree (valency) of each of the ten vertices A, B, …, J.

b Hence find a route of minimum length. You should clearly state, with reasons, which, if any, paths will be covered twice.

c State the total length of your shortest route. ***E***

Solution:

a

vertex	A	B	C	D	E	F	G	H	I	J
degree	2	2	2	5	4	3	2	3	3	4

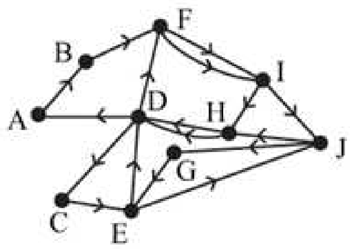
b $DF + HI = 19 + 36 = 55$

$DH + FI = 22 + 27 = 49 \leftarrow$ least weight

$DI + HF = 46 + 41 = 87$

Shortest routes
D to I is DFI
H to F is HDF

Repeat DH and FI
Add these to the network to get



A possible route is
A – B – F – I – J – G – E – J – H – D – F – I – H – D – C – E – D – A.

c length $= 725 + 49 = 774$

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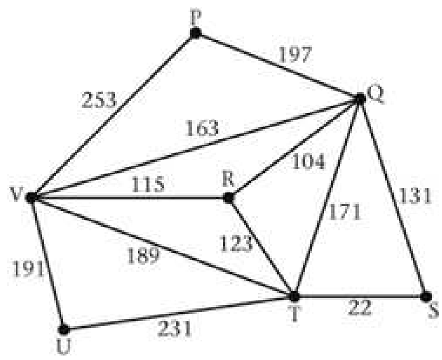
Exercise C, Question 2

Question:



Starting and finishing at P, solve the route inspection (Chinese postman) problem for the network shown above. You must make your method and working clear.
State:

a your route, using vertices to describe the arcs

b the total length of your route. ***E***

Solution:

Odd vertices Q, R, T, V.
Considering all possible pairings and their weight

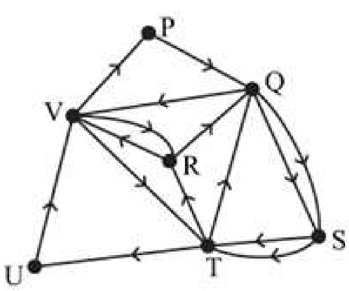
$QR + TV = 104 + 189 = 293$

$QT + RV = 153 + 115 = 268 \leftarrow$ least weight

$QV + RT = 163 + 123 = 286$

Repeat arcs QS, ST and RV.
Add these to the network

Shortest route from Q to T is QST.



A possible route is
P – Q – S – T – Q – S – T – R – Q – V – R – V – T – U – V – P.

length of route $= 1890 + 268$

$= 2158$

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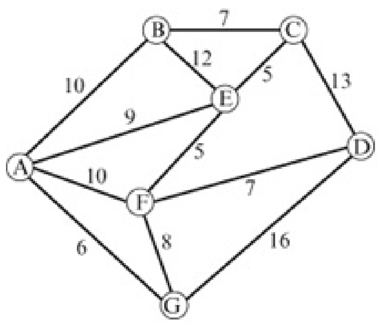
Exercise C, Question 3

Question:



The diagram shows the network of paths in a garden to be opened to the public. The number on each path gives its length in metres. The gardener wishes to inspect each of the paths to check for broken paving slabs, so that they can be repaired before the garden is opened.

a Write down the degree (valency) of the seven vertices A, B, C, D, E, F and G.

b Use an appropriate algorithm to find a route of minimum length which starts and finishes at A and which traverses each path at least once. Write down which paths, if any, will be traversed twice.

c Calculate the total length of your shortest route. ***E***

Solution:

a

vertex	A	B	C	D	E	F	G
degree	4	3	3	3	4	4	3

Odd valencies at B, C, D and G.

b Considering all possible pairings and their weight

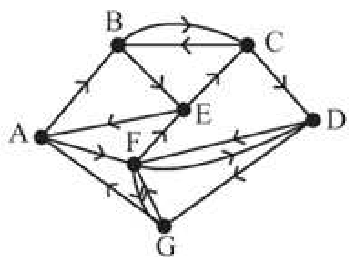
BC + DG = 7 + 15 via F = 22

BD + CG = 20 + 18 = 38

BG + CD = 16 + 13 = 29

BC, DF and FG should be repeated.
Adding these arcs to the network gives

Shortest routes
DG is DFG
BD is BCD
CG is CEFG
BG is BAG



A possible route is
A – B – C – B – E – C – D – F – D – G – F – G – A – F – E – A.

c Length = 108 + 22 = 130

Solutionbank D1

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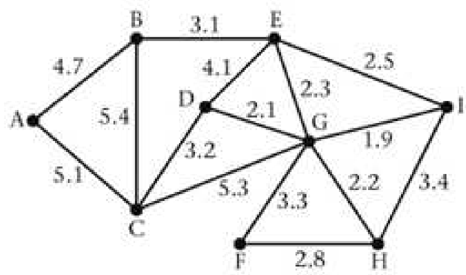
Exercise C, Question 4

Question:

The network shows the major roads that are to be gritted by a council in bad weather. The number on each arc is the length of the road in kilometres.



a List the valency of each of the vertices.

b Starting and finishing at A, use an algorithm to find a route of minimum length that covers each road at least once. You should clearly state, with reasons, which (if any) roads will be traversed twice.

c Obtain the total length of your shortest route.

There is a minor road BD (not shown) between B and D of length 6.4 km. It is not a major road so it does not need gritting urgently.

d Decide whether or not it is sensible to include BD as a part of the main gritting route, giving your reasons. (You may ignore the cost of the grit.) ***E***

Solution:

a

vertex	A	B	C	D	E	F	G	H	I
valency	2	3	4	3	4	2	6	3	3

Odd valencies at B, D, H and I.

b Considering all possible complete pairings and their weight

$BD + HI = 7.2 + 3.4 = 10.6$

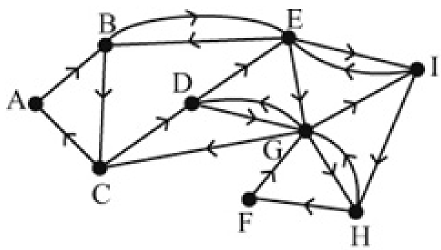
$BH + DI = 7.6 + 4 = 11.6$

$BI + DH = 5.6 + 4.3 = 9.9 \leftarrow$ least weight

Repeat BE, EI and DG, GH.

Adding these arcs to the network gives

Shortest routes
BD is BED
BH is BEGH
DI is DGI
BI is BEI
DH is DGH



A possible route is:

A – B – E – I – H – G – I – E – B – C – D – G – D – E – G – H – F – G – C – A.

c length $= 51.4 + 9.9 = 61.3$ km

d If BD is included B and D now have even valency.

Only H and I have odd valency.

So the shortest path from H to I needs to be repeated.

$$\text{Length of new route} = 51.4 + BD + \text{path from H to I}$$
$$= 51.4 + 6.4 + 3.4$$
$$= 61.2 \text{ km}$$

This is (slightly) shorter than the previous route so choose to grit BD since it saves 0.1 km.

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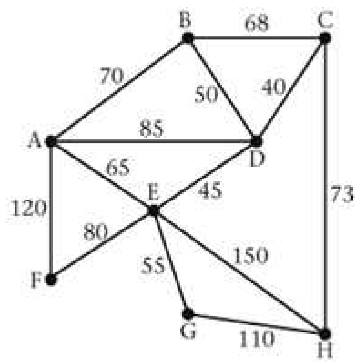
Exercise C, Question 5

Question:

The network opposite represents the streets in a village. The number on each arc represents the length of the street in metres.

The junctions have been labelled A, B, C, D, E, F, G and H.

An aerial photographer has taken photographs of the houses in the village. A salesman visits each house to see if the occupants would like to buy a photograph of their house. He needs to travel along each street at least once. He parks his car at A and starts and finishes there. He wishes to minimise the total distance he has to walk.



a Describe an appropriate algorithm that can be used to find the minimum distance the salesman needs to walk.

b Apply the algorithm and hence find a route that the salesman could take, stating the total distance he has to walk.

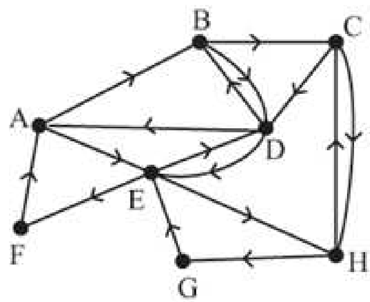
c A friend offers to drive the salesman to B at the start of the day and collect him from C later in the day.
Explaining your reasoning, carefully determine whether this would increase or decrease the total distance the salesman has to walk. ***E***

Solution:

a The route inspection algorithm
(method as shown in main text page 69)

b Odd valencies B, C, E, H.
Considering all possible complete pairings and their weight
$BC + EH = 68 + 150 = 218$
$BE + CH = 95 + 73 = 168 \leftarrow$ least weight
$BH + CE = 141 + 85 = 226$
Repeat BD, DE and CH
Adding these arcs to the network gives

Shortest routes
BE is BDE
BH is BCH, CE is CDE



A possible route is:
A − B − D − B − C − H − C − D − E − D − A − E − H − G − E − F − A.
$\text{length} = 1011 + 168$
$= 1179 \text{ m}$

c This would make B the start and C the finish.
We would have to repeat the shortest path between E and H only.
$\text{New route} = 1011 + 150 = 1161 \text{ m}$.
$1161 < 1179$
So this would decrease the total distance by 18 m.

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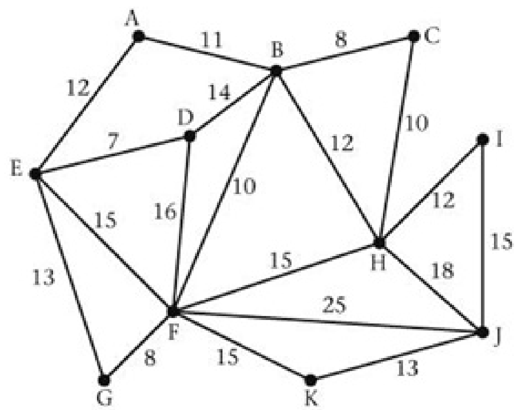
Exercise C, Question 6

Question:



a Describe an algorithm that is used to solve the route inspection (Chinese postman) problem.

b Apply an algorithm and find a route, starting and finishing at A, that solves the route inspection problem for the network shown.

c State the total length of your route.

The situation is now altered so that instead of starting and finishing at A, the route starts at one vertex and finishes at another vertex.

d **i** State the starting vertex and the finishing vertex which minimises the total length of the route. Give a reason for your selections.

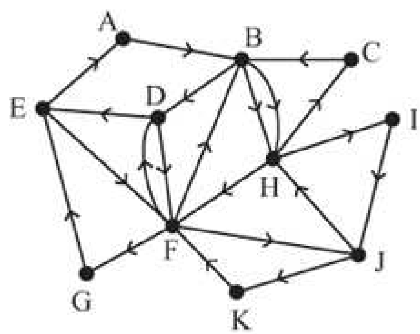
ii State the length of your route.

e Explain why, in any network, there is always an even number of vertices of odd degree. ***E***

Solution:

a The route inspection algorithm – description in main text on page 69.

b Odd vertices B, D, F, H
Considering all complete pairings
BD + FH = 14 + 15 = 29
BF + DH = 10 + 26 = 36 ← The shortest route DH is DBH.
BH + DF = 12 + 16 = 28 ← least weight
Repeat BH and DF.
Adding these arcs to the network gives



A possible route is:
A – B – H – C – B – H – I – J – H – F – J – K – F – B – D – F – D – E – F – G – E – A.

c length of route = 249 + 28 = 277

d **i** We will still have to repeat the shortest path between a pair of the odd nodes.
We will choose the pair that requires the shortest path.
The shortest path of the six is BF (10).
We will use D and H as the start and finish nodes.

ii 249 + 10 = 259

e Each edge, having two ends, contributes two to the sum of valencies for the network.
Therefore the sum = 2 × number of edges
The sum is even so any odd valencies must occur in pairs.